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Existence of Solutions for Modified Schrödinger-Poisson System with Critical Nonlinearity in $\mathbb{R}^3$

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Abstract. In this paper, we study the existence and multiplicity of semiclassical solutions of a modified version of the Schrödinger-Poisson system with critical nonlinearity in $\mathbb{R}^3$. Under some given conditions which are given in Section 1, we prove that the problem has at least one nontrivial solution provided that $\epsilon \le \varepsilon$ and that for any $n^* \in \mathbb{N}$, it has at least n^* pairs of solutions if $\epsilon \le \varepsilon_{n^*}$, where ε and ε_{n^*} are sufficiently small positive numbers. Moreover, these solutions $u_\epsilon \to 0$ in $H^1(\mathbb{R}^3)$ as $\epsilon \to 0$.

1. Introduction and main result

In this paper, we consider the following modified Schrödinger-Poisson system

$$(1.1)\qquad \begin{cases} -\epsilon^2\Delta u + V(x)u - \epsilon^2\Delta(u^2)u + \Phi(x)u = K(x)|u|^{22^*-2}u + h(x,u), & x \in \mathbb{R}^3, \\ -\Delta\Phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\epsilon > 0$, $22^* = \frac{4N}{N-2} = 12$, $V(x)$ is a nonnegative potential, and $K(x)$ is a bounded positive function. We assume that $V(x)$, $K(x)$ and $h(x,u)$ satisfy the following conditions:

(V_1) $V \in C(\mathbb{R}^3,\mathbb{R})$ and there is $b > 0$ such that the set $V^b := \left\{x \in \mathbb{R}^3 : V(x) < b\right\}$ has finite Lebesgue measure.

(V_2) $0 = V(0) = \min_{x\in\mathbb{R}^3} V(x) \le V(x) < M$.

(K) $K \in C(\mathbb{R}^3,\mathbb{R})$, $0 < K_1 := \inf_{x\in\mathbb{R}^3} K(x) \le K_2 := \sup_{x\in\mathbb{R}^3} K(x) < \infty$.

(h_1) $h \in C(\mathbb{R}^3 \times \mathbb{R},\mathbb{R})$, $h(x,u) = o(|u|)$ uniformly in x as $u \to 0$.

(h_2) There are $c_0 > 0$ and $q \in (2, 2^*)$ such that

$$|h(x,u)| \le c_0\left(1 + |u|^{2q-1}\right) \quad \text{for all } (x,u).$$

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(h_3) There are $a_0 > 0$ and $4 < \mu, l < 2^*$ such that $H(x,u) \geq a_0(|u|^2 + |u|)^l$ and $2\mu H(x,u) \leq h(x,u)u$ for all (x,u), where $H(x,u) = \displaystyle\int_0^u h(x,s)\,ds$.

The modified Schrödinger-Poisson system appears in an interesting physical context. According to a classical model, the interaction of a charge particle with an electro-magnetic field can be described by coupling the nonlinear Schrödinger's and Poisson's equations.

In recent years, there has been a lot of works dealing with the following Schrödinger-Poisson equations:

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u + \lambda \Phi(x)u = f(u), & x \in \mathbb{R}^3, \\ -\Delta \Phi = u^2, \quad \lim\limits_{|x|\to\infty} \Phi(x) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

In [33], Zhang studied the existence and behavior of bound states for (1.2) with $\lambda > 0$ and small $\epsilon > 0$. For $f(u) = |u|^{p-2}u$, $p \in (1,5)$, there are some results in the literature. In the case of $\epsilon = 1$, $V(x) \equiv 1$, the existence of radially symmetric positive solutions of system (1.2) was obtained by D'Aprile and Mugnai in [7] and Ruiz in [27] for $p \in (2,5)$. Azzollini and Pomponio in [3] obtained the existence of ground state solutions for $p \in (2,5)$. When $p \in (1,2)$, Ruiz in [27] obtained a nonexistence result. For $f(u) = u^p$, $\lambda \equiv 1$, the authors proved that there exist radially symmetric solutions concentrate on the spheres in [13, 16] and a positive bound state solution concentrates on the local minimum of the potential V in [15]. Ruiz and Vaira in [28] proved the existence of multi-bump solutions whose bumps concentrated around the local minimum of the potential V. The proofs explored in [28] are based on a singular perturbation, essentially a Lyapunov-Schmidt reduction method. In [14], Ianni and Ruiz have been concerned with the existence of ground and bound states for (1.2) with $\epsilon = 1$, $V(x) = 0$, $f(u) = \mu u^p$. For more related results, one can refer to [2, 5, 11, 12, 17, 18, 34] and the references therein.

Some authors researched the following modified nonlinear Schrödinger equation:

$$-\Delta u + V(x)u - \Delta(u^2)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

In [26], the existence of positive ground state solution of (1.3) with $f(x,u) = \lambda|u|^{p-1}u$ and $N = 1$ was considered by using a constrained minimization argument, with λ being the Lagrange multiplier. In [24], by a change of variables, the quasilinear problem was transformed to a semilinear one and an Orlicz space framework was used as the working space, and they were able to prove the existence of positive solutions of (1.3) with $f(x,u) = \lambda|u|^{p-1}u$ by the mountain-pass theorem, where $4 \leq p < 22^*$. In [23], by utilizing the Nehari method, Liu treated more general quasilinear problems and obtained positive and sign-changing solutions. It was shown in [23] that (1.3) has no positive solutions in $H^1(\mathbb{R}^N)$ with $u^2|\nabla u|^2 \in L^1(\mathbb{R}^N)$ if $p \geq 22^*$ and V satisfies $\nabla V(x) \cdot x \geq 0$ in $\mathbb{R}^N$. Liu and Wang in [19] extended (1.3) to the general quasilinear elliptic equations. For more related results, one can refer to [9, 10, 19–23, 25, 30, 32] and the references therein.

In [31], Yang and Ding have concerned with the existence and multiplicity of semiclassical solutions of the following quasilinear Schrödinger equation:

$$-\epsilon^2\Delta u + V(x)u - \epsilon^2\Delta(u^2)u = K(x)|u|^{22^*-2}u + h(x,u), \quad x \in \mathbb{R}^N.$$

Inspired by [31], we consider the modified Schrödinger-Poisson system with critical nonlinearity in $\mathbb{R}^3$. To the best of our knowledge, these results are new. In order to prove all the results, we mainly follow the ideas in [31]. Our proofs are based on variational methods.

The main results of this paper are as follows:

Theorem 1.1. *Let* (V$_1$), (V$_2$), (K), (h$_1$), (h$_2$) *and* (h$_3$) *hold. Then for any* $\delta > 0$*, there is* $\varepsilon_\delta > 0$ *such that if* $\epsilon \le \varepsilon_\delta$*, problem* (1.1) *has at least one nontrivial solution* u_ϵ *satisfying:*

(i) $\displaystyle \frac{1}{24}\int_{\mathbb{R}^3} K(x)|u_\epsilon|^{22^*} + \frac{\mu-4}{4}\int_{\mathbb{R}^3} H(x,u_\epsilon) \le \delta\epsilon^3$, *and*

(ii) $\displaystyle \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left(\epsilon^2|\nabla u_\epsilon|^2 + V(x)|u_\epsilon|^2\right) \le \delta\epsilon^3.$

Theorem 1.2. *Assume that* (V$_1$), (V$_2$), (K), (h$_1$), (h$_2$) *and* (h$_3$) *hold, and* $h(x,-u) = -h(x,u)$*. Then for any* $n^* \in \mathbb{N}$ *and* $\delta > 0$ *there is* $\varepsilon_{n^*\delta}$ *such that if* $\epsilon \le \varepsilon_{n^*\delta}$*, problem* (1.1) *has at least* n^* *pairs of solutions* u_ϵ*, which satisfies the estimates* (i) *and* (ii) *in Theorem* 1.1.

