TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 2, pp. 473–489, April 2016
DOI: 10.11650/tjm.20.2016.6150
This paper is available online at http://journal.tms.org.tw

Radial Symmetry and Asymptotic Estimates for Positive Solutions to a Singular Integral Equation

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Abstract. In this paper, we are concerned with the nonlinear singular integral equation

$$u(x) = |x|^\sigma \int_{R^n} \frac{u^p(y)\,dy}{|x-y|^{n-\alpha}},$$

where $\alpha \in (0,n)$, $\sigma \in (\max\{-\alpha, \frac{\alpha-n}{2}\}, 0]$. Such an integral equation appears in the study of sharp constants of the Hardy-Sobolev inequality and the Hardy-Littlewood-Sobolev inequality. It is often used to describe the shapes of the extremal functions. If $0 < p \leq \frac{n}{n-\alpha-\sigma}$, there is not any positive solution to this equation. Under the assumption of $p = \frac{n+\alpha+2\sigma}{n-\alpha}$, we obtain an integrability result for the integrable solution u (i.e., $u \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$) of the integral equation. Such an integrable solution is radially symmetric and decreasing about $x_0 \in \mathbb{R}^n$. Furthermore, x_0 is also the origin if $\sigma \neq 0$. In addition, this integrable solution is blowing up with the rate $-\sigma$ when $|x| \to 0$. Moreover, if $n + p\sigma > 0$, then u decays fast with the rate $n - \alpha - \sigma$ when $|x| \to \infty$.

1. Introduction

In this paper, we are concerned with the singular integral equation

$$u(x) = |x|^\sigma \int_{\mathbb{R}^n} \frac{u^p(y)\,dy}{|x-y|^{n-\alpha}}, \quad u > 0 \text{ in } \mathbb{R}^n. \tag{1.1}$$

Here $n \geq 3$, $\sigma \in (-\alpha, 0]$, $n - \alpha + 2\sigma > 0$ and $p > 0$.

This equation is related to the sharp constant of the Hardy-Sobolev inequality

$$\int_{\mathbb{R}^n} \frac{v^p(x)\,dx}{|x|^a} \leq C \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{m}{2}} v\right|^2 dx$$

for each v belonging to the homogeneous Sobolev space $\mathcal{D}^{m,2}(\mathbb{R}^n)$, where $n > 2m$, $\max\{0, \frac{4mn-n^2}{2m}\} < a < 2m$ and $p = \frac{2(n-a)}{n-2m}$. To find the extremal functions, one can investigate the Euler-Lagrange equation

$$v(x) = \int_{\mathbb{R}^n} \frac{v^p(y)\,dy}{|y|^a|x-y|^{n-2m}}. \tag{1.2}$$

Received April 14, 2015, accepted August 21, 2015.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification.* 45E10, 45G05.
Key words and phrases. Riesz potential, Singular integral equation, Decay rates, Integrable solution, Bounded decaying solution.

The radial symmetry and the regularity of the integrable solutions were proved (cf. [19] and the references therein). Afterwards, [13] generalized those results to the case of $p > \frac{n-a}{n-2m}$. Set $u(x) = |x|^\sigma v(x)$ with $\sigma = -a/p$ and $\alpha = 2m$. Then the integral equation becomes (1.1).

Such an equation (1.1) is also related to the sharp constant in the weighted Hardy-Littlewood-Sobolev (WHLS) inequality (cf. [23])

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}\,dxdy\right| \le C_{\alpha,\beta,s,\lambda,n}\,\|f\|_r\,\|g\|_s \tag{1.3}$$

where $1 < r, s < \infty$, $0 < \lambda < n$, $\alpha+\beta \ge 0$ and $\alpha+\beta+\lambda \le n$. In addition,

$$1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r} \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{n} = 2.$$

Another form of WHLS inequality is (cf. [11])

$$\left\|\int_{\mathbb{R}^n} \frac{g(y)\,dy}{|x|^\alpha|x-y|^\lambda|y|^\beta}\right\|_p \le C\,\|g\|_s\,, \tag{1.4}$$

where $1 < s, p < \infty$, $\alpha+\beta \ge 0$, $0 < \lambda < n$, $1 + \frac{1}{p} = \frac{1}{s} + \frac{\lambda+\alpha+\beta}{n}$ and $\frac{1}{p} - \frac{\lambda}{n} < \frac{\alpha}{n} < \frac{1}{p}$. To find the extremal functions, [17] considered the following Euler-Lagrange system and proved that the ground state solutions exist

$$\begin{cases} U(x) = \dfrac{1}{|x|^\alpha}\displaystyle\int_{\mathbb{R}^n} \frac{V(y)^q}{|y|^\beta|x-y|^\lambda}\,dy \\ V(x) = \dfrac{1}{|x|^\beta}\displaystyle\int_{\mathbb{R}^n} \frac{U(y)^p}{|y|^\alpha|x-y|^\lambda}\,dy. \end{cases} \tag{1.5}$$

Setting $u(x) = |x|^{\alpha-\beta(1+1/q)}U(x)$ and $v(x) = |x|^{\beta-\alpha(1+1/p)}V(x)$, we get

$$\begin{cases} u(x) = \dfrac{1}{|x|^{\beta(1+1/q)}}\displaystyle\int_{\mathbb{R}^n} \frac{V(y)^q}{|x-y|^\lambda}\,dy \\ v(x) = \dfrac{1}{|x|^{\alpha(1+1/p)}}\displaystyle\int_{\mathbb{R}^n} \frac{U(y)^p}{|x-y|^\lambda}\,dy. \end{cases}$$

Let $p = q$, $\alpha = \beta$, $u \equiv v$ and $\sigma = -\alpha(1+1/p)$. Then the system above is also reduced to (1.1) with $n - \alpha = \lambda$.

Jin and Li [11] proved the radial symmetry of the positive solutions to (1.5) by using the method of moving planes in integral forms. Later, they established the optimal integrability of those solutions (cf. [12]). Based on these results, [14] obtained the fast decay rates when $|x| \to 0$ and $|x| \to \infty$.

