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L-series for Vector-valued Modular Forms

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Abstract. Motivated by the recent works of Bringmann, Guerzhoy, Kent, and Ono [4] and Bringmann, Fricke, and Kent [3], we introduce L-series for vector-valued weakly holomorphic cusp forms, and mock modular period polynomials for vector-valued harmonic weak Maass forms. In particular, we will discuss an integral representation of this new L-series and the limiting behavior of special values. Moreover, we also give relations between mock modular periods and L-series for vector-valued harmonic weak Maass forms.

1. Introduction

Special values of L-series have played many important roles in analytic number theory. In particular, after L. Mordell proved Ramanujan's conjecture on τ function

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}},$$

many researchers have studied roles of L-series associated to modular forms in that $\tau(n)$ is n-th coefficient of the unique cusp form of weight 12 on the full modular group. The famous Birch-Swinnerton-Dyer conjecture is one of many examples showing that special values of modular L-series are deep and important subject in number theory.

These L-series associated to modular forms have been defined for holomorphic modular forms for the convergence of L-series. However, in a recent work of K. Bringmann, K. Fricke, and Z. Kent [3], they defined a modified L-series for weakly holomorphic modular forms, which overcomes the difficulty arose from exponential growth of Fourier coefficients of weakly holomorphic modular forms. Their main idea is using formal Eichler integral and regularized period integral to define an L-series for weakly holomorphic cusp form $F(z) \in S_k^!$. More precisely, Bringmann, P. Guerzhoy, Kent and K. Ono [4] introduced the

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Eichler integral for weakly holomorphic cusp forms to study Eichler-Shimura theory for harmonic weak Maass forms by defining the Eichler integral as a formal power series

$$\mathcal{E}_F(z) := \sum_{n \gg 0} a(n) n^{1-k} q^n,$$

provided $F(z)$ has the Fourier expansion of the form $\sum_{n\gg 0} a(n)q^n$, where $q = \exp(2\pi i z)$ with $z \in \mathbb{H}$, the upper half complex plane. Motivated from the fact that the period polynomial for a cusp form encodes special values of L-series, Bringmann, Fricke, and Kent [3] were able to extend L-series by regularizing the period integral.

On the other hand, vector-valued modular forms have played prominent role in number theory like the classical modular forms have done. For example, R. Borcherds [2] used vector-valued modular forms associated to the Weil representation to provide an elegant description of the Fourier expansion of various theta liftings. M. Eichler and D. Zagier in [9] showed that vector-valued modular forms are closely related with holomorphic Jacobi forms through the theta expansion. In this light, it is natural to study L-series for vector-valued modular forms. Bruinier and Stein [7] defined L-series for vector-valued Hecke eigenforms associated to the Weil representation by using Hecke eigenvalues. In the note we define vector-valued L-series for vector-valued modular forms using their Fourier coefficients and study their basic properties. As an application, we show that the results of Bringmann, Frick and Kent [3] can be extended to L-series associated to vector-valued weakly holomorphic modular forms. This type of L-series has more applications including the infinitude of sign changes for Fourier coefficients of Jacobi cusp forms [10] and a relation between cuspidality and Hecke bound for vector-valued modular forms [11].

We also define Eichler integral for vector-valued harmonic weak Maass forms with Weil representation, which extend Eichler integral for harmonic weak Maass forms in [4] to vector-valued harmonic weak Maass forms. Let k be an even integer and let $\rho\colon\ \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_p(\mathbb{C})$ be a p-dimensional unitary complex representation such that $\rho(T)$ is diagonal, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For any integer $p \geq 1$ let $\boldsymbol{e}_j$ be the standard basis of $\mathbb{C}^p$. Let $S^!_{k,\rho}$ denote the space of weight k vector-valued weakly holomorphic cusp forms, i.e., those weakly holomorphic modular forms whose Fourier expansion

$$F(z) = \sum_{j=1}^{p} \sum_{n \geq -M_j} a_n(j) q^{n/N_j} \boldsymbol{e}_j \tag{1.1}$$

satisfies $a_0(j) = 0$ for all $j \in \{1, 2, \dots, p\}$. Here and throughout the paper, $q = e^{2\pi i z}$, and N_j is a certain positive integer depending on ρ and j. For $t_0 > 0$ and $F(z) \in S^!_{k,\rho}$ with Fourier expansion as in (1.1) we define L-series by

$$L(F, s) := \frac{(2\pi)^s}{\Gamma(s)} L^*(F, s) \tag{1.2}$$

with

$$L^*(F,s) := \sum_{j=1}^{p}(N_j)^s \sum_{n\geq -M_j} \frac{a_n(j)\Gamma(s, 2\pi n t_0/N_j)}{(2\pi n)^s}\boldsymbol{e}_j$$
$$+ \rho^{-1}(S)\sum_{j=1}^{p} i^{-k}(N_j)^{k-s} \sum_{n\geq -M_j} \frac{a_n(j)\Gamma\left(k-s, \frac{2\pi n}{t_0 N_j}\right)}{(2\pi n)^{k-s}}\boldsymbol{e}_j.$$

Here the incomplete gamma function, $\Gamma(s,z)$, is given by the analytic continuation (to an entire function with respect to s and fixed $z \neq 0$) of $\int_z^\infty e^{-t}t^{s-1}\,dt$. Absolute convergence of $L^*(F,s)$ is guaranteed since $\Gamma(s,x) \sim x^{s-1}e^{-x}$ as $x \to \infty$ and Lemma 3.5 implies that

$$a_n(j) = O\left(e^{C\sqrt{|n|}}\right), \text{ as } n \to \infty.$$

Using the modularity of $F(z)$ we can show that (1.2) is independent of t_0 (see the proof of Theorem 1.1).