Our paper is organized as follows. In Section 2, we describe the analytic setting where we restate the problems in equivalent form by replacing ϵ^2 with $\frac{1}{\lambda}$ other than the usual scaling [1], because of the non-autonomy of nonlinearities. In Section 3, we show that the corresponding energy functional satisfies the Cerami condition at the levels less than $\alpha_0\lambda^{-\frac{1}{2}}$ with some $\alpha_0 > 0$ independent of λ. Then, we construct minimax levels less than $\delta\lambda^{-\frac{1}{2}}$ for all λ large enough in Section 4. We prove our main results in Section 5.

Notations:

1. The ordinary inner product between two vectors $a, b \in \mathbb{R}^3$ will be denoted by $a \cdot b$.
2. $C, \widetilde{C}, c_i$ denote generic constants, which may vary inside a chain of inequalities.
3. $|u|_p$ denotes the usual $L^p(\mathbb{R}^3)$ norm $(\int_{\mathbb{R}^3}|u|^p\,dx)^{\frac{1}{p}}$.
4. We use $O(t), o(t)$ to mean $|O(t)| \le C|t|$, $\frac{o(t)}{t} \to 0$ as $t \to 0$; $o(1)$ denotes quantities that tend to 0 as $|t| \to \infty$.

2. An equivalent variational problem

We introduce the space

$$E=\left\{u:u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}V(x)u^2<\infty\right\},$$

which is a Banach space under the scalar product

$$\langle v_1,v_2\rangle=\int_{\mathbb{R}^3}\nabla v_1\nabla v_2+\int_{\mathbb{R}^3}V(x)v_1v_2.$$

The norm induced by the product $\langle\cdot,\cdot\rangle$ is

$$\|u\|=\sqrt{\langle u,u\rangle},\quad u\in H^1(\mathbb{R}^3).$$

The space

$$D^{1,2}(\mathbb{R}^3)=\left\{u\in L^{2^*}(\mathbb{R}^3):\nabla u\in L^2(\mathbb{R}^3)\right\},$$

with the norm

$$\|u\|_{D^{1,2}}=\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^{\frac{1}{2}}.$$

By the assumption (V_1), we know that the embedding $E\hookrightarrow H^1(\mathbb{R}^3)$ is continuous (see [8, 29]). Note that the norm $\|\cdot\|$ is equivalent to the one $\|\cdot\|_\lambda$ defined by

$$\|u\|_\lambda=\left(\int_{\mathbb{R}^3}|\nabla u|^2+\lambda\int_{\mathbb{R}^3}V(x)u^2\right)^{\frac{1}{2}},$$

for each $\lambda>0$. It is obvious that for each $p\in[2,2^*]$, there is $c_p>0$ such that if $\lambda\geq 1$

$$|u|_p\leq c_p\|u\|\leq c_p\|u\|_\lambda. \tag{2.1}$$

It is well known that problem (1.1) can be reduced to a single equation with a nonlocal term. Actually, for each $u\in E\subset H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\Phi_u\in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\Phi_u=u^2$ and Φ_u can be represented by

$$\Phi_u(x)=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}\,dy. \tag{2.2}$$

Furthermore, one has

$$\int_{\mathbb{R}^3}\Phi_u u^2\leq c|u|^4_{\frac{12}{5}},$$

where $c>0$.

Substituting (2.2) into elliptic system (1.1), we can rewrite system (1.1) as the following equivalent equation

$$-\epsilon^2\Delta u+V(x)u-\epsilon^2\Delta(u^2)u+\Phi_u u=K(x)|u|^{22^*-2}u+h(x,u). \tag{2.3}$$

Let $\lambda = \epsilon^{-2}$, then (2.3) becomes

$$-\Delta u + \lambda V(x)u - \Delta(u^2)u + \lambda\Phi_u u = \lambda K(x)|u|^{22^*-2}u + \lambda h(x,u). \tag{2.4}$$

Denote

$$g(x,u) = K(x)|u|^{22^*-2}u + h(x,u) \tag{2.5}$$

and

$$G(x,u) = \int_0^u g(x,s)\,ds = \frac{1}{22^*}K(x)|u|^{22^*} + H(x,u). \tag{2.6}$$

We notice that the natural functional associated with (2.4)

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(1+2|u|^2)|\nabla u|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)|u|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3}\Phi_u|u|^2 - \lambda\int_{\mathbb{R}^3} G(x,u)$$

is not well defined in general in function space E. Because the presence of the second-order nonhomogeneous term $\Delta(u^2)u$ prevents us to work directly with the functional I_λ, which is not well defined in general in $H^1(\mathbb{R}^3)$. The other difficulty in treating this equations is the possible lack of compactness due to the unboundedness of the domain and the critical exponent growth. To overcome these difficulties that have arisen from these features, we apply an argument developed by Liu in [24] and Colin, Jeanjean in [6]. We make the change of variables by $v = f^{-1}(u)$, where f is defined by

$$\begin{aligned} f'(t) &= \frac{1}{(1+2|f(t)|^2)^{\frac{1}{2}}} && \text{on } [0,+\infty), \\ f(t) &= -f(-t) && \text{on } (-\infty,0]. \end{aligned}$$

Below we summarize the properties of f.

Lemma 2.1 (Lemma 2.3, [6,24]). *The function $f(t)$ and its derivative satisfy the following properties:*

(1) *f is uniquely defined, C^∞ and invertible;*

(2) *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$;*

(3) *$|f(t)f'(t)| < 1$ for all $t \in \mathbb{R}$;*

(4) *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*

(5) *$\frac{f(t)}{t} \to 1$ as $t \to 0$;*

(6) *$|f(t)| \le 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$;*

(7) $\frac{f(t)}{2} \leq tf'(t) \leq f(t)$ *for all* $t \geq 0$*;*

(8) $\frac{f^2(t)}{2} \leq f(t)f'(t)t \leq f^2(t)$ *for all* $t \in \mathbb{R}$*;*

(9) *there exists a positive constant* C *such that*

$$|f(t)| \geq \begin{cases} C|t|, & \text{if } |t| \leq 1, \\ C|t|^{\frac{1}{2}}, & \text{if } |t| \geq 1. \end{cases}$$

Then, after the change of variables, we obtain the following functional

$$\begin{aligned} J_\lambda(v) &:= I_\lambda(f(v)) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)|f(v)|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(v)}|f(v)|^2 - \lambda\int_{\mathbb{R}^3} G(x, f(v)), \end{aligned} \tag{2.7}$$

which is well defined in the space E and belongs C^1. Moreover, the critical points of J_λ are the weak solutions of the Euler-Lagrange equation associated with the functional J_λ given by

$$\begin{aligned} -\Delta v = {} & \lambda K(x)|f(v)|^{22^*-2} f(v)f'(v) + \lambda h(x, f(v))f'(v) - \lambda V(x)f(v)f'(v) \\ & - \lambda \Phi_{f(v)} f(v)f'(v). \end{aligned} \tag{2.8}$$

Now we can restate Theorems 1.1 and 1.2 as follows:

Theorem 2.2. *Let* (V_1), (V_2), (K), (h_1), (h_2) *and* (h_3) *hold. Then for any* $\delta > 0$, *there is* $\Lambda_\delta > 0$ *such that if* $\lambda \geq \Lambda_\delta$, *problem* (2.8) *has at least one nontrivial solution* v_λ *satisfying:*

(i) $\displaystyle \frac{1}{24}\int_{\mathbb{R}^3} K(x)|f(v_\lambda)|^{22^*} + \frac{\mu - 4}{4}\int_{\mathbb{R}^3} H(x, f(v_\lambda)) \leq \delta\lambda^{-\frac{3}{2}}$, *and*

(ii) $\displaystyle \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \left(|\nabla v_\lambda|^2 + \lambda V(x)|f(v_\lambda)|^2\right) \leq \delta\lambda^{-\frac{1}{2}}$.