Let $\sigma = 0$ in (1.1), then

$$u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)\,dy}{|x-y|^{n-\alpha}}, \quad u > 0 \text{ in } \mathbb{R}^n. \tag{1.6}$$

Lieb [17] used this integral equation to obtain an extremal function of the Hardy-Littlewood-Sobolev inequality. Papers [8] and [16] classified the positive solutions to (1.6), and hence proved that such an extremal function is also the unique critical point up to translations and scalings.

When $\alpha = 2m$ is even, then (1.6) becomes

$$(-\Delta)^m u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n.$$

Here $m \in [1, n/2)$ is an integer. The classification of the solutions has provided an important ingredient in the study of the prescribing scalar curvature problem. This scalar equation and its corresponding positive solutions were studied rather extensively (cf. [3, 5–7, 9, 15, 18, 24] and the references therein).

When $\alpha = 2m$ in (1.1), it becomes the higher-order Lane-Emden type equation with weight

$$(-\Delta)^m(|x|^{-\sigma}u(x)) = u^p(x), \quad u > 0 \text{ in } \mathbb{R}^n. \tag{1.7}$$

If $\sigma > 0$, this equation is associated with the Hénon model to study spherically symmetric clusters of stars. If $\sigma < 0$, it is related to the study of the sharp constant of the integer-order Hardy-Sobolev inequality. The quantitative properties of this type equation are also interesting in critical point theory and nonlinear elliptic equations (cf. [1, 2, 4, 20, 22]).

When $\sigma \neq 0$, it seems difficult to classify the positive solutions to (1.1). Thus, it is important to understand the shapes of solutions. In this paper, we try to show the radial symmetry, the integrability and the asymptotic rates of the positive solutions, which play key roles to describe the shapes of the positive solutions.

When $m = 1$, it was proved by Mitidieri and Pokhozhaev [21] that (1.7) has no positive solution as long as $0 < p \leq \frac{n+\sigma}{n-2}$ (see also [22]). In general, we have the following result.

Theorem 1.1. *Let $\alpha \in (0, n)$ and $\sigma \in (-\alpha, 0]$. If $0 < p \leq \frac{n}{n-\alpha-\sigma}$, then (1.1) has no positive solution.*

The positive solution u of (1.1) is called the *integrable solution* if $u \in L^{\frac{n(p-1)}{\alpha+\sigma}}(\mathbb{R}^n)$. Moreover, if p is equal to the critical exponent $\frac{n+\alpha+2\sigma}{n-\alpha}$, then $\frac{n(p-1)}{\alpha+\sigma} = \frac{2n}{n-\alpha}$. According to the results in [19], the extremal function of the Hardy-Sobolev inequality is an integrable solution.

Theorem 1.1 shows that if $u > 0$ solves (1.1), then $p > \frac{n}{n-\alpha-\sigma}$. This implies $n-\alpha-\sigma > \frac{\alpha+\sigma}{p-1}$. Therefore, the exponent $n - \alpha - \sigma$ of $|x|$ is called the fast decay rate of u, and the exponent $\frac{\alpha+\sigma}{p-1}$ is called the slow decay rate.

The following result shows that the integrable solution decays fast.

Theorem 1.2. *Let* $p = \frac{n+\alpha+2\sigma}{n-\alpha}$, *and* $u \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ *be a positive solution to* (1.1) *with*

$$\alpha \in (0, n), \quad \sigma \in (\max\left\{-\alpha, \tfrac{\alpha-n}{2}\right\}, 0]. \tag{1.8}$$

Then

(1) $u \in L^r(\mathbb{R}^n)$ *for all* $\frac{1}{r} \in (-\frac{\sigma}{n}, \frac{n-\alpha-\sigma}{n})$. *In addition, if* $n + p\sigma > 0$, *then* $u \in L^r(\mathbb{R}^n)$ *for all* $\frac{1}{r} \in (0, \frac{n-\alpha-\sigma}{n})$.

(2) $\lim\limits_{|x|\to 0} |x|^{-\sigma} u(x) = \int_{\mathbb{R}^n} |y|^{\alpha-n} u^p(y)\, dy$. *In addition, if* $n+p\sigma > 0$, *then* $\lim\limits_{|x|\to\infty} |x|^{n-\alpha-\sigma} u(x) = \int_{\mathbb{R}^n} u^p(y)\, dy$.

(3) $u(x)$ *is radially symmetric and decreasing about some point* x_0. *Moreover, if* $\sigma \neq 0$, *then* x_0 *is the origin.*

Remark 1.3. The condition $n + p\sigma > 0$ is natural in studying (1.2), if we notice $a < n$ there (cf. [19]).

Remark 1.4. Theorem 1.2 is not the simple corollary of the corresponding results of (1.2) and (1.5) (cf. [12–14, 19]). The reason is that the transformation (multiplying the power functions of $|x|$) changes not only the integrability of solutions, but also the asymptotic rates near the origin and infinity.

2. Exponents

We show a necessary condition for the existence of positive solutions to (1.1), which implies the lower bound of the exponent p. It can be called Serrin type exponent.

Theorem 2.1. *If* $p \in (0, \frac{n}{n-\alpha-\sigma}]$, *then* (1.1) *has no positive solution.*

Proof. Step 1. Suppose u is a positive solution to (1.1). We deduce a contradiction if $p \in (0, \frac{n}{n-\alpha-\sigma})$.