Moreover, $L^*(F,s)$ has an integral representation. For this we require certain regularized integrals motivated from [3]. Let $F(z) = \sum_{j=1}^{p} F_j(z)\boldsymbol{e}_j$ be a continuous function satisfying a growth condition: There is a $c \in \mathbb{R}^+$ such that

$$F_j(z) = O(e^{cy}) \tag{1.3}$$

for each j uniformly in x as $y \to \infty$. Then we define the integral

$$\int_{z_0}^{i\infty} e^{uiw}F(w)\,dw, \tag{1.4}$$

for each $z_0 \in \mathbb{H}$, where the path of integration lies within a vertical strip. The integral (1.4) is convergent for $u \in \mathbb{C}$ with $\mathrm{Re}(u) \gg 0$. If the integral (1.4) has an analytic continuation to $u = 0$, we define the regularized integral by

$$R.\int_{z_0}^{i\infty} F(w)\,dw := \left[\int_{z_0}^{i\infty} e^{uiw}F(w)\,dw\right]_{u=0},$$

where the right hand side means that we take the value at $u = 0$ of the analytic continuation of the integral. Similarly, we define integrals at other cusps $\mathfrak{a}$. Specifically, suppose that $\mathfrak{a} = \sigma_{\mathfrak{a}}(i\infty)$ for a scaling matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{Z})$. If $F(\sigma_{\mathfrak{a}}z)$ satisfies (1.3), then we define

$$R.\int_{z_0}^{\mathfrak{a}} F(w)\,dw := R.\int_{\sigma_{\mathfrak{a}}^{-1}z_0}^{i\infty} (F|_{2,\rho}\sigma_{\mathfrak{a}})(w)\,dw,$$

where the slash operator $|_{k,\rho}$ is defined by

$$(F|_{k,\rho}\gamma)(z) := (cz+d)^{-k}\rho^{-1}(\gamma)F(\gamma z)$$

for functions F on $\mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. For cusps $\mathfrak{a}, \mathfrak{b}$, we set

$$R. \int_{\mathfrak{a}}^{\mathfrak{b}} F(w)\, dw := R. \int_{z_0}^{\mathfrak{b}} F(w)\, dw - R. \int_{z_0}^{\mathfrak{a}} F(w)\, dw$$

for any $z_0 \in \mathbb{H}$. This integral is independent of $z_0 \in \mathbb{H}$. We now prove an integral representation of L-series associated to a vector-valued weakly holomorphic cusp form (c.f. Theorem 2.3).

Theorem 1.1. *If $F(z) \in S^!_{k,\rho}$, then*

$$L^*(F, s) = R. \int_0^{\infty} F(iy) y^{s-1}\, dy. \tag{1.5}$$

Furthermore, $L^*(F, s)$ satisfies the functional equation

$$L^*(F, k-s) = i^k \rho(S) L^*(F, s).$$

Moreover, special values of $L^*(F, s)$ manifest a very interesting limiting behavior, which is an analog of [3, Proposition 5.1].

Theorem 1.2. *Let $\delta = 0$ or 1, and $F(z) = \sum_{j=1}^{p} \sum_{n \geq -M_j} a_n(j) q^{n/N_j} \boldsymbol{e}_j \in S^!_{k,\rho}$. Then,*

$$\lim_{\substack{n \to \infty \\ n \equiv \delta \pmod 2}} \frac{2\pi (m_j/N_j)^n}{(n-1)!} L^*(F, n) = a_{m_j}(j) + (-1)^{\delta} a_{-m_j}(j),$$

for all $j = \{1, 2, \ldots, p\}$, where $m_j \in \mathbb{N}$ is minimal with $a_{m_j}(j) + (-1)^{\delta} a_{-m_j}(j) \neq 0$.

Before stating the next result, we recall properties of vector-valued harmonic weak Maass forms. For details, look at Section 3 and the references therein. Every vector-valued harmonic weak Maass form $\mathcal{F}(z)$ has the unique decomposition $\mathcal{F}(z) = \mathcal{F}^+(z) + \mathcal{F}^-(z)$, where $\mathcal{F}^+(z)$ (respectively $\mathcal{F}^-(z)$) is holomorphic (respectively non-holomorphic) on $\mathbb{H}$. Following Zagier, the holomorphic part $\mathcal{F}^+(z)$ is called a *mock modular form.* The differential operators D^{k-1} with $D := \frac{1}{2\pi i} \frac{\partial}{\partial z}$ and $\xi_{2-k} := 2iy^{2-k} \overline{\frac{\partial}{\partial \bar{z}}}$ play important roles in the theory. In particular, they define maps

$$D^{k-1} \colon H_{2-k,\rho} \to S^!_{k,\rho} \quad \text{and} \quad \xi_{2-k} \colon H_{2-k,\rho} \to S^!_{k,\bar{\rho}}.$$

For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define the *$\gamma$-mock modular period function* for $\mathcal{F}^+(z)$ by

$$\mathbb{P}(\mathcal{F}^+, \gamma; z) := \frac{(4\pi)^{k-1}}{(k-2)!} (\mathcal{F}^+ - \mathcal{F}^+|_{2-k,\rho}\gamma)(z).$$

We now consider vector-valued weakly holomorphic cusp form $F(z) \in S^!_{k,\rho}$ with Fourier expansion as in (1.1). We use the formal *Eichler integral* of $F(z)$ given by

$$\mathcal{E}_F(z) := \sum_{j=1}^{p} \sum_{n \geq -M_j} \frac{a_n(j)}{(n/N_j)^{k-1}} q^{n/N_j} \boldsymbol{e}_j.$$

Then the period polynomial of $F(z)$ is defined by

$$r(F;z) := c_k(\mathcal{E}_F - \mathcal{E}_F|_{2-k,\rho}S)(z),$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $c_k := -\frac{\Gamma(k-1)}{(2\pi i)^{k-1}}$. We then obtain the following theorem, which is the vector-valued modular form version of [3, Theorem 1.2] and [4, Theorem 1.4].