Theorem 2.3. *Let* (V_1), (V_2), (K), (h_1), (h_2) *and* (h_3) *hold, and* $h(x, -u) = -h(x, u)$. *Then for any* $n^* \in \mathbb{N}$ *and* $\delta > 0$, *there is* $\Lambda_{n^*\delta} > 0$ *such that if* $\lambda \geq \Lambda_{n^*\delta}$, *problem* (2.8) *has at least* n^* *pairs of solutions* v_λ, *which satisfies the estimates* (i) *and* (ii) *in Theorem* 2.2.

3. Behaviors of Cerami sequences

Let E be a real Banach space and $J_\lambda \colon E \to \mathbb{R}$ be a function of class C^1. We say that $\{v_n\} \subset E$ is a Cerami sequence at c ($(C)_c$-sequence, for short) for J_λ if $\{v_n\}$ satisfies $J_\lambda(v_n) \to c$ and $(1 + \|v_n\|_\lambda)J'_\lambda(v_n) \to 0$, as $n \to \infty$. J_λ is said to satisfy the Cerami condition if any Cerami sequence contains a convergent subsequence. The main result of this section is the following compactness result.

Lemma 3.1. *Let* (V_1), (V_2), (K), (h_1), (h_2) *and* (h_3) *hold. Let* $\{v_n\}$ *be a* $(C)_c$ *sequence for* J_λ. *Then* $c \geq 0$ *and* $\{v_n\}$ *is bounded in* E.

Proof. In order to prove $\{v_n\}$ is bounded in E, we use the same argument in the proof of Lemma 3.1 in [31], and we only need to prove (3.2), because other estimates are the same as in [31].

Since $\{v_n\} \in E$ is a Cerami sequence for J_λ, we have

$$J_\lambda(v_n) - \frac{1}{\mu} J'_\lambda(v_n)v_n = c + o(1), \tag{3.1}$$

as $n \to \infty$. By (K), (h_3) and Lemma 2.1(7), we have

$$\begin{aligned}
& J_\lambda(v_n) - \frac{1}{\mu} J'_\lambda(v_n)v_n \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + \lambda V(x)|f(v_n)|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(v_n)}|f(v_n)|^2 \\
&\quad - \lambda \int_{\mathbb{R}^3} G(x, f(v_n)) - \frac{1}{\mu}\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + \lambda V(x) f(v_n) f'(v_n) v_n\right) \\
&\quad - \frac{\lambda}{\mu}\int_{\mathbb{R}^3} \Phi_{f(v_n)} f(v_n) f'(v_n) v_n + \frac{\lambda}{\mu}\int_{\mathbb{R}^3} g(x, f(v_n)) f'(v_n) v_n \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + \lambda V(x)|f(v_n)|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(v_n)}|f(v_n)|^2 \\
&\quad - \frac{\lambda}{\mu}\int_{\mathbb{R}^3} \Phi_{f(v_n)} f(v_n) f'(v_n) v_n - \frac{1}{\mu}\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + \lambda V(x) f(v_n) f'(v_n) v_n\right) \\
&\quad + \lambda \int_{\mathbb{R}^3} \left(\frac{1}{\mu} h(x, f(v_n)) f'(v_n) v_n - H(x, f(v_n))\right) \\
&\quad + \lambda \int_{\mathbb{R}^3} \left(\frac{1}{\mu} K(x)|f(v_n)|^{22^*-2} f(v_n) f'(v_n) v_n - \frac{1}{22^*} K(x)|f(v_n)|^{22^*}\right) \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + \lambda V(x)|f(v_n)|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(v_n)}|f(v_n)|^2 \\
&\quad - \frac{\lambda}{\mu}\int_{\mathbb{R}^3} \Phi_{f(v_n)} f(v_n) f'(v_n) v_n - \frac{1}{\mu}\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + \lambda V(x) f(v_n) f'(v_n) v_n\right) \\
&\quad + \lambda \int_{\mathbb{R}^3} \left(\frac{1}{2\mu} - \frac{1}{22^*}\right) K(x)|f(v_n)|^{22^*} \\
&\quad + \lambda \int_{\mathbb{R}^3} \left(\frac{1}{2\mu} h(x, f(v_n)) f(v_n) - H(x, f(v_n))\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + \lambda V(x)|f(v_n)|^2\right) + \lambda\left(\frac{1}{4} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \Phi_{f(v_n)}|f(v_n)|^2 \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + \lambda V(x)|f(v_n)|^2\right).
\end{aligned} \tag{3.2}$$

□

From Lemma 3.1, we know that every Cerami sequence is bounded, hence, without loss of generality, we may assume $v_n \rightharpoonup v$ in E, $L^r(\mathbb{R}^3)$ and $L^{2^*}(\mathbb{R}^3)$, $v_n \to v$ in $L^t_{\text{loc}}(\mathbb{R}^3)$ for $2 \leq t < 2^*$ and $v_n \to v$ a.e. for $x \in \mathbb{R}^3$. Obviously, v is a critical point of J_λ.

Lemma 3.2 (Lemma 3.2, [31]). *Let $s \in [2, 22^*)$, and $\{v_n\}$ be a bounded Cerami sequence. Then there is a subsequence $\{v_{n_j}\}$ such that, for each $\epsilon > 0$, there exists $r_\epsilon > 0$ such that*

$$\limsup_{j\to\infty} \int_{B_j \setminus B_r} |f(v_{n_j})|^s \le \epsilon,$$

for all $r \ge r_\epsilon$, where $B_k = \{x \in \mathbb{R}^3 : |x| \le k\}$.

For the proof of the above lemma, we refer the reader to [31].

Remark 3.3. From the proof of Lemma 3.2, we can find the same subsequence $\{v_{n_j}\}$ such that the result of Lemma 3.2 holds for both $s = 2$ and $s = q$.

Let $\eta\colon [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge 2$. Define

$$\widetilde{v}_j(x) = \eta\left(\frac{2|x|}{j}\right) v(x).$$

Clearly,

$$\|v - \widetilde{v}_j\|_\lambda \to 0 \quad \text{as } j \to \infty. \tag{3.3}$$

Then we have the following lemma which was proved in [31].

Lemma 3.4 (Lemma 3.4, [31]). *Let $\{v_{n_j}\}$ be defined in Lemma 3.2. Then we have*

$$\lim_{j\to\infty} \left| \int_{\mathbb{R}^3} \left(h(x, f(v_{n_j}))f'(v_{n_j}) - h(x, f(v_{n_j} - \widetilde{v}_j))f'(v_{n_j} - \widetilde{v}_j) - h(x, f(\widetilde{v}_j))f'(\widetilde{v}_j) \right) \phi \right| = 0$$

uniformly in $\phi \in E$ with $\|\phi\|_\lambda \le 1$.

Lemma 3.5. *Let $\{v_n\}$ be defined in Lemma 3.2. Then we have*

(i) $J_\lambda(v_n - \widetilde{v}_n) \to c - J_\lambda(v)$;

(ii) $J'_\lambda(v_n - \widetilde{v}_n) \to 0$.