When $y \in B_1(0)$ and $|x| > 1$, $|x-y| < 2|x|$. In addition, $\int_{B_1(0)} u^p(y)\, dy > c > 0$. Thus, for $|x| > 1$,

$$u(x) \geq c|x|^{\alpha+\sigma-n} \int_{B_1(0)} u^p(y)\, dy \geq c|x|^{\alpha+\sigma-n}. \tag{2.1}$$

Denoting $n - \alpha - \sigma$ by b_0, for $|x| > 2$ we have

$$u(x) \geq |x|^\sigma \int_{B(x,|x|/2)} \frac{u^p(y)\, dy}{|x-y|^{n-\alpha}} \geq c|x|^{-b_1}, \quad b_1 = pb_0 - \alpha - \sigma.$$

By induction, when $|x| > R$ for some large $R > 0$, we have

$$u(x) \geq c|x|^{-b_j}, \quad b_0 = n - \alpha - \sigma, \quad b_j = pb_{j-1} - \alpha - \sigma, \quad j = 1, 2, \ldots.$$

We claim that there must be j_0 such that $b_{j_0} < 0$. In fact,

$$\begin{aligned} b_j &= p^2 b_{j-2} - (\alpha+\sigma)(p+1) = \cdots \\ &= p^j b_0 - (\alpha+\sigma)(1+p+\cdots+p^{j-1}). \end{aligned}$$

When $p = 1$, $b_j = b_0 - (\alpha+\sigma)j$. Then there exists large j_0 such that $b_{j_0} < 0$. When $p \neq 1$,

$$b_j = \left(b_0 - \frac{\alpha+\sigma}{p-1}\right) p^j + \frac{\alpha+\sigma}{p-1}.$$

If $p \in (0,1)$, $b_j \to -\frac{\alpha+\sigma}{1-p} < 0$ as $j \to \infty$. So there exists j_0 such that $b_{j_0} < 0$. If $p \in (1, \frac{n}{n-\alpha-\sigma})$. Noting $b_0 - \frac{\alpha+\sigma}{p-1} < 0$, we can find a large j_0 such that $b_{j_0} < 0$.

Hence, for fixed $x \in B_{R/2}(0)$,

$$u(x) \geq |x|^\sigma \int_{\mathbb{R}^n \setminus B_R(0)} \frac{u^p(y)\,dy}{|x-y|^{n-\alpha}} \geq C|x|^\sigma \int_R^\infty r^{\alpha - p b_{j_0}} \frac{dr}{r} = \infty.$$

It is impossible.

Step 2. Suppose u is a positive solution to (1.1). We deduce a contradiction if $p = \frac{n}{n-\alpha-\sigma}$.

For $R > 0$, denote $B_R(0)$ by B. From (1.1) we have

$$u(x) \geq \frac{|x|^\sigma}{(R+|x|)^{n-\alpha}} \int_B u^p(y)\,dy. \tag{2.2}$$

Therefore, by $p = \frac{n}{n-\alpha-\sigma}$, it follows that

$$\int_B u^p(x)\,dx \geq \int_B \frac{|x|^{p\sigma}\,dx}{(R+|x|)^{p(n-\alpha)}} \left(\int_B u^p(y)\,dy\right)^p \geq c\left(\int_B u^p(y)\,dy\right)^p. \tag{2.3}$$

Here c is independent of R. Letting $R \to \infty$, we have

$$\int_{\mathbb{R}^n} u^p(x)\,dx < \infty. \tag{2.4}$$

By (2.2), we get

$$u^p(x) \geq \frac{|x|^{p\sigma}}{(R+|x|)^{p(n-\alpha)}} \left(\int_B u^p(y)\,dy\right)^p.$$

Integrating on $A_R = B_{2R}(0) \setminus B_R(0)$ yields

$$\int_{A_R} u^p(x)\,dx \geq c\int_{A_R} \frac{|x|^{p\sigma}\,dx}{(R+|x|)^{p(n-\alpha)}} \left(\int_B u^p(y)\,dy\right)^p.$$

By $p = \frac{n}{n-\alpha-\sigma}$, it follows

$$\int_{A_R} u^p(x)\,dx \geq c\left(\int_B u^p(y)\,dy\right)^p,$$

where c is independent of R. Letting $R \to \infty$, and noting (2.4), we obtain $\int_{\mathbb{R}^n} u^p(y)\,dy = 0$, which implies $u \equiv 0$. $\square$

Next, we introduce an exponent of Sobolev type $p = \frac{n+\alpha+2\sigma}{n-\alpha}$, which ensures that the equation has *some* property of conformal invariant.

Theorem 2.2. *Let $\lambda > 0$. Eq.* (1.1) *and the norm $\|u\|_{\frac{2n}{n-\alpha}}$ is invariant under the scaling transformation $u_\lambda(x) = \lambda^\theta u(\lambda x)$, if and only if $\theta = \frac{n-\alpha}{2}$ and $p = \frac{n+\alpha+2\sigma}{n-\alpha}$.*

Proof. By (1.1),

$$u_\lambda(x) = \lambda^\theta |\lambda x|^\sigma \int_{\mathbb{R}^n} \frac{u^p(y)\, dy}{|\lambda x - y|^{n-\alpha}} = \lambda^{\alpha+\sigma-(p-1)\theta} \int_{\mathbb{R}^n} \frac{u_\lambda^p(z)\, dz}{|x-z|^{n-\alpha}|x|^{-\sigma}}.$$

Thus, u_λ solves (1.1) if and only if

$$\alpha + \sigma - (p-1)\theta = 0. \tag{2.5}$$

In addition,

$$\int_{\mathbb{R}^n} [u_\lambda(x)]^{\frac{2n}{n-\alpha}}\, dx = \lambda^{\frac{2n\theta}{n-\alpha}-n} \int_{\mathbb{R}^n} u^{\frac{2n}{n-\alpha}}(y)\, dy.$$

Thus, the norm $\|u_\lambda\|_{\frac{2n}{n-\alpha}} = \|u\|_{\frac{2n}{n-\alpha}}$ if and only if $\frac{2n\theta}{n-\alpha} - n = 0$. Combining with (2.5), we see that $\theta = \frac{n-\alpha}{2}$ and $p = \frac{n+\alpha+2\sigma}{n-\alpha}$. □

3. Integrability

Hereafter, we always assume $p = \frac{n+\alpha+2\sigma}{n-\alpha}$.