Theorem 1.3. *Suppose that $\mathcal{F}(z) \in H_{2-k,\rho}$.*

(1) $\overline{\mathbb{P}(\mathcal{F}^+, S; \overline{z})} = \sum_{n=0}^{k-2} \frac{L(\xi_{2-k}(\mathcal{F}), n+1)}{(k-2-n)!}(2\pi i z)^{k-2-n}$.

(2) $r(\xi_{2-k}(\mathcal{F}; z)) \equiv -\frac{(4\pi)^{k-1}}{(k-2)!}\overline{r(D^{k-1}(\mathcal{F}), \overline{z})} \pmod{z^{k-2}-1}$, *where equivalence modulo $z^{k-2}-1$ means that the difference of the two functions is of the form $c(z^{k-2}-1)$ for a constant vector $c \in \mathbb{C}^p$.*

(3) $L(\xi_{2-k}(\mathcal{F}), n+1) = (-1)^n \frac{(4\pi)^{k-1}}{(k-2)!}\overline{L(D^{k-1}(\mathcal{F}), n+1)}$ *for integers* $0 < n < k-2$.

The rest of the paper is organized as follows. In Section 2, we introduce basic notion of vector-valued modular forms and L-series associated to vector-valued cusp forms. In Section 3, we introduce vector-valued harmonic weak Maass forms and review properties of ξ_{2-k} and D^{k-1} operators. In Section 4, we prove Theorems 1.1, 1.2 and 1.3. Besides the proofs, we also introduce some implications of the results.

2. Vector-valued modular forms and their L-series

In this section, we introduce the basic notion of vector-valued modular forms following [12], and define L-series associated to vector-valued cusp forms. We start with defining the vector-valued modular forms.

Definition 2.1. Let k be an even integer and let $\rho\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_p(\mathbb{C})$ be a p-dimensional unitary complex representation such that $\rho(T)$ is diagonal, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For any integer $p \geq 1$, let $\boldsymbol{e}_j$ be the standard basis of $\mathbb{C}^p$. A vector-valued weakly holomorphic modular form of weight k with respect to ρ on $\mathrm{SL}_2(\mathbb{Z})$ is a sum $F(z) = \sum_{j=1}^{p} F_j(z)\boldsymbol{e}_j$ of functions holomorphic in $\mathbb{H}$ satisfying the following conditions:

(1) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$(F|_{k,\rho}\gamma)(z) = F(z).$$

(2) Each function $F_j(z)$ has a convergent q-expansion holomorphic at infinity:

$$F_j(z) = \sum_{n \gg -\infty} a_n(j) q^{n/N_j}.$$

We write $M^!_{k,\rho}$ for the space of vector-valued weakly holomorphic modular forms of weight k with respect to ρ on $\mathrm{SL}_2(\mathbb{Z})$. There are subspaces $M_{k,\rho}$ and $S_{k,\rho}$ of vector-valued holomorphic modular forms and vector-valued cusp forms, respectively, for which we require that each $a_n(j) = 0$ when n is negative, respectively, non-positive. We also let $S^!_{k,\rho}$ denote the space of vector-valued weakly holomorphic cusp forms, which are weakly holomorphic modular forms $F(z) = \sum_{j=1}^{p} \sum_{n\ge -M_j} a_n(j) q^{n/N_j} \boldsymbol{e}_j$ satisfying $a_0(j) = 0$ for all $j \in \{1, 2, \ldots, p\}$.

The following theorem is an analog of the elementary estimates for classical modular forms of weight k, namely $a(n) = O(n^k)$ for holomorphic modular forms and $a(n) = O(n^{k/2})$ for cusp forms.

Theorem 2.2. [12, Section 1] *Let $F(z) = \sum_{j=1}^{p} F_j(z)\boldsymbol{e}_j$ be a vector-valued holomorphic modular form of weight k associated to a representation ρ of $\mathrm{SL}_2(\mathbb{Z})$. Then the Fourier coefficients $a_n(j)$ satisfy the growth condition $a_n(j) = O(n^k)$ for every $1 \le j \le p$, as $n \to \infty$. If $F(z)$ is cuspidal then $a_n(j) = O(n^{k/2})$ for every $1 \le j \le p$, as $n \to \infty$.*

For a vector-valued cusp form $F(z) = \sum_{j=1}^{p} \sum_{n\ge 1} a_n(j) q^{n/N_j} \boldsymbol{e}_j$ we define an L-series

$$L(F, s) = \sum_{j=1}^{p} \sum_{n\ge 1} \frac{a_n(j)}{(n/N_j)^s} \boldsymbol{e}_j.$$

By Theorem 2.2 we know that this series converges absolutely for $\mathrm{Re}(s) \gg 0$.