Proof.

$$\begin{aligned} J_\lambda(v_n - \widetilde{v}_n) = {} & J_\lambda(v_n) - J_\lambda(\widetilde{v}_n) + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n - \nabla \widetilde{v}_n|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \widetilde{v}_n|^2 \\ & - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)|f(v_n - \widetilde{v}_n)|^2 \\ & + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)|f(\widetilde{v}_n)|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)|f(v_n)|^2 \\ & + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(v_n - \widetilde{v}_n)}|f(v_n - \widetilde{v}_n)|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(\widetilde{v}_n)}|f(\widetilde{v}_n)|^2 \\ & - \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(v_n)}|f(v_n)|^2 \\ & + \frac{\lambda}{22^*}\int_{\mathbb{R}^3} K(x)\left(|f(v_n)|^{22^*} - |f(v_n - \widetilde{v}_n)|^{22^*} - |f(\widetilde{v}_n)|^{22^*} \right) \\ & + \lambda \int_{\mathbb{R}^3} \left(H(x, f(v_n)) - H(x, f(v_n - \widetilde{v}_n)) - H(x, f(\widetilde{v}_n)) \right). \end{aligned} \tag{3.4}$$

From assumptions (h_1), (h_2), (h_3) and Lemma 2.1, similar to the proof of Lemma 3.4, it is not difficult to check that

$$\int_{\mathbb{R}^3} (H(x, f(v_n)) - H(x, f(v_n - \widetilde{v}_n)) - H(x, f(\widetilde{v}_n))) \to 0.$$

Since $V(x)$ and $K(x)$ are bounded, using Lemma 3.5 in [31] and Proposition 2.1 in [33], we obtain

$$\int_{\mathbb{R}^3} V(x)|f(v_n - \widetilde{v}_n)|^2 + \int_{\mathbb{R}^3} V(x)|f(\widetilde{v}_n)|^2 - \int_{\mathbb{R}^3} V(x)|f(v_n)|^2 \to 0, \tag{3.5}$$

$$\int_{\mathbb{R}^3} K(x) \left(|f(v_n)|^{22^*} - |f(v_n - \widetilde{v}_n)|^{22^*} - |f(\widetilde{v}_n)|^{22^*}\right) \to 0 \tag{3.6}$$

and

$$\int_{\mathbb{R}^3} \Phi_{f(v_n - \widetilde{v}_n)}|f(v_n - \widetilde{v}_n)|^2 + \int_{\mathbb{R}^3} \Phi_{f(\widetilde{v}_n)}|f(\widetilde{v}_n)|^2 - \int_{\mathbb{R}^3} \Phi_{f(v_n)}|f(v_n)|^2 \to 0. \tag{3.7}$$

From (3.4) to (3.7), together the facts that $J_\lambda(v_n) \to c$ and $J_\lambda(\widetilde{v}_n) \to J_\lambda(v)$ as $n \to \infty$, we obtain Lemma 3.5(i). To prove (ii), note that, for any $\phi \in E$,

$$\begin{aligned}
&J'_\lambda(v_n - \widetilde{v}_n)\phi \\
&= J'_\lambda(v_n)\phi - J'_\lambda(\widetilde{v}_n)\phi + \int_{\mathbb{R}^3} (\nabla(v_n - \widetilde{v}_n) + \nabla\widetilde{v}_n - \nabla v_n) \nabla\phi \\
&\quad + \lambda \int_{\mathbb{R}^3} V(x) \left(f(v_n - \widetilde{v}_n)f'(v_n - \widetilde{v}_n) + f(\widetilde{v}_n)f'(\widetilde{v}_n) - f(v_n)f'(v_n)\right) \phi \\
&\quad + \lambda \int_{\mathbb{R}^3} \left(\Phi_{f(v_n - \widetilde{v}_n)} f(v_n - \widetilde{v}_n)f'(v_n - \widetilde{v}_n) + \Phi_{f(\widetilde{v}_n)} f(\widetilde{v}_n)f'(\widetilde{v}_n) - \Phi_{f(v_n)} f(v_n)f'(v_n)\right) \phi \\
&\quad + \lambda \int_{\mathbb{R}^3} K(x) \Big(|f(v_n)|^{22^*-2} f(v_n)f'(v_n) \\
&\qquad\qquad - |f(v_n - \widetilde{v}_n)|^{22^*-2} f(v_n - \widetilde{v}_n)f'(v_n - \widetilde{v}_n) - |f(\widetilde{v}_n)|^{22^*-2} f(\widetilde{v}_n)f'(\widetilde{v}_n)\Big) \phi \\
&\quad + \lambda \int_{\mathbb{R}^3} \left(h(x, f(v_n))f'(v_n) - h(x, f(v_n - \widetilde{v}_n))f'(v_n - \widetilde{v}_n) - h(x, f(\widetilde{v}_n))f'(\widetilde{v}_n)\right) \phi.
\end{aligned}$$

Using Lemma 3.5 in [31] and Proposition 2.1 in [33], we induce $I'_\lambda(u_n - \widetilde{u}_n) \to 0$. □

Proposition 3.6. *Assume that* (V_1), (V_2), (K), (h_1), (h_2) *and* (h_3) *hold. Then there exists a constant* $\alpha_0 > 0$ *independent of* λ *such that, for any Cerami sequence* $\{v_n\}$ *for* J_λ*, either* $v_n \to v$ *in* E*, or*

$$c - J_\lambda(v) \geq \alpha_0 \lambda^{-\frac{1}{2}}.$$

Proof. Taking $w_n := v_n - \widetilde{v}_n$. Then $v_n - v = w_n + (\widetilde{v}_n - v)$, and by (3.3), $v_n \to v$ if and only if $w_n \to 0$. From Lemma 3.5, we have $J_\lambda(w_n) \to c - J_\lambda(v)$ and $J'_\lambda(w_n) \to 0$. By (h_3)

and Lemma 2.1(7), we have

$$
\begin{aligned}
&J_\lambda(w_n) - \frac{1}{4}J'_\lambda(w_n)w_n \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla w_n|^2 + \lambda V(x)|f(w_n)|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3}\Phi_{f(w_n)}|f(w_n)|^2 \\
&\quad - \lambda\int_{\mathbb{R}^3}G(x,f(w_n)) - \frac{1}{4}\int_{\mathbb{R}^3}\left(|\nabla w_n|^2 + \lambda V(x)f(w_n)f'(w_n)w_n\right) \\
&\quad - \frac{\lambda}{4}\int_{\mathbb{R}^3}\Phi_{f(w_n)}f(w_n)f'(w_n)w_n + \frac{\lambda}{4}\int_{\mathbb{R}^3}g(x,f(w_n))f'(w_n)w_n \\
&\geq \lambda\int_{\mathbb{R}^3}\left(\frac{1}{4}h(x,f(w_n))f'(w_n)w_n - H(x,f(w_n))\right) \qquad (3.8)\\
&\quad + \lambda\int_{\mathbb{R}^3}\left(\frac{1}{4}K(x)|f(w_n)|^{22^*-2}f(w_n)f'(w_n)w_n - \frac{1}{22^*}K(x)|f(w_n)|^{22^*}\right) \\
&\geq \lambda\int_{\mathbb{R}^3}\left(\frac{1}{4}K(x)|f(w_n)|^{22^*-2}f(w_n)f'(w_n)w_n - \frac{1}{22^*}K(x)|f(w_n)|^{22^*}\right) \\
&\geq \lambda\int_{\mathbb{R}^3}\left(\frac{1}{8} - \frac{1}{22^*}\right)K(x)|f(w_n)|^{22^*} \\
&\geq \frac{\lambda K_1}{24}\int_{\mathbb{R}^3}|f(w_n)|^{22^*}.
\end{aligned}
$$

Therefore,

$$
|f(w_n)|_{22^*}^{22^*} \leq \frac{24(c - J_\lambda(v))}{\lambda K_1} + o(1). \qquad (3.9)
$$

Now, using Lemma 3.6 in [31], we can complete the proof. □

By Proposition 3.6, we get the following compactness result.