Theorem 3.1. *The integrable solution $u \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ of* (1.1) *with* (1.8) *has the following integrability*

$$u \in L^s(\mathbb{R}^n) \quad \text{for all} \quad \frac{1}{s} \in \left(\frac{-\sigma}{n}, \frac{n-\alpha-\sigma}{n}\right).$$

Proof. For $A > 0$, define

$$\begin{cases} u_A(x) = u(x), & \text{if } u(x) > A \text{ or } |x| > A, \\ u_A(x) = 0, & \text{otherwise.} \end{cases}$$

Let

$$\frac{1}{s} \in \left(\frac{-\sigma}{n}, \frac{n-\alpha-\sigma}{n}\right). \tag{3.1}$$

For $f \in L^s(\mathbb{R}^n)$, define

$$Tf(x) = \int_{\mathbb{R}^n} \frac{u_A^{p-1}(y) f(y)\, dy}{|x-y|^{n-\alpha}|x|^{-\sigma}}, \quad F(x) = \int_{\mathbb{R}^n} \frac{(u-u_A)^p(y)\, dy}{|x-y|^{n-\alpha}|x|^{-\sigma}}.$$

Clearly, u solves

$$f = Tf + F.$$

Applying the WHLS inequality (1.4) and the Hölder inequality, we get

$$\|Tf\|_s \le C \left\| u_A^{p-1} f \right\|_{\frac{ns}{n+(\alpha+\sigma)s}} \le C \|u_A\|^{p-1}_{\frac{2n}{n-\alpha}} \|f\|_s .$$

In view of $u \in L^{2n/(n-\alpha)}(\mathbb{R}^n)$, if we take A suitably large, then $C \|u_A\|^{p-1}_{\frac{2n}{n-\alpha}} < 1$. Thus, T is a contraction map from $L^s(\mathbb{R}^n)$ to itself as long as s satisfies (3.1). Moreover, $n-\alpha+2\sigma > 0$ implies $\frac{n-\alpha}{2n} \in (\frac{-\sigma}{n}, \frac{n-\alpha-\sigma}{n})$. Thus, T is also a contraction map from $L^{2n/(n-\alpha)}(\mathbb{R}^n)$ to itself.

Next, the WHLS inequality (1.4) leads to $\|F\|_s \le C \|u - u_A\|^p_{\frac{nsp}{n+(\alpha+\sigma)s}}$. According to the definition of u_A, we have $F \in L^s(\mathbb{R}^n)$ as long as s satisfies (3.1).

Using the lifting lemma (Lemma 2.1 in [12]) on the regularity, we can obtain $u \in L^s(\mathbb{R}^n)$ as long as s satisfies (3.1). □

Remark 3.2. The proof of Theorem 3.1 shows that, in order to use the WHLS inequality and the Hölder inequality, we need to choose the initial integrability $u \in L^{\frac{n(p-1)}{\alpha+\sigma}}(\mathbb{R}^n)$ even if p is not the critical exponent $\frac{n+\alpha+2\sigma}{n-\alpha}$. Such an initial number $\frac{\alpha+\sigma}{n(p-1)}$ is smaller than the right hand end point $\frac{n-\alpha-\sigma}{n}$. Otherwise, $p \le \frac{n}{n-\alpha-\sigma}$. According to Theorem 2.1, the equation has no positive solution.

Moreover, the right end point of the integrability interval in Theorem 3.1 is optimal, since $\|u\|_r = \infty$ when $\frac{1}{s} \ge \frac{n-\alpha-\sigma}{n}$. In fact, there exists $c > 0$ such that for suitably large $|x| > 2$,

$$\begin{aligned} u(x) &\ge |x|^\sigma \int_{B_1(0)} \frac{u^p(y)\,dy}{|x-y|^{n-\alpha}} \\ &\ge \frac{c}{|x|^{n-\alpha-\sigma}} \int_{B_1(0)} u^p(y)\,dy \ge \frac{c}{|x|^{n-\alpha-\sigma}}. \end{aligned} \tag{3.2}$$

This implies that

$$\int_{\mathbb{R}^n} u^s(x)\,dx \ge c \int_2^\infty \rho^{n-s(n-\alpha-\sigma)} \frac{d\rho}{\rho} = \infty, \quad \forall \frac{1}{s} \ge \frac{n-\alpha-\sigma}{n}.$$

On the other hand, the left end point of the integrability interval $(-\frac{\sigma}{n}, \frac{n-\alpha-\sigma}{n})$ may not be optimal. In fact, we have the following result.

Theorem 3.3. *If* $n+p\sigma > 0$, *then the integrable solution to* (1.1) *with* (1.8) *has the better integrability*

$$u \in L^s(\mathbb{R}^n) \quad \textit{for all} \quad \frac{1}{s} \in \left(0, \frac{n-\alpha-\sigma}{n}\right).$$

Proof. Step 1. Noting $n + p\sigma > 0$, we see $\frac{-p\sigma}{n} - \frac{\alpha+\sigma}{n} < \frac{n-\alpha-\sigma}{n}$. For

$$\frac{1}{r} \in \left(\frac{-p\sigma}{n} - \frac{\alpha+\sigma}{n}, \frac{n-\alpha-\sigma}{n}\right), \tag{3.3}$$

by Theorem 3.1 we obtain $\|u\|_{\frac{nrp}{n+r(\alpha+\sigma)}} < \infty$. In addition, (3.3) shows $\frac{nr}{n+(\alpha+\sigma)r} > 1$. We can apply the WHLS inequality (1.4) to get

$$\|u\|_r \leq C\,\|u^p\|_{\frac{nr}{n+(\alpha+\sigma)r}} = C\,\|u\|^p_{\frac{nrp}{n+r(\alpha+\sigma)}} < +\infty. \tag{3.4}$$

Thus, $u \in L^r(\mathbb{R}^n)$ for all r satisfying (3.3).

From $n-\alpha+2\sigma > 0$, we see that the integrability interval in (3.3) is larger than that in (3.1).