Theorem 2.3. *If $F(z) \in S_{k,\rho}$ is a vector-valued cusp form of weight k, then*

$$\frac{\Gamma(s)}{(2\pi)^s} L(F, s) = \int_0^\infty F(iy) y^s \frac{dy}{y}.$$

Furthermore, $L(F, s)$ has an analytic continuation and satisfies a functional equation

$$\xi(F, s) = i^k \rho(S) \xi(F, k - s),$$

where $\xi(F, s) = \frac{\Gamma(s)}{(2\pi)^s} L(F, s)$.

We omit the proof of Theorem 2.3 since it follows from the same argument for classical modular forms.

For a cusp form $F(z)$ of weight k, we define the period polynomial

$$r(F, z) := \int_0^{i\infty} F(\tau)(z - \tau)^{k-2}\, d\tau.$$

A computation reveals that

$$\begin{aligned} r(F,z) &= \int_0^{i\infty} F(\tau)(z-\tau)^{k-2}\,d\tau = \int_0^{\infty} F(iy)(z-iy)^{k-2} i\,dy \\ &= \int_0^{\infty} F(iy)\sum_{n=0}^{k-2}\binom{k-2}{n}(-iy)^n z^{k-2-n} i\,dy \\ &= \sum_{n=0}^{k-2} i^{n+1}\binom{k-2}{n} r_n(F) z^{k-2-n}, \end{aligned}$$

where $r_n(F) = \int_0^\infty F(iy)y^n\,dy$ is the nth period of $F(z)$. The nth period $r_n(F)$ encodes the critical value of $L(F,s)$

$$r_n(F) = \int_0^\infty F(iy)y^n\,dy = \xi(F,n+1) = \frac{\Gamma(n+1)}{(2\pi)^{n+1}} L(F,n+1).$$

Therefore, we can rewrite the period polynomial $r(F,z)$ in terms of the critical values of $L(F,s)$

$$\begin{aligned} r(F,z) &= \sum_{n=0}^{k-2} i^{n+1}\binom{k-2}{n} r_n(F) z^{k-2-n} \\ &= \sum_{n=0}^{k-2} i^{n+1}\binom{k-2}{n}\frac{\Gamma(n+1)}{(2\pi)^{n+1}} L(F,n+1) z^{k-2-n} \\ &= (k-2)!\, i^{k+3}(2\pi)^{-k+1}\sum_{n=0}^{k-2}(-1)^n \frac{L(F,n+1)}{(k-2-n)!}(2\pi i z)^{k-2-n}. \end{aligned}$$

G. Bol [1] stated the following theorem known as Bol's identity.

Theorem 2.4. *If $f(z)$ is a smooth complex-valued function on $\mathbb{H}$, then*

$$\frac{\partial^{k-1}}{\partial z^{k-1}}(cz+d)^{k-2} f(\gamma z) = (cz+d)^{-k} f^{(k-1)}(\gamma z),$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ where $\gamma z = \frac{az+b}{cz+d}$ as usual.

This implies that if $f(z)$ is a (scalar-valued) modular form of weight $2-k$ then $f^{(k-1)}(z)$ is a (scalar-valued) modular form of weight k. This identity can be written in terms of slash operator and extended to the vector-valued functions: If $F(z) = \sum_{j=1}^p F_j(z)\boldsymbol{e}_j$ is a smooth vector-valued function, then we have the identity

$$\frac{\partial^{k-1}}{\partial z^{k-1}}(F|_{2-k,\rho}\gamma)(z) = (F^{(k-1)}|_{k,\rho}\gamma)(z),$$

where $\frac{\partial}{\partial z}F(z) = \sum_{j=1}^p (\frac{\partial}{\partial z}F_j)(z)\boldsymbol{e}_j$. So we can say that if $F(z)$ is a vector-valued modular form of weight $2-k$ then $F^{(k-1)}(z)$ is a vector-valued modular form of weight k. As in the scalar-valued modular form case (see [8]) we can define Eichler integrals for vector-valued modular forms by using Bol's identity.

Definition 2.5. Eichler integral of weight k is a holomorphic vector-valued function $G(z)$ on $\mathbb{H}$ such that $\left(\frac{\partial^{k-1}}{\partial z^{k-1}}G\right)(z)$ is modular of weight k where *modular* means that it is invariant under the slash operator. In particular, if $F(z)$ is modular of weight k, then we say that $G(z)$ is an Eichler integral of $F(z)$ if $\left(\frac{\partial^{k-1}}{\partial z^{k-1}}G\right)(z) = F(z)$.

For example, let $F(z) = \sum_{j=1}^{p} \sum_{n>0} a_n(j) q^{n/N_j} \boldsymbol{e}_j$ be a vector-valued cusp form of weight $k \geq 2$. Then the function

$$\mathcal{E}_F(z) = \frac{1}{c_k} \int_z^{i\infty} F(\tau)(z-\tau)^{k-2}\, d\tau$$

is an Eichler integral of $F(z)$ and its Fourier expansion is given by

$$\sum_{j=1}^{p} \sum_{n>0} a_n(j) \left(\frac{n}{N_j}\right)^{1-k} q^{n/N_j} \boldsymbol{e}_j. \tag{2.1}$$

We can prove that $\mathcal{E}_F(z)$ is an Eichler integral by using induction on $k-2$. Furthermore, it is easy to check that the function (2.1) is also an Eichler integral of $F(z)$. Since both $\mathcal{E}_F(z)$ and (2.1) are Eichler integrals of $F(z)$, their difference is given by a polynomial whose degree is at most $k-2$. But this polynomial should be zero since $\mathcal{E}_F(i\infty) = 0$. This completes the proof about Fourier expansion of $\mathcal{E}_F(z)$.