Corollary 3.7. *Under the assumptions of Proposition* 3.6*,* J_λ *satisfies the Cerami condition for all* $c < \alpha_0\lambda^{-\frac{1}{2}}$*.*

4. The mountain-pass structure

In the sequel, we always assume $\lambda \geq 1$. First of all, the following two lemmas are standard, which imply that J_λ possesses the mountain-pass structure.

Lemma 4.1. *Let* (V_1), (V_2), (K), (h_1), (h_2) *and* (h_3) *hold. For each* λ *there is a closed subset* S_λ *of* E *which disconnects (arcwise)* E *into distinct connected components* E_1 *and* E_2*. Then the functional* J_λ *satisfies:* $0 \in E_1$ *and there is* α_λ *such that* $J_\lambda|_{S_\lambda} \geq \alpha_\lambda > 0$*.*

Proof. First note that, for each λ, $J_\lambda(0) = 0$. Now, for every $\rho > 0$, define

$$
S_{\lambda,\rho} = \left\{v \in E : \int_{\mathbb{R}^3}\left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right) = \rho^2\right\}.
$$

Since the functional $\int_{\mathbb{R}^3} \left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right)$ is continuous, $S_{\lambda,\rho}$ is a closed subset which disconnects the space E.

By (h_1), (h_2), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\int_{\mathbb{R}^3} H(x, f(v)) \leq \delta \int_{\mathbb{R}^3} |f(v)|^2 + C_\delta \int_{\mathbb{R}^3} |f(v)|^{2q}.$$

From Lemma 2.1(4), we know $|f(v)|, |f(v)|^2 \in E$. And since the embedding from E to $L^s(\mathbb{R}^3)$, $2 \leq s \leq 2^*$, is continuous, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |f(v)|^2 &\leq C_2^2 \int_{\mathbb{R}^3} \left(|\nabla f(v)|^2 + \lambda V(x)|f(v)|^2\right) \\
&\leq C_2^2 \int_{\mathbb{R}^3} \left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right) \\
&= C_2^2 \rho^2.
\end{aligned} \tag{4.1}$$

Taking $0 < \tau < 1$ such that $q = \tau + 2^*(1-\tau)$, by the Hölder inequality and the Sobolev Embedding Theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} |f(v)|^{2q} &\leq \left(\int_{\mathbb{R}^3} |f(v)|^2\right)^{\tau} \left(\int_{\mathbb{R}^3} |f(v)|^{22^*}\right)^{1-\tau} \\
&\leq 2^{\frac{2^*(1-\tau)}{2}} \left(\int_{\mathbb{R}^3} |f(v)|^2\right)^{\tau} \left(\int_{\mathbb{R}^3} |v|^{2^*}\right)^{1-\tau} \\
&\leq 2^{\frac{2^*(1-\tau)}{2}} C_2^{2\tau} \rho^{2\tau} S^{\frac{2^*(\tau-1)}{2}} \left(\int_{\mathbb{R}^3} |\nabla v|^2\right)^{\frac{2^*(1-\tau)}{2}} \\
&\leq 2^{\frac{2^*(1-\tau)}{2}} C_2^{2\tau} \rho^{2\tau+2^*(1-\tau)} S^{\frac{2^*(\tau-1)}{2}}.
\end{aligned} \tag{4.2}$$

Furthermore, since $K(x)$ is bounded, by Lemma 2.1(6) and the Sobolev Embedding Theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)|f(v)|^{22^*} &\leq 2^{\frac{2^*}{2}} K_2 \int_{\mathbb{R}^3} |v|^{2^*} \\
&\leq 2^{\frac{2^*}{2}} K_2 S^{-\frac{2^*}{2}} \left(\int_{\mathbb{R}^3} |\nabla v|^2\right)^{\frac{2^*}{2}} \\
&\leq 2^{\frac{2^*}{2}} K_2 S^{-\frac{2^*}{2}} \rho^{2^*}.
\end{aligned} \tag{4.3}$$

From the above inequalities, we know that

$$\begin{aligned}
J_\lambda(v) &= \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right) + \frac{\lambda}{4} \int_{\mathbb{R}^3} \Phi_{f(v)} |f(v)|^2 - \lambda \int_{\mathbb{R}^3} G(x, f(v)) \\
&\geq \left(\frac{1}{2} - \lambda\delta C_2^2\right) \rho^2 - \lambda C_\delta 2^{\frac{2^*(1-\tau)}{2}} C_2^{2\tau} \rho^{2\tau+2^*(1-\tau)} S^{\frac{2^*(\tau-1)}{2}} - \frac{\lambda}{22^*} 2^{\frac{2^*}{2}} K_2 S^{-\frac{2^*}{2}} \rho^{2^*},
\end{aligned} \tag{4.4}$$

for every $v \in S_{\lambda,\rho}$. Since $2\tau + 2^*(1-\tau) > 2$, we conclude that there are $\alpha_\lambda > 0$ and ρ_λ such that $J_\lambda|_{S_\lambda} \geq \alpha_\lambda > 0$. □

Lemma 4.2. *Under the assumptions of Lemma* 4.1*, for any finite-dimensional subspace* $F \subset E$*,*

$$J_\lambda(v) \to -\infty \quad \text{as } v \in F,\ \|v\|_\lambda \to \infty.$$

Proof. By (h_3) and Lemma 2.1(4), we have

$$
\begin{aligned}
J_\lambda(v) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(v)}|f(v)|^2 - \lambda\int_{\mathbb{R}^3} G(x, f(v)) \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_v|v|^2 - \lambda\int_{\mathbb{R}^3} H(x, f(v)) \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right) + C\frac{\lambda}{4}\|v\|_\lambda^4 - \lambda\int_{\mathbb{R}^3} a_0(|f(v)|^2 + |f(v)|)^l \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right) + C\frac{\lambda}{4}\|v\|_\lambda^4 - \lambda C\int_{\mathbb{R}^3} |v|^l,
\end{aligned}
\tag{4.5}
$$

for all $u \in E$ since all norms in a finite-dimensional space are equivalent and $l > 4$. This completes the proof of Lemma 4.2. □

By Lemma 4.1 and Lemma 4.2, if J_λ satisfies the Cerami condition for all $c > 0$, then Theorem 2.2 follows from standard critical theory. However, in general, we do not know if J_λ satisfies Cerami condition for c large. By Corollary 3.7, J_λ satisfies Cerami condition for λ large and c_λ sufficiently small. Therefore, we will find special finite-dimensional subspaces by which we construct sufficiently small minimax levels.

Recall that

$$\inf\left\{\int_{\mathbb{R}^3} |\nabla\phi|^2 : \phi \in C_0^\infty(\mathbb{R}^3) : |\phi|_r = 1\right\} = 0, \quad 2 < r < 2^*.$$

For any $\sigma > 0$, we can choose $\phi_\sigma \in C_0^\infty(\mathbb{R}^3)$ with $|\phi_\sigma|_r = 1$ and $\operatorname{supp}\phi_\sigma \subset B_{r_\sigma}(0)$ so that $|\nabla\phi_\sigma|_2^2 < \sigma$. Denote

$$e_\lambda(x) = \phi_\sigma(\lambda^{\frac{1}{2}}x). \tag{4.6}$$

Then $\operatorname{supp} e_\lambda \subset B_{\lambda^{-\frac{1}{2}}r_\sigma}(0)$.