Step 2. Set

$$\beta_1 = -\frac{\sigma}{n}, \quad \beta_{j+1} = \beta_j p - \frac{\alpha+\sigma}{n}, \quad j = 1, 2, \ldots.$$

We claim that β_j is a monotonically decreasing sequence. First, $n-\alpha+2\sigma > 0$ implies $\beta_2 < \beta_1$. Next, suppose $\beta_k < \beta_{k-1} < \cdots < \beta_1$. We verify $\beta_{k+1} < \beta_k$. Clearly, $n-\alpha+2\sigma > 0$ also implies

$$\beta_k - \beta_{k-1} = \beta_{k-1}\frac{2(\alpha+\sigma)}{n-\alpha} - \frac{\alpha+\sigma}{n} < \beta_1\frac{2(\alpha+\sigma)}{n-\alpha} - \frac{\alpha+\sigma}{n} < 0.$$

So $\beta_{k+1} - \beta_k < 0$ established. Thus, we obtain by induction that β_j is decreasing.

Suppose this sequence has a positive lower bound, then $\lim_{j\to\infty}\beta_j$ exists and is denoted by A. Letting $j \to \infty$ in $\beta_{j+1} = \beta_j p - \frac{\alpha+\sigma}{n}$, we have $A = Ap - \frac{\alpha+\sigma}{n}$, which leads to $A = \frac{n-\alpha}{2n}$. Since $\frac{n-\alpha}{2n} > \frac{-\sigma}{n} = \beta_1$, it contradicts with the monotonicity. So sequence β_j has not the positive lower bound.

Step 3. Step 1 shows that if $u \in L^r(\mathbb{R}^n)$ for all $\frac{1}{r} \in (\beta_j, \frac{n-\alpha-\sigma}{n})$, then $u \in L^r(\mathbb{R}^n)$ for all $\frac{1}{r} \in (\beta_{j+1}, \frac{n-\alpha-\sigma}{n})$ as long as $\beta_{j+1} \geq 0$. Thus, we know the left end point of the integrability interval is zero by finite steps. $\square$

Remark 3.4. By a simple calculation, $n+p\sigma > 0$ holds as long as either $\alpha \in (0, (4\sqrt{2}-5)n]$, or $\alpha \in ((4\sqrt{2}-5)n, n)$ and $\sigma \in (\frac{-(n+\alpha)+\sqrt{\Delta}}{4}, 0]$, where $\Delta = \alpha^2 + 10n\alpha - 7n^2$.

4. Radial symmetry

In this section, we prove the radial symmetry by using the method of moving planes in integral forms which was established by Chen-Li-Ou [8].

Theorem 4.1. *Let $u \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ be a solution to (1.1) with (1.8). Then it must be radially symmetric and decreasing about some point $x_0 \in \mathbb{R}^n$. Moreover, if $\sigma \neq 0$, then $x_0 = 0$.*

Proof. For a given real number λ, define $\Sigma_\lambda = \{x = (x_1, \ldots, x_n) \mid x_1 \geq \lambda\}$. Let $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$, $u_\lambda(x) = u(x^\lambda)$.

To prove this theorem, we compare $u(x)$ with $u_\lambda(x)$ on Σ_λ. The proof consists of two steps.

Step 1. We show that there exists an $N > 0$ such that for $\lambda \leq -N$, we have

$$u(x) \geq u_\lambda(x) \quad \text{a.e. in } \Sigma_\lambda. \tag{4.1}$$

Thus we can start moving the plane from $\lambda \leq N$ to the right as long as (4.1) holds.

It is not difficult to obtain

$$\begin{aligned} u_\lambda(x) - u(x) = &\int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} - \frac{1}{|x^\lambda - y|^{n-\alpha}|x^\lambda|^{-\sigma}} \right) \left(u_\lambda^p - u^p\right) dy \\ &+ E(x,\lambda), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} E(x,\lambda) = &-\int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} - \frac{1}{|x-y|^{n-\alpha}|x^\lambda|^{-\sigma}} \right) u_\lambda^p \, dy \\ &- \int_{\Sigma_\lambda} \left(\frac{1}{|x^\lambda-y|^{n-\alpha}|x|^{-\sigma}} - \frac{1}{|x^\lambda - y|^{n-\alpha}|x^\lambda|^{-\sigma}} \right) u_\lambda^p \, dy. \end{aligned} \tag{4.3}$$

Define $\Sigma_\lambda^u = \{x \in \Sigma_\lambda \mid u(x) < u_\lambda(x)\}$, $\Sigma_\lambda^- = \Sigma_\lambda \setminus \Sigma_\lambda^u$. We show that for sufficiently negative values of λ, Σ_λ^u must be empty. Clearly, when $y \in \Sigma_\lambda$, $E(x,\lambda) \leq 0$. Thus, we obtain

$$\begin{aligned} u_\lambda(x) - u(x) &\leq \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} - \frac{1}{|x^\lambda - y|^{n-\alpha}|x^\lambda|^{-\sigma}} \right) \left(u_\lambda^p - u^p\right) dy \\ &= \int_{\Sigma_\lambda^u} \frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} (u_\lambda^p - u^p)(y) \, dy + I \\ &\leq \int_{\Sigma_\lambda^u} \frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} (u_\lambda^p - u^p)(y) \, dy, \end{aligned} \tag{4.4}$$

since

$$\begin{aligned} I = &\int_{\Sigma_\lambda^-} \left(\frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} - \frac{1}{|x^\lambda - y|^{n-\alpha}|x^\lambda|^{-\sigma}} \right) (u_\lambda^q - u^q)(y) \, dy \\ &- \int_{\Sigma_\lambda^u} \frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} (u_\lambda^p - u^p)(y) \, dy \leq 0. \end{aligned} \tag{4.5}$$

By the mean value theorem, it is easy to verify that

$$u_\lambda(x) - u(x) \leq C \int_{\Sigma_\lambda^u} \frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} u_\lambda^{p-1} (u_\lambda - u)(y) \, dy. \tag{4.6}$$