3. Vector-valued harmonic weak Maass forms

Vector-valued harmonic weak Maass forms are studied by Bruinier and Funke [5] when ρ is a Weil representation. In this section we define those for the general representation and study their properties. For the most part, this is the review of [5] and [6].

Definition 3.1. A vector-valued harmonic weak Maass form of weight k is a real-analytic vector-valued function $F(z) = \sum_{j=1}^{p} F_j(z)\boldsymbol{e}_j$ which satisfies the followings

(1) $F(z)$ is modular of weight k.

(2) $F(z)$ is annihilated by the weight k hyperbolic Laplacian Δ_k where

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and $z = x + iy$.

(3) There is a $C > 0$ such that the function $F_j(z) = O(e^{Cy})$ as $y \to \infty$ (uniformly in x).

(4) Each function $F_j(z)$ has a convergent q-expansion at infinity:

$$F_j(z) = \sum_{n\in\mathbb{Z}} a_n(j,y) e(nx/N_j)$$

for positive integers N_j. Here and below, $e(x) = e^{2\pi i x}$ as usual.

We write $H_{k,\rho}$ for the space of vector-valued harmonic weak Maass forms of weight k with respect to ρ.

Let $F(z) \in H_{k,\rho}$. Because of property (2) in Definition 3.1, the coefficients $a_n(j,y)$ satisfy the second order differential equation $\Delta_k a_n(j,y)e(nx/N_j) = 0$ as functions in y. Solving this differential equation we find that

$$a_n(j,y) = \begin{cases} a_0^+(j) + a_0^-(j)y^{1-k} & \text{if } n = 0, \\ a_n^+(j)e^{-2\pi ny/N_j} + a_n^-(j)H(2\pi ny/N_j) & \text{if } n \neq 0, \end{cases}$$

with complex coefficients $a_n^\pm(j)$ where $H(w) = e^{-w}\int_{-2w}^{\infty} e^{-t}t^{-k}\,dt$. The function $H(w)$ has the asymptotic behavior

$$H(w) \sim \begin{cases} (2|w|)^{-k}e^{-|w|} & \text{for } w \to -\infty, \\ (-2w)^{-k}e^{w} & \text{for } w \to +\infty. \end{cases}$$

Then any vector-valued harmonic weak Maass form $F(z)$ of weight k has a unique decomposition $F(z) = F^+(z) + F^-(z)$, where

$$\begin{aligned} F^+(z) &= \sum_{j=1}^{p}\sum_{n\in\mathbb{Z}} a_n^+(j)q^{n/N_j}\boldsymbol{e}_j, \\ F^-(z) &= \sum_{j=1}^{p}\left(a_0^-(j)y^{1-k} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n^-(j)H(2\pi ny/N_j)e(nx)\right)\boldsymbol{e}_j. \end{aligned} \tag{3.1}$$

Note that if $F(z)$ satisfies property (3) in Definition 3.1, then all but finitely many $a_n^+(j)$ (respectively $a_n^-(j)$) with negative (respectively positive) index n vanish.

Let us briefly recall the Maass raising and lowering operators on non-holomorphic modular forms of weight k. They are defined as the differential operators

$$R_k = 2i\frac{\partial}{\partial z} + ky^{-1} \quad \text{and} \quad L_k = -2iy^2\frac{\partial}{\partial \bar{z}}.$$

The raising operator R_k maps $H_{k,\rho}$ to $H_{k+2,\rho}$, and the lowering operator L_k maps to $H_{k,\rho}$ to $H_{k-2,\rho}$. The Laplacian Δ_k can be expressed in terms of R_k and L_k by $-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k$. The following lemma is proved by a straightforward computation.

Lemma 3.2. *Let $F(z) \in H_{k,\rho}$ be a vector-valued harmonic weak Maass form of weight k. Then*

$$\begin{aligned} L_kF(z) &= L_kF^-(z) \\ &= -2v^{2-k}\sum_{j=1}^{p}\left((k-1)a_0^-(j) + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n^-(j)(-4\pi n/N_j)^{1-k}e(n\bar{z}/N_j)\right)\boldsymbol{e}_j. \end{aligned}$$

Using this lemma, we can prove the following result.

Proposition 3.3. *The assignment* $F(z) \mapsto \xi_k(F)(z) := y^{k-2}\overline{L_k F(z)} = R_{-k} y^k \overline{F(z)}$ *define an anti-linear mapping*

$$\xi_k \colon H_{k,\rho} \to M^!_{2-k,\overline{\rho}}.$$

Its kernel is $M^!_{k,\rho} \subset H_{k,\rho}$.

We let $H^+_{k,\rho}$ denote the inverse image of the space of vector-valued cusp forms $S_{2-k,\overline{\rho}}$ under the mapping ξ_k. Hence, if $F(z) \in H^+_{k,\rho}$, then the Fourier coefficients $a^-_n(j)$ with non-negative index n vanish, so $F^-(z)$ is rapidly decreasing for $y \to \infty$. Clearly $M^!_{k,\rho} \subset H^+_{k,\rho}$.