Observe that

$$
\begin{aligned}
J_\lambda(te_\lambda) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(t^2|\nabla e_\lambda|^2 + \lambda V(x)|f(te_\lambda)|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(te_\lambda)}|f(te_\lambda)|^2 - \lambda\int_{\mathbb{R}^3} G(x, f(te_\lambda)) \\
&\le \frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla e_\lambda|^2 + \lambda V(x)|e_\lambda|^2\right) + \frac{\lambda}{4}t^4\int_{\mathbb{R}^3} \Phi_{e_\lambda}|e_\lambda|^2 - \lambda\int_{\mathbb{R}^3} H(x, f(te_\lambda)) \\
&\le \frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla e_\lambda|^2 + \lambda V(x)|e_\lambda|^2\right) + \frac{\lambda}{4}t^4\int_{\mathbb{R}^3} \Phi_{e_\lambda}|e_\lambda|^2 - \lambda a_0\int_{\mathbb{R}^3} \left(|f(te_\lambda)|^2 + |f(te_\lambda)|\right)^l \\
&\le \frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla e_\lambda|^2 + \lambda V(x)|e_\lambda|^2\right) + \frac{\lambda}{4}t^4\int_{\mathbb{R}^3} \Phi_{e_\lambda}|e_\lambda|^2 - \lambda a_0 C^l t^l\int_{\mathbb{R}^3} |e_\lambda|^l \\
&= \lambda^{-\frac{1}{2}}\left(\frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla\phi_\sigma|^2 + V(\lambda^{-\frac{1}{2}}x)|\phi_\sigma|^2\right) + \frac{t^4}{4}\lambda^{-1}\int_{\mathbb{R}^3} \Phi_{\phi_\sigma}|\phi_\sigma|^2 - a_0 C^l t^l\int_{\mathbb{R}^3} |\phi_\sigma|^l\right) \\
&= \lambda^{-\frac{1}{2}}\Psi_\lambda(t\phi_\sigma),
\end{aligned}
$$

where $\Psi_\lambda \in C^1(E,\mathbb{R})$ defined by

$$\Psi_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla v|^2 + V(\lambda^{-\frac{1}{2}}x)|v|^2\right) + \frac{1}{4}\lambda^{-1}\int_{\mathbb{R}^3}\Phi_v|v|^2 - a_0C^l\int_{\mathbb{R}^3}|v|^l.$$

Since $l>4$, thus there exists finite number $t_0 \in [0,+\infty)$ such that

$$\begin{aligned}\max_{t\geq 0}\Psi_\lambda(t\phi_\sigma) &= \frac{t_0^2}{2}\int_{\mathbb{R}^3}\left(|\nabla\phi_\sigma|^2 + V(\lambda^{-\frac{1}{2}}x)|\phi_\sigma|^2\right)\\ &\quad + \frac{t_0^4}{4}\lambda^{-1}\int_{\mathbb{R}^3}\Phi_{\phi_\sigma}|\phi_\sigma|^2 - a_0C^lt_0^l\int_{\mathbb{R}^3}|\phi_\sigma|^l\\ &\leq \frac{t_0^2}{2}\int_{\mathbb{R}^3}\left(|\nabla\phi_\sigma|^2 + V(\lambda^{-\frac{1}{2}}x)|\phi_\sigma|^2\right) + \frac{t_0^4}{4}C\lambda^{-1}\left(\int_{\mathbb{R}^3}|\phi_\sigma|^{\frac{12}{5}}\right)^{\frac{5}{3}}.\end{aligned}$$

On the other hand, since $V(0)=0$ and $\operatorname{supp}\phi_\sigma \subset B_{r_\sigma}(0)$, there exists Λ_σ such that

$$V(\lambda^{-\frac{1}{2}}x) \leq \frac{\sigma}{|\phi_\sigma|_2^2} \quad \text{for all } |x|\leq r_\sigma \text{ and } \lambda \geq \Lambda_\sigma,$$

and $\lambda^{-1}<\sigma$. Then

$$\max_{t\geq 0}\Psi_\lambda(t\phi_\sigma) \leq \widetilde{C}\sigma.$$

Hence, for any $\lambda \geq \Lambda_\sigma$,

$$\max_{t\geq 0} J_\lambda(te_\lambda) \leq \widetilde{C}\sigma\lambda^{-\frac{1}{2}}. \tag{4.7}$$

Therefore, we have the following lemma.

Lemma 4.3. *Under the assumptions of Lemma* 4.1*, for any $\delta>0$ there exists $\overline{\Lambda}_\delta>0$ such that, for each $\lambda \geq \overline{\Lambda}_\delta$, there is $\overline{e}_\lambda$ with $\|\overline{e}_\lambda\|>\rho_\lambda$, $J_\lambda(\overline{e}_\lambda)\leq 0$ and*

$$\max_{t\geq 0} J_\lambda(t\overline{e}_\lambda) \leq \delta\lambda^{-\frac{1}{2}},$$

where ρ_λ is given by Lemma 4.1*.*

Proof. Choose $\sigma>0$ so small that $\widetilde{C}\sigma \leq \delta$, and let $e_\lambda \in E$ be the function defined by (4.6). Take $\overline{\Lambda}_\delta = \Lambda_\sigma$. Let $\overline{t}_\lambda>0$ be such that $\overline{t}_\lambda\|e_\lambda\|_\lambda > \rho_\lambda$ and $J_\lambda(te_\lambda)\leq 0$ for all $t\geq \overline{t}_\lambda$. Let $\overline{e}_\lambda := \overline{t}_\lambda e_\lambda$. Then, by (4.7), we know the conclusion of Lemma 4.3 holds. □

For any $n^* \in \mathbb{N}$, we can choose n^* functions $\phi_\sigma^j \in C_0^\infty(\mathbb{R}^3)$ with $|\phi_\sigma^j|_r = 1$ and $\operatorname{supp}\phi_\sigma^j \cap \operatorname{supp}\phi_\sigma^k = \emptyset$, $j\neq k$ so that $|\nabla\phi_\sigma^j|_2^2 < \sigma$. Let $r_\sigma^{n^*}>0$ be such that $\operatorname{supp}\phi_\sigma^j \subset B_{r_\sigma^{n^*}}(0)$ for $j=1,2,\ldots,n^*$. Let

$$e_\lambda^j(x) = \phi_\sigma^j(\lambda^{\frac{1}{2}}x) \quad \text{for } j=1,2,\ldots,n^* \tag{4.8}$$

and

$$H_{\lambda\sigma}^{n^*} = \operatorname{span}\left\{e_\lambda^1,\ldots,e_\lambda^{n^*}\right\}.$$

Observe that for each

$$v = \sum_{j=1}^{n^*} c_j e_\lambda^j \in H_{\lambda\sigma}^{n^*},$$

we have

$$J_\lambda(v) \le C \sum_{j=1}^{n^*} J_\lambda(c_j e_\lambda^j),$$

for some constant $C > 0$. By the similar argument as before, we know that

$$J_\lambda(c_j e_\lambda^j) \le \lambda^{-\frac{1}{2}} \Psi_\lambda(|c_j| e_\lambda^j).$$

Denote

$$\beta_\sigma := \max \left\{ |\phi_\sigma^j|_2^2 \mid j = 1, 2, \ldots, n^* \right\},$$

and choose $\Lambda_{n^*\sigma}$ such that

$$V(\lambda^{-\frac{1}{2}} x) \le \frac{\sigma}{\beta_\sigma} \quad \text{for all } |x| \le r_\sigma^{n^*} \text{ and } \lambda \ge \Lambda_{n^*\sigma}.$$

Similarly, we have

$$\max_{u \in H_{\lambda\sigma}^{n^*}} J_\lambda(v) \le \widetilde{C} \sigma \lambda^{-\frac{1}{2}}, \tag{4.9}$$

for all $\lambda \ge \Lambda_{n^*\sigma}$.