Applying the WHLS inequality (1.4) and the Hölder inequality, we get

$$\begin{aligned} \|u_\lambda(x) - u(x)\|_{L^s(\Sigma_\lambda^u)} &\leq C \left\| u_\lambda^{p-1}(u_\lambda - u) \right\|_{L^{\frac{ns}{n+s(\alpha+\sigma)}}(\Sigma_\lambda^u)} \\ &\leq C \left\| u_\lambda \right\|_{L^{\frac{2n}{n-\alpha}}(\Sigma_\lambda^u)}^{p-1} \|u_\lambda - u\|_{L^s(\Sigma_\lambda^u)}. \end{aligned} \tag{4.7}$$

Since $u \in L^s(\mathbb{R}^n)$, we can choose a sufficiently large $|N|$ with $N < 0$ such that for $\lambda \leq N < 0$, $C\|u_\lambda\|^{p-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma^u_\lambda)} \leq \frac{1}{4}$. This implies that $\|u_\lambda(x) - u(x)\|_{L^s(\Sigma^u_\lambda)} = 0$. Therefore Σ^u_λ must be of measure zero and (4.1) holds. This completes Step 1.

Step 2. Step 1 shows that we can start moving the plane continuously from $\lambda \leq R$ to the right as long as (4.1) holds. If the plane stops at $x_1 = \lambda_0$ for some $\lambda_0 < 0$, then u must be symmetric and decreasing about the plane $x_1 = \lambda_0$. Otherwise, we can move the plane all the way to $x_1 = 0$. More precisely, suppose that at a point $\lambda_0 < 0$, we have $u(x) \geq u_{\lambda_0}(x)$ but $u(x) \not\equiv u_{\lambda_0}(x)$ on Σ_{λ_0}. By the same argument as in (4.7), one can find ϵ sufficiently small, so that $u(x) \not\equiv u_\lambda(x)$ on Σ_λ for all λ in $[\lambda_0, \lambda_0 + \epsilon)$. This implies the plane can be moved further to the right.

Finally, we claim $\lambda_0 = 0$ if $\sigma \neq 0$. In fact, if $\lambda_0 < 0$, then (4.2) implies

$$
\begin{aligned}
0 &= u_{\lambda_0}(x) - u(x) \\
&= -\int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\alpha}|x|^{-\sigma}} - \frac{1}{|x-y|^{n-\alpha}|x^{\lambda_0}|^{-\sigma}} \right) u^p_{\lambda_0}\, dy \\
&\quad - \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x^{\lambda_0}-y|^{n-\alpha}|x|^{-\sigma}} - \frac{1}{|x^{\lambda_0}-y|^{n-\alpha}|x^{\lambda_0}|^{-\sigma}} \right) u^p_{\lambda_0}\, dy < 0.
\end{aligned} \tag{4.8}
$$

It is impossible. The claim is verified.

Since x_1 can be chosen arbitrarily, u is radially symmetric and decreasing. □

5. Blow up near the origin

Theorem 5.1. *Let $u \in L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n)$ be a solution to* (1.1) *with* (1.8). *Then $|x|^{-\sigma}u$ is bounded when $|x|$ is small.*

Proof. By changing the order of the integral variables, we can use the Wolff type potential to write $|x|^{-\sigma}u(x)$. Namely,

$$
\begin{aligned}
|x|^{-\sigma}u(x) &= \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}\, dy \\
&= (n-\alpha)\int_{\mathbb{R}^n} u^p(y) \left(\int_{|x-y|}^{\infty} t^{\alpha-n}\frac{dt}{t} \right) dy \\
&= (n-\alpha)\int_0^\infty \left(\frac{\int_{B_t(x)} u^p(y)\, dy}{t^{n-\alpha}} \right) \frac{dt}{t}.
\end{aligned} \tag{5.1}
$$

Thus,

$$
\begin{aligned}
|x|^{-\sigma}u(x) &= (n-\alpha)\left(\int_0^d \frac{\int_{B_t(x)} u^p(y)\, dy}{t^{n-\alpha}}\frac{dt}{t} + \int_d^\infty \frac{\int_{B_t(x)} u^p(y)\, dy}{t^{n-\alpha}}\frac{dt}{t} \right) \\
&:= (n-\alpha)(K_1 + K_2).
\end{aligned} \tag{5.2}
$$

Take $\frac{1}{k} = \frac{\epsilon-\sigma}{n}p$ with $\epsilon \in (0, \frac{1}{2})$ small enough, then $\frac{1}{pk} \in (\frac{-\sigma}{n}, \frac{n-\alpha-\sigma}{n})$. Using the Höler inequality and Theorem 3.1, we obtain

$$K_1 \leq C \int_0^d \frac{\|u\|_{kp}^p t^{n(1-\frac{1}{k})}}{t^{n-\alpha}} \frac{dt}{t} \leq C \int_0^d t^{\alpha-\frac{n}{k}} \frac{dt}{t}.$$

Noting (1.8), we can verify

$$2\sigma^2 + \sigma(n+\alpha) + n\alpha - \alpha^2 > 0. \tag{5.3}$$

This shows $\alpha k - n > 0$, and hence $K_1 < \infty$.

For $z \in B_d(x)$, by virtue of $B_t(x) \subset B_{t+\delta}(x)$, we get

$$\begin{aligned} K_2 &\leq \int_d^\infty \frac{\int_{B_{t+d}(z)} u^p(y)\,dy}{(t+d)^{n-\alpha}} \left(\frac{t+d}{t}\right)^{n-\alpha+1} \frac{d(t+d)}{t+d} \\ &\leq C \int_0^\infty \frac{\int_{B_t(z)} u^p(y)\,dy}{t^{n-\alpha}} \frac{dt}{t} \leq Cu(z)|z|^{-\sigma}. \end{aligned} \tag{5.4}$$

Combining the estimates of K_1 and K_2, for $z \in B_d(x)$, we have $u(x)|x|^{-\sigma} \leq C + Cu(z)|z|^{-\sigma}$. Then we get

$$u^s(x)|x|^{-s\sigma} \leq C + Cu^s(z)|z|^{-s\sigma}.$$

Integrating on $B_d(x)$ and noticing $u \in L^s(\mathbb{R}^n)$, we obtain

$$\int_{B_d(x)} u^s(x)|x|^{-s\sigma}\,dx \leq C \int_{B_d(x)} dz + C \int_{B_d(x)} u^s(z)|z|^{-s\sigma}\,dz.$$

So, we get

$$u^s(x)|x|^{-s\sigma} \leq C + C(|x|+d)^{-s\sigma}.$$

Thus, $|x|^{-\sigma}u(x)$ is bounded when $|x|$ is small. □

This result shows that when $|x| \to 0$, the rate of $u(x) \to \infty$ is not larger than $|x|^\sigma$. Furthermore, we also have a more accurate estimate.