Now let $F(z) \in H_{k,\rho}$ and write its Fourier expansion as in (3.1). Then we call the Fourier polynomial

$$P(F)(z) = \sum_{j=1}^{p} \sum_{\substack{n \in \mathbb{Z} \\ n \leq 0}} a^+_n(j) q^{n/N_j} \boldsymbol{e}_j$$

the principal part of $F(z)$. Observe that if $F(z) \in H^+_{k,\rho}$, then $F(z) - P(F)(z)$ is exponentially decreasing as $y \to \infty$. From the transformation behavior and the growth of $F(z)$, we prove the following lemma.

Lemma 3.4. *If* $F(z) \in H_{k,\rho}$, *then there is a constant* $C > 0$ *such that* $F(z) = O(e^{C/y})$ *as* $y \to 0$, *uniformly in* x.

Later, we will need the following growth estimate for the Fourier coefficients of vector-valued harmonic weak Maass forms.

Lemma 3.5. *Let* $F(z) \in H_{k,\rho}$ *and write its Fourier expansion as in* (3.1). *Then there is a constant* $C > 0$ *such that the Fourier coefficients satisfy*

$$a^+_n(j) = O\left(e^{C\sqrt{|n|}}\right), \quad n \to +\infty,$$
$$a^-_n(j) = O\left(e^{C\sqrt{|n|}}\right), \quad n \to -\infty.$$

If $F(z) \in H^+_{k,\rho}$, *then the* $a^-_n(j)$ *actually satisfy the stronger bound* $a^-_n(j) = O(|n|^{k/2})$ *as* $n \to -\infty$.

Proof. To prove the asymptotic for $a^-_n(j)$ we consider the vector-valued weakly holomorphic modular form $\xi_k(F)(z) \in M^!_{2-k,\overline{\rho}}$. By Lemma 3.2 and the formula for the Fourier coefficients we have

$$2a^-_n(j)(-4\pi n)^{1-k} = -\int_0^{N_j} y^{k-2}\overline{L_k F_j(z)} e(nz/N_j)\, dx. \tag{3.2}$$

Thus, according to Lemma 3.4, we get

$$a_n^-(j) \ll |n|^{k-1} \int_0^{N_j} e^{C/y} e^{-2\pi n y/N_j}\, dx$$

for all positive $0 < y \le 1$ with some positive constant C (independent of y and n). If we take y equal to $1/\sqrt{|n|}$, we see that

$$a_n^-(j) \ll |n|^{k-1} e^{C\sqrt{|n|}} e^{2\pi\sqrt{|n|}/N_j}$$

for all $n < 0$, proving the first assertion on the $a_n^-(j)$.

From $\xi_k(F)(z) \in M^!_{2-k,\overline{\rho}}$ it can be deduced that the individual functions $F^+(z)$ and $F^-(z)$ in the splitting $F(z) = F^+(z) + F^-(z)$ also satisfy the estimate of Lemma 3.4. We may apply the above argument to

$$a_n^+(j) = \int_0^{N_j} F_j^+(z) e(-nz/N_j)\, dx$$

to derive the estimate for the $a_n^+(j)$ as $n \to +\infty$ where $F_j^+(z)$ is the jth component of $F^+(z)$.

If $F(z) \in H^+_{k,\rho}$, then $\xi_k(F)(z) \in S_{2-k,\overline{\rho}}$ is a vector-valued cusp form. Hence Theorem 2.2 implies that the left hand side of (3.2) is bounded by some constant time $|n|^{1-k/2}$ for all $n < 0$. Thus $a_n^-(j) = O(|n|^{k/2})$ as $n \to -\infty$. $\square$

By Bol's identity, we see that the differential operator $D := \frac{1}{2\pi i}\frac{\partial}{\partial z}$ define a linear map

$$D^{k-1} \colon M^!_{2-k,\rho} \to M^!_{k,\rho}.$$

This map may be extended to harmonic weak Maass forms.

Theorem 3.6. *If $2 \le k \in \mathbb{Z}$ and $F(z) \in H^+_{2-k,\rho}$ with Fourier expansion as in* (3.1)*, then*

$$D^{k-1}(F)(z) \in M^!_{k,\rho}.$$

Moreover, assuming the notation in (3.1)*, we have*

$$D^{k-1}(F)(z) = D^{k-1}(F^+)(z) = \sum_{j=1}^{p} \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_n^+(j) \left(\frac{n}{N_j}\right)^{k-1} q^{n/N_j} \boldsymbol{e}_j.$$

Proof. To prove this theorem, we compute the Fourier expansion of $D^{k-1}F(z)$. Since we have

$$\begin{aligned} D^{k-1}(F^+)(z) &= \left(\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{k-1} \left(\sum_{j=1}^{p}\sum_{n\in\mathbb{Z}} a_n^+(j) q^{n/N_j} \boldsymbol{e}_j\right) \\ &= \sum_{j=1}^{p} \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_n^+(j) \left(\frac{n}{N_j}\right)^{k-1} q^{n/N_j} \boldsymbol{e}_j, \end{aligned}$$

it suffices to check that $D^{k-1}(F^-)(z)=0$. Note that

$$\begin{aligned}H(2\pi ny/N_j)e(nx) &= \Gamma(k-1,4\pi|n|y/N_j)e^{2\pi inz/N_j}\\ &= \Gamma(k-1,4\pi|n|y/N_j)e^{-4\pi ny/N_j}e^{2\pi in\overline{z}/N_j}\end{aligned}$$

for $n<0$ where $\Gamma(l,x)=\int_x^\infty e^{-t}t^{l-1}\,dt$ is the incomplete gamma function (Here $F(z)$ is of weight $2-k$). By partial integration, we obtain

$$\Gamma(l+1,4\pi|n|y) = -e^{-4\pi|n|y}(4\pi|n|y)^l + l\Gamma(l,4\pi|n|y)$$

for $l>0$. By the iteration, we have

$$\begin{aligned}\Gamma(l+1,4\pi|n|y) &= -e^{-4\pi|n|y}(4\pi|n|y)^l - le^{-4\pi|n|y}(4\pi|n|y)^{l-1}\\ &\quad - l(l-1)e^{-4\pi|n|y}(4\pi|n|y)^{l-2}+\cdots\\ &\quad + l(l-1)\cdots 3e^{-4\pi|n|y}(4\pi|n|y)^2 + l!\Gamma(1,4\pi|n|y).\end{aligned}$$