Using this estimate, we have the following lemma.

Lemma 4.4. *Under the assumptions of Lemma* 4.1*, for any* $n^* \in \mathbb{N}$ *and* $\delta > 0$ *there exists* $\overline{\Lambda}_{n^*\delta} > 0$ *such that, for each* $\lambda \ge \overline{\Lambda}_{n^*\delta}$*, there exists an* n^**-dimensional subspace* $F_{\lambda n^*}$ *satisfying*

$$\max_{u \in F_{\lambda n^*}} J_\lambda(v) \le \delta \lambda^{-\frac{1}{2}}.$$

Proof. Choose $\sigma > 0$ so small that $\widetilde{C}\sigma \le \delta$, and take $F_{\lambda n^*} = H_{\lambda\sigma}^{n^*}$. From (4.9), we know that Lemma 4.4 holds. □

5. The proofs of our main results

Now, we are in the position to prove our main results.

Proof of Theorem 2.2. Consider the functional J_λ. For any $0 < \delta < \alpha_0$, by Lemma 4.3 we choose Λ_δ and define for $\lambda \ge \Lambda_\delta$ the minimax value

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(t \overline{e}_\lambda),$$

where

$$\Gamma_\lambda := \{\gamma \in C([0,1],E) : \gamma(0) = 0 \text{ and } \gamma(1) = \overline{e}_\lambda\}.$$

It follows from Lemma 4.1 and (4.9) that

$$\alpha_\lambda \le c_\lambda \le \delta\lambda^{-\frac{1}{2}}.$$

Since by Corollary 3.7, J_λ satisfies the Cerami condition, the mountain-pass theorem implies that there is $v_\lambda \in E$ such that $J'_\lambda(v_\lambda) = 0$ and $J_\lambda(v_\lambda) = c_\lambda$. From the elliptic regularity theory, we know v_λ is of C^2. Then we know that $u = f(v_\lambda)$ must solve equation (2.4).

Because v_λ is a critical point of J_λ, for $\nu \in [2, 2^*]$,

$$\begin{aligned}
\delta\lambda^{-\frac{1}{2}} &\ge J_\lambda(v_\lambda) = J_\lambda(v_\lambda) - \frac{1}{\nu} J'_\lambda(v_\lambda)v_\lambda \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v_\lambda|^2 + \lambda V(x)|f(v_\lambda)|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_{f(v_\lambda)}|f(v_\lambda)|^2 - \lambda\int_{\mathbb{R}^3} G(x, f(v_\lambda)) \\
&\quad - \frac{1}{\nu}\int_{\mathbb{R}^3} \left(|\nabla v_\lambda|^2 + \lambda V(x) f(v_\lambda) f'(v_\lambda)v_\lambda\right) - \frac{\lambda}{\nu}\int_{\mathbb{R}^3} \Phi_{f(v_\lambda)} f(v_\lambda) f'(v_\lambda)v_\lambda \\
&\quad + \frac{\lambda}{\nu}\int_{\mathbb{R}^3} g(x, f(v_\lambda)) f'(v_\lambda)v_\lambda \\
&\ge \left(\frac{1}{2} - \frac{1}{\nu}\right)\int_{\mathbb{R}^3} \left(|\nabla v_\lambda|^2 + \lambda V(x)|f(v_\lambda)|^2\right) + \lambda\left(\frac{1}{4} - \frac{1}{\nu}\right)\int_{\mathbb{R}^3} \Phi_{f(v_\lambda)}|f(v_\lambda)|^2|f(v_\lambda)|^2 \\
&\quad + \lambda\left(\frac{1}{2\nu} - \frac{1}{22^*}\right)\int_{\mathbb{R}^3} K(x)|f(v_\lambda)|^{22^*} + \lambda\left(\frac{\mu}{\nu} - 1\right)\int_{\mathbb{R}^3} H(x, f(v_\lambda)),
\end{aligned}$$

where μ is the constant in (h_3). Taking $\nu = \mu$, we obtain

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \left(|\nabla v_\lambda|^2 + \lambda V(x)|f(v_\lambda)|^2\right) \le \delta\lambda^{-\frac{1}{2}}$$

and taking $\nu = 4$ we obtain

$$\frac{1}{24}\int_{\mathbb{R}^3} K(x)|f(v_\lambda)|^{22^*} + \frac{\mu - 4}{4}\int_{\mathbb{R}^3} H(x, f(v_\lambda)) \le \delta\lambda^{-\frac{3}{2}}. \qquad \square$$

Proof of Theorem 2.3. Using Lemma 4.4, for any $n^* \in \mathbb{N}$ and $\delta \in [0, \alpha_0]$ there is $\Lambda_{n^*\delta}$ such that for each $\lambda \ge \Lambda_{n^*\delta}$, we can choose n^*-dimensional subspace $F_{\lambda n^*}$ with max $J_\lambda(F_{\lambda n^*}) \le \delta\lambda^{-\frac{1}{2}}$.

Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ. Let gen(Z) be the Krasnoselski genus and let

$$i(Z) := \min_{h\in\Gamma_{n^*}} \operatorname{gen}(h(Z) \cap S_\lambda),$$

where Γ_{n^*} is the set of all odd homeomorphisms $h \in C(E, E)$ and S_λ is the closed symmetric set

$$S_\lambda = \left\{v \in E : \int_{\mathbb{R}^3} \left(|\nabla v|^2 + \lambda V(x)|f(v)|^2\right) = \rho^2\right\},$$

such that $J_\lambda|_{S_\lambda} \geq \alpha_\lambda > 0$. Then, i is a version of Benci's pseudo-index [4]. Let

$$c_{\lambda_j} := \inf_{i(Z)\geq j} \sup_{v\in Z} J_\lambda(v), \quad 1 \leq j \leq n^*.$$

Because by Lemma 4.1, we know that $J_\lambda|_{S_\lambda}(v) \geq \alpha_\lambda > 0$ and since $i(F_{\lambda n^*}) = \dim F_{\lambda n^*} = n^*$,

$$\alpha_\lambda \leq c_{\lambda_1} \leq \cdots \leq c_{\lambda_{n^*}} \leq \sup_{v\in F_{\lambda n^*}} J_\lambda(v) \leq \delta\lambda^{-\frac{1}{2}}.$$

By Corollary 3.7, J_λ satisfies the Cerami condition at all level c_{λ_j}, $j = 1, 2, \ldots, n^*$. By the usual critical point theory, all c_{λ_j}, $j = 1, 2, \ldots, n^*$, are critical levels, and J_λ has at least n^* pairs of nontrivial critical points satisfying

$$\alpha_\lambda \leq J_\lambda(v_\lambda) \leq \delta\lambda^{-\frac{1}{2}}.$$

Therefore, (2.8) has at least n^* pairs of solutions and $u = f(v_\lambda)$ must solve the problem (2.4). □

References

[1] C. O. Alves and G. M. Figueiredo, *Existence and multiplicity of positive solutions to a p-Laplacian equation in* $\mathbb{R}^N$, Differential Integral Equations **19** (2006), no. 2, 143–162.