Theorem 5.2. *We have*

$$\lim_{|x|\to 0} |x|^{-\sigma}u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^{n-\alpha}}\,dy.$$

Here $\int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^{n-\alpha}}\,dy$ *is a constant.*

Proof. Step 1. We claim $\int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^{n-\alpha}}\,dy < \infty$. In fact, take $\frac{1}{k} = p\frac{\varepsilon-\sigma}{n}$ with $\varepsilon \in (0, \frac{1}{2})$ small enough. Theorem 3.1 implies $\|u\|_{kp} < \infty$.

From (5.3), it follows that $1 + p\frac{\sigma}{n} > \frac{n-\alpha}{n}$. Therefore, $\int_{B_R} |y|^{k'(\alpha-n)}\,dy < \infty$. Here $\frac{1}{k'} = 1 - \frac{1}{k}$.

Thus, using the Höler inequality, we deduce that

$$\int_{B_R} \frac{u^p(y)\,dy}{|y|^{n-\alpha}} \le \left(\int_{B_R} u^{pk}(y)\,dy\right)^{\frac{1}{k}} \left(\int_{B_R} |y|^{k'(\alpha-n)}\,dy\right)^{\frac{1}{k'}} < \infty.$$

On the other hand, take $\frac{1}{pk} = \frac{n-\alpha-\sigma-\varepsilon}{n}$, then $\frac{1}{k'} = 1 - \frac{1}{k} = 1 - \frac{n-\alpha-\sigma-\varepsilon}{n}p$. According to Theorem 3.1, $\|u\|_{pk} < \infty$.

In addition, (1.8) leads to

$$2\sigma^2 + \sigma(3\alpha - n) + n\alpha - n^2 < 0,$$

which is equivalent to $1 - p\frac{n-\alpha-\sigma}{n} > \frac{n-\alpha}{n}$. Hence, $\int_{\mathbb{R}^n \setminus B_R} |y|^{k'(\alpha-n)}\,dy < \infty$.

Using the Höler inequality, we have

$$\int_{\mathbb{R}^n \setminus B_R} \frac{u^p(y)\,dy}{|y|^{n-\alpha}} \le \left(\int_{\mathbb{R}^n \setminus B_R} u^{pk}(y)\,dy\right)^{\frac{1}{k}} \left(\int_{\mathbb{R}^n \setminus B_R} |y|^{k'(\alpha-n)}\,dy\right)^{\frac{1}{k'}} < \infty.$$

Combining the estimates above, we see $\int_{\mathbb{R}^n} |y|^{\alpha-n} u^p(y)\,dy < \infty$.

Step 2. We claim that as $|x| \to 0$,

$$\left|\int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}\,dy - \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^{n-\alpha}}\,dy\right| \to 0. \tag{5.5}$$

For any fixed δ, denote $B_\delta(0)$ by B_δ. Clearly,

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}\,dy - \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^{n-\alpha}}\,dy\right| \\ &\le \int_{B_\delta} \left(\frac{u^p(y)}{|x-y|^{n-\alpha}} + \frac{u^p(y)}{|y|^{n-\alpha}}\right) dy + \int_{\mathbb{R}^n \setminus B_\delta} \left|\frac{u^p(y)}{|x-y|^{n-\alpha}} - \frac{u^p(y)}{|y|^{n-\alpha}}\right| dy \\ &=: J_1 + J_2. \end{aligned} \tag{5.6}$$

Since $|x|$ is small, $B(x, \frac{|x|}{2}) \subset B_\delta$. Using the Höler inequality, we have

$$\int_{B(x,\frac{|x|}{2})} \frac{u^p(y)}{|x-y|^{n-\alpha}}\,dy \le C\left(\int_{B(x,\frac{|x|}{2})} u^{pk}(y)\,dy\right)^{\frac{1}{k}} \left(\int_0^{\frac{|x|}{2}} \frac{\mathbb{R}^n}{r^{(n-\alpha)k'}} \frac{dr}{r}\right)^{\frac{1}{k'}},$$

where $\frac{1}{k} + \frac{1}{k'} = 1$. Take $\frac{1}{pk} = \frac{\varepsilon-\sigma}{n}$. Theorem 3.1 shows $\|u\|_{pk} < \infty$. By (5.3) we have $\frac{1}{k'} = 1 - p\frac{\varepsilon-\sigma}{n} > \frac{n-\alpha}{n}$. Then we get

$$\int_{B(x,\frac{|x|}{2})} \frac{u^p(y)}{|x-y|^{n-\alpha}}\,dy \le C|x|^{n-k'(n-\alpha)} \to 0, \quad \text{as } |x| \to 0.$$

In $B_\delta \setminus B(x, \frac{|x|}{2})$, since $|x-y| > |y| - |x| > |y| - 2|x-y|$, there holds $3|x-y| > |y|$. Then from Step 1,

$$\int_{B_\delta \setminus B(x,\frac{|x|}{2})} \frac{u^p(y)}{|x-y|^{n-\alpha}}\,dy \le C\int_{B_\delta} \frac{u^p(y)}{|y|^{n-\alpha}}\,dy \le C(\delta) \to 0, \quad \text{as } \delta \to 0.$$

Letting $|x| \to 0$ and then $\delta \to 0$, we obtain $J_1 \to 0$.