Since $\Gamma(1,4\pi|n|y)=-e^{-4\pi|n|y}$, one can see that $\Gamma(l+1,4\pi|n|y)e^{4\pi|n|y}$ is a degree l polynomial of y. This shows that $\Gamma(k-1,4\pi|n|y/N_j)e^{-4\pi|n|y/N_j}$ is a degree $k-2$ polynomial of y. Since $\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$, we have $D^{k-1}(y^{k-2})=0$. From this, one can see that $D^{k-1}(H(2\pi ny/N_j)e(nx))=0$ for every $n<0$, which implies that $D^{k-1}(F^-)(z)=0$. □

4. L-series of vector-valued weakly holomorphic modular forms

In this section, we prove Theorems 1.1, 1.2 and 1.3 and give implications of them. We start with the proof of Theorem 1.1.

Proof of Theorem 1.1. To prove (1.5), we split the integral into two ranges. Inserting the Fourier expansion of $F(z)=\sum_{j=1}^p\sum_{n\geq -M_j}a_n(j)q^{n/N_j}\boldsymbol{e}_j$ yields that, for arbitrary $t_0>0$, we have

$$\begin{aligned}R.\int_{t_0}^\infty F(iy)y^{s-1}\,dy &= \sum_{j=1}^p\sum_{n\geq -M_j}a_n(j)\left[\int_{t_0}^\infty e^{-2\pi ny/N_j-ty}y^{s-1}\,dy\right]_{t=0}\boldsymbol{e}_j\\ &= \left[\sum_{j=1}^p\sum_{n\geq -M_j}\frac{a_n(j)}{(t+2\pi n/N_j)^s}\int_{2\pi nt_0/N_j+tt_0}^\infty e^{-y}y^{s-1}\,dy\,\boldsymbol{e}_j\right]_{t=0}\\ &= \sum_{j=1}^p\sum_{n\geq -M_j}\frac{a_n(j)}{(2\pi n)^s}(N_j)^s\Gamma(s,2\pi nt_0/N_j)\boldsymbol{e}_j.\end{aligned}$$

Similarly we compute that

$$R.\int_0^{t_0}F(iy)y^{s-1}\,dy = \rho^{-1}(S)i^{-k}\sum_{j=1}^p(N^j)^{k-s}\sum_{n\geq -M_j}\frac{a_n(j)\Gamma(k-s,\frac{2\pi n}{t_0N_j})}{(2\pi n)^{k-s}}\boldsymbol{e}_j.$$

This yields the integral representation of $L^*(F,s)$. Since $t_0 > 0$ is chosen arbitrarily, one can see that the definition of $L^*(F,s)$ in (1.2) is independent of t_0.

Note that, by the definition of $L^*(F,s)$ in (1.2), $L^*(F,k-s)$ is equal to

$$\begin{aligned}
&\sum_{j=1}^{p}(N_j)^{k-s}\sum_{n\geq -M_j}\frac{a_n(j)\Gamma(k-s,2\pi n t_0/N_j)}{(2\pi n)^{k-s}}\boldsymbol{e}_j\\
&+\rho^{-1}(S)\sum_{j=1}^{p} i^{-k}(N_j)^{s}\sum_{n\geq -M_j}\frac{a_n(j)\Gamma\left(s,\frac{2\pi n}{t_0N_j}\right)}{(2\pi n)^{s}}\boldsymbol{e}_j\\
&=\rho(S)i^kR.\int_0^{1/t_0}F(iy)y^{s-1}\,dy+\rho^{-1}(S)i^{-k}R.\int_{1/t_0}^{\infty}F(iy)y^{s-1}\,dy.
\end{aligned}$$

Since $F(z)$ is modular of weight k with respect to ρ and $-I\in \mathrm{SL}_2(\mathbb{Z})$, we have $\rho(-I)F(z) = F(z)$, where $I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Therefore, we obtain

$$R.\int_{1/t_0}^{\infty}F(iy)y^{s-1}\,dy=\rho(-I)R.\int_{1/t_0}^{\infty}F(iy)y^{s-1}\,dy.$$

Moreover, $i^{2k}=1$ since k is an even integer. Therefore, we have

$$\begin{aligned}
\rho^{-1}(S)i^{-k}R.\int_{1/t_0}^{\infty}F(iy)y^{s-1}\,dy&=\rho^{-1}(S)\rho(-I)i^{2k}i^{-k}R.\int_{1/t_0}^{\infty}F(iy)y^{s-1}\,dy\\
&=\rho(S)i^kR.\int_{1/t_0}^{\infty}F(iy)y^{s-1}\,dy.
\end{aligned}$$

This implies that

$$\begin{aligned}
L^*(F,k-s)&=\rho(S)i^k\left(R.\int_0^{1/t_0}F(iy)y^{s-1}\,dy+R.\int_{1/t_0}^{\infty}F(iy)y^{s-1}\,dy\right)\\
&=\rho(S)i^kR.\int_0^{i\infty}F(iy)y^{s-1}\,dy\\
&=\rho(S)i^kL^*(F,s).
\end{aligned}$$

This completes the proof. □

We define Eichler integral formally

$$\mathcal{E}_F(z):=\sum_{j=1}^{p}\sum_{n\geq -M_j}\frac{a_n(j)}{(n/N_j)^{k-1}}q^{n/N_j}\boldsymbol{e}_j$$

for $F(z)\in S^!_{k,\rho}$. This Eichler integral also has an integral representation.