[2] C. O. Alves and M. A. S. Souto, *Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains*, Z. Angew. Math. Phys. **65** (2014), no. 6, 1153–1166. http://dx.doi.org/10.1007/s00033-013-0376-3

[3] A. Azzollini and A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. **345** (2008), no. 1, 90–108.

[4] V. Benci, *On critical point theory for indefinite functionals in the presence of symmetries*, Trans. Amer. Math. Soc. **274** (1982), no. 2, 533–572. http://dx.doi.org/10.1090/s0002-9947-1982-0675067-x

[5] S. Chen and C. Wang, *Existence of multiple nontrivial solutions for a Schrödinger-Poisson system*, J. Math. Anal. Appl. **411** (2014), no. 2, 787–793. http://dx.doi.org/10.1016/j.jmaa.2013.10.008

[6] M. Colin and L. Jeanjean, *Solutions for a quasilinear Schrödinger equation: a dual approach*, Nonlinear Anal. **56** (2004), no. 2, 213–226. http://dx.doi.org/10.1016/j.na.2003.09.008

[7] T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 5, 893–906. http://dx.doi.org/10.1017/s030821050000353x

[8] Y. Ding and A. Szulkin, *Bound states for semilinear Schrödinger equations with sign-changing potential*, Calc. Var. Partial Differential Equations **29** (2007), no. 3, 397–419. http://dx.doi.org/10.1007/s00526-006-0071-8

[9] X.-D. Fang and A. Szulkin, *Multiple solutions for a quasilinear Schrödinger equation*, J. Differential Equations **254** (2013), no. 4, 2015–2032.

[10] Y. Guo and Z. Tang, *Ground state solutions for the quasilinear Schrödinger equation*, Nonlinear Anal. **75** (2012), no. 6, 3235–3248. http://dx.doi.org/10.1016/j.na.2011.12.024

[11] X. He and W. Zou, *Existence and concentration of ground states for Schrödinger-Poisson equations with critical growth*, J. Math. Phys. **53** (2012), no. 2, 023702, 19 pp. http://dx.doi.org/10.1063/1.3683156

[12] I. Ianni, *Sign-changing radial solutions for the Schrödinger-Poisson-Slater problem*, Topol. Methods Nonlinear Anal. **41** (2013), no. 2, 365–386.

[13] ———, *Solutions of the Schrödinger-Poisson problem concentrating on spheres II: Existence*, Math. Models Methods Appl. Sci. **19** (2009), no. 6, 877–910. http://dx.doi.org/10.1142/s0218202509003656

[14] I. Ianni and D. Ruiz, *Ground and bound states for a static Schrödinger-Poisson-Slater problem*, Commun. Contemp. Math. **14** (2012), no. 1, 1250003, 22 pp. http://dx.doi.org/10.1142/s0219199712500034

[15] I. Ianni and G. Vaira, *On concentration of positive bound states for the Schrödinger-Poisson problem with potentials*, Adv. Nonlinear Stud. **8** (2008), no. 3, 573–595.

[16] ———, *Solutions of the Schrödinger-Poisson problem concentrating on spheres I: Necessary conditions*, Math. Models Methods Appl. Sci. **19** (2009), no. 5, 707–720. http://dx.doi.org/10.1142/s0218202509003589

[17] ———, *Non-radial sign-changing solutions for the Schrödinger-Poisson problem in the semiclassical limit*, NoDEA Nonlinear Differential Equations Appl. **22** (2015), no. 4, 741–776. http://dx.doi.org/10.1007/s00030-014-0303-0

[18] S. Kim and J. Seok, *On nodal solutions of the nonlinear Schrödinger-Poisson equations*, Commun. Contemp. Math. **14** (2012), no. 6, 1250041, 16 pp. http://dx.doi.org/10.1142/s0219199712500411

[19] X.-Q Liu, J.-Q Liu and Z.-Q Wang, *Quasilinear elliptic equations with critical growth via perturbation method*, J. Differential Equations **254** (2013), no. 1, 102–124. http://dx.doi.org/10.1016/j.jde.2012.09.006

[20] ______, *Quasilinear elliptic equations via perturbation method*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 253–263. http://dx.doi.org/10.1090/s0002-9939-2012

[21] ______, *Ground states for quasilinear Schrödinger equations with critical growth*, Calc. Var. Partial Differential Equations **46** (2013), no. 3-4, 641–669. http://dx.doi.org/10.1007/s00526-012-0497-0

[22] J. Liu and Z.-Q Wang, *Soliton solutions for quasilinear Schrödinger equations I*, Proc. Amer. Math. Soc. **131** (2003), no. 2, 441–448.

[23] J.-Q Liu, Y.-Q Wang and Z.-Q Wang, *Solutions for quasilinear Schrödinger equations via the Nehari method*, Comm. Partial Differential Equations **29** (2004), no. 5-6, 879–901. http://dx.doi.org/10.1081/pde-120037335

[24] ______, *Soliton solutions for quasilinear Schrödinger equations II*, J. Differential Equations **187** (2003), no. 2, 473–493. http://dx.doi.org/10.1016/s0022-0396(02)00064-5

[25] J. Nie and X. Wu, *Existence and multiplicity of non-trivial solutions for a class of modified Schrödinger-Poisson systems*, J. Math. Anal. Appl. **408** (2013), no. 2, 713–724. http://dx.doi.org/10.1016/j.jmaa.2013.06.011

[26] M. Poppenberg, K. Schmitt and Z.-Q Wang, *On the existence of soliton solutions to quasilinear Schrödinger equations*, Calc. Var. Partial Differential Equations **14** (2002), no. 3, 329–344. http://dx.doi.org/10.1007/s005260100105

[27] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), no. 2, 655–674. http://dx.doi.org/10.1016/j.jfa.2006.04.005

[28] D. Ruiz and G. Vaira, *Cluster solutions for the Schrödinger-Poisson-Slater problem around a local minimum of the potential*, Rev. Mat. Iberoam. **27** (2011), no. 1, 253–271. http://dx.doi.org/10.4171/rmi/635

[29] B. Sirakov, *Standing wave solutions of the nonlinear Schrödinger equations in* $\mathbb{R}^N$, Ann. Mat. Pura Appl. (4) **181** (2002), no. 1, 73–83. http://dx.doi.org/10.1007/s1023102

[30] E. A. B. Silva and G. F. Vieira, *Quasilinear asymptotically periodic Schrödinger equations with subcritical growth*, Nonlinear Anal. **72** (2010), no. 6, 2935–2949. http://dx.doi.org/10.1016/j.na.2009.11.037

[31] M. Yang and Y. Ding, *Existence of semiclassical states for a quasilinear Schrödinger equation with critical exponent in* $\mathbb{R}^N$, Ann. Mat. Pura Appl. (4) **192** (2013), no. 5, 783–804.

[32] ______, *Existence and multiplicity of semiclassical states for a quasilinear Schrödinger equation in* $\mathbb{R}^N$, Commun. Pure Appl. Anal. **12** (2013), no. 1, 429–449.

[33] J. Zhang, *The existence and concentration of positive solutions for a nonlinear Schrödinger-Poisson system with critical growth*, J. Math. Phys. **55** (2014), no. 3, 031507, 14 pp.

[34] L. Zhao, H. Liu and F. Zhao, *Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential*, J. Differential Equations **255** (2013), no. 1, 1–23.

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