On the other hand, let $y \in \mathbb{R}^n \setminus B_\delta$. Clearly, $||y|^{n-\alpha} - |x-y|^{n-\alpha}| \to 0$ as $|x| \to 0$. In addition, $|x-y| > |y| - |x| > \delta - \frac{\delta}{2} = \frac{\delta}{2}$. Hence,

$$\frac{u^p(y)(|y|^{n-\alpha} - |x-y|^{n-\alpha})}{|x-y|^{n-\alpha}|y|^{n-\alpha}} \leq C\frac{u^p(y)}{|y|^{n-\alpha}}$$

Noting Step 1, we see that $\lim_{|x|\to 0} J_2 = 0$ by Lebesgue's dominated convergence theorem. Thus, letting $|x| \to 0$, and then $\delta \to 0$, we get (5.5). Thus, we complete the proof. □

6. Decay at infinity

Theorem 5.2 shows that the rate of $\lim_{|x|\to 0} u(x) = \infty$ is $-\sigma$. A natural problem is whether $-\sigma$ is also the decay rate of $u(x)$ when $|x| \to \infty$.

Noting (1.8), by Theorem 2.1 we have

$$-\sigma < \frac{\alpha+\sigma}{p-1} < n - \alpha - \sigma.$$

Step 1 in the proof of Theorem 2.1 shows that the rate can not be smaller than $\frac{\alpha+\sigma}{p-1}$. Therefore, $-\sigma$ is not the decay rate. Furthermore, the following theorem shows that the decay rate is the fast one $n-\alpha-\sigma$.

Theorem 6.1. *If $n + p\sigma > 0$, then*

$$\lim_{|x|\to\infty} |x|^{n-\alpha-\sigma} u(x) = \|u\|_p^p.$$

Here $\|u\|_p^p = \int_{\mathbb{R}^n} u^p(y)\,dy$ is a finite constant.

Proof. Step 1. We claim that $\|u\|_p < \infty$. In fact, from $n + p\sigma > 0$ it follows that $\frac{1}{p} > \frac{-\sigma}{n}$. On the other hand, (1.8) implies

$$\frac{1}{p} < \frac{n-\alpha-\sigma}{n}. \tag{6.1}$$

Therefore, we can deduce $u \in L^p(\mathbb{R}^n)$ by Theorem 3.1. In addition, we can also apply Theorem 3.3 to deduce $u \in L^p(\mathbb{R}^n)$.

Step 2. For fixed $R > 0$, we write

$$L_1 = \int_{B_R} u^p(y)\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}\,dy.$$

When $y \in B_R$ and $|x| \to \infty$, by Step 1,

$$u^p(y)\left(\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}} - 1\right) \leq 2u^p(y) \in L^1(\mathbb{R}^n). \tag{6.2}$$

Using Lebesgue's dominated convergence theorem, we obtain

$$\left|\int_{B_R} v^p(y)u^q(y)\left(\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}-1\right)dy\right| \to 0, \quad \text{as } |x|\to\infty.$$

This result leads to

$$\lim_{R\to\infty}\lim_{|x|\to\infty} L_1 = \int_{\mathbb{R}^n} u^p(y)\,dy.$$

Next, we write

$$L_2 = \int_{(\mathbb{R}^n\setminus B_R)\setminus B(x,|x|/2)} u^p(y)\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}\,dy.$$

Clearly, if $y\in(\mathbb{R}^n\setminus B_R)\setminus B(x,|x|/2)$, then $|x-y|\geq |x|/2$. Therefore, when $R\to\infty$,

$$L_2 \leq C\int_{\mathbb{R}^n\setminus B_R} w^{p+q}(y)\,dy \to 0.$$

We write finally

$$L_3 = \int_{B(x,|x|/2)} u^p(y)\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}\,dy.$$

According to Theorem 4.1, u is radially symmetric and decreasing about x_0. When $\sigma\neq 0$, $x_0=0$. When $\sigma=0$, we still view x_0 as the origin since x is sufficiently large. Therefore, if we denote the point $\overline{ox}\cap\partial B(x,|x|/2)$ by x_*, then

$$\frac{L_3}{|x|^{n-\alpha}} \leq u^p(x_*)\int_{B(x,|x|/2)}\frac{dy}{|x-y|^{n-\alpha}} \leq \frac{Cu^p(x_*)}{|x|^{-\alpha}}.$$

Define $\widetilde{w}(r)=u(x)$, $r=|x|$. Hence,

$$\frac{L_3}{|x|^{n-\alpha}} \leq \frac{C\widetilde{w}^p(r-|x|/2)}{|x|^{-\alpha}}. \tag{6.3}$$

On the other hand, Theorem 3.1 shows that $u\in L^t(\mathbb{R}^n)$ with $\frac{1}{t}=\frac{n-\alpha-\sigma-\epsilon}{n}$. Here $\varepsilon>0$ is sufficiently small. This integrability result, together with the decreasing property of $\widetilde{w}$, implies

$$\widetilde{w}^t(r-|x|/2)(r-3|x|/4)^n \leq C\int_{B(x_0,r-\frac{|x|}{2})\setminus B(x_0,r-\frac{3|x|}{4})} w^t(y)\,dy \leq C.$$

This inequality implies $\widetilde{w}^p(r-|x|/2)\leq C|x|^{-np/t}$. Inserting this into (6.3) and noting (6.1), we deduce that

$$L_3 \leq C|x|^{n(1-\frac{p}{t})} \to 0, \quad \text{as } |x|\to\infty.$$

Combining all the estimates of L_1, L_2 and L_3, we get

$$|x|^{n-\alpha-\sigma}u(x) = L_1+L_2+L_3 \to \int_{\mathbb{R}^n} u^p(y)\,dy$$

when $|x|\to\infty$. Thus, the proof of Theorem 6.1 is complete. $\square$

Acknowledgments

The author is grateful to Yuanyuan Kong for many discussions. This work was supported by the NSFC (No. 11471164), the NSF of Jiangsu (No. BK2012846), and the PAPD of Jiangsu Higher Education Institutions.

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