Proposition 4.1. *For $F(z)\in S^!_{k,\rho}$, we have*

$$\mathcal{E}_F(z)=c_k^{-1}R.\int_z^{i\infty}F(\tau)(\tau-z)^{k-2}\,d\tau.$$

Proof. In the range of integration, the only possible pole of the integrand is at $\tau \to i\infty$. Thus we compute, writing $F(z) = \sum_{j=1}^{p} \sum_{n \geq -M_j} a_n(j) q^{n/N_j} \boldsymbol{e}_j$,

$$\begin{aligned} R.\int_z^{i\infty} F(\tau)(\tau - z)^{k-2}\, d\tau &= \left[\int_z^{i\infty} e^{iu\tau} F(\tau)(\tau - z)^{k-2}\, d\tau\right]_{u=0} \\ &= \sum_{j=1}^{p} \sum_{n \geq -M_j} a_n(j) \left[\int_z^{i\infty} e^{iu\tau + 2\pi i n\tau/N_j} (\tau - z)^{k-2}\, d\tau\right]_{u=0} \boldsymbol{e}_j \\ &= \sum_{j=1}^{p} \sum_{n \geq -M_j} a_n(j) q^{n/N_j} \left[\int_0^{\infty} e^{i\tau(u + 2\pi n/N_j)} \tau^{k-2}\, d\tau\right]_{u=0} \boldsymbol{e}_j. \end{aligned}$$

The integral now converges at $u = 0$ and inserting the integral representation of the gamma function yields the claim. □

Now we give a proof of Theorem 1.2. Since our proof will proceed as [3, Proposition 5.1], we will not give every detail.

Proof of Theorem 1.2. Throughout the proof, we choose $t_0 = 1$ and fix $j \in \{1, 2, \ldots, p\}$, therefore we drop j from the notation. First, we examine the second part of (1.2). Since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x, and

$$\lim_{n\to\infty} \sum_{m \geq -M} a_m \frac{\Gamma(k - n, 2\pi m/N)}{(2\pi m/N)^{k-n}} = \lim_{n\to\infty} \sum_{m \geq -M} a_m \Gamma(-n, 2\pi m/N) \left(\frac{2\pi m}{N}\right)^n, \tag{4.1}$$

it suffices to show that the above limit exists. Let $x = 2\pi m/N$. By [3, Proof of Proposition 5.1], $|\Gamma(-n, x)x^n| \ll e^x$ if $|x| > 1$. On the other hand, when $|x| < 1$, by [3, eqn. 5.3], we see that

$$\begin{aligned} |\Gamma(-n, x)x^n| &= \left|(-1)^n \frac{\Gamma(0, x)x^n}{n!} + \frac{(-1)^{n+1} e^{-x} x^{n-1}}{n!} \sum_{t=0}^{n-1} \frac{(-1)^t t!}{x^t}\right| \\ &< \frac{1}{n!} + \frac{1}{n} \sum_{t=0}^{n-1} \binom{n}{m}^{-1} \frac{x^{n-m-1}}{(n - m - 1)!} \\ &< \frac{1}{n!} + \frac{e^x}{n}. \end{aligned}$$

Therefore, we can split the negative indexes from (4.1). Moreover, since $|a_m| \ll e^{C\sqrt{|m|}}$ for some positive number C, by adopting the same argument in [3], we see that

$$\lim_{n\to\infty} \sum_{m \geq 1} a_m \Gamma(-n, 2\pi m/N) \left(\frac{2\pi m}{N}\right)^n = 0.$$

Now we examine the first sum of (1.2). For $M > C^2N^2/\pi^2$, we observe that

$$\begin{aligned}
\sum_{m\geq M} \frac{a_m\Gamma(n,2\pi m/N)}{(2\pi m/N)^n} &\ll \left(\frac{N}{2\pi}\right)^n \sum_{m\geq M} e^{C\sqrt{m}} m^{-n} \int_{2\pi m/N}^{\infty} e^{-t}t^{n-1}\,dt \\
&\ll \left(\frac{N}{2\pi}\right)^n \sum_{m\geq M} m^{-n} \int_{2\pi m/N}^{\infty} e^{-t/2}t^{n-1}\,dt \\
&\ll \left(\frac{N}{\pi}\right)^n (n-1)! \sum_{m\geq M} m^{-n} \ll (n-2)! \left(\frac{N}{\pi M}\right)^n.
\end{aligned}$$

Therefore, for $M > 2m_j$, the contribution for the limit is zero. For the remaining terms, by combining mth and $-m$th terms, we obtain

$$\begin{aligned}
&\frac{(2\pi m_j)^n}{(n-1)!}\left(a_m\frac{\Gamma(n,2\pi m/N)}{(2\pi m)^n} + a_{-m}\frac{\Gamma(n,-2\pi m/N)}{(-2\pi m)^n}\right) \\
&= \left(\frac{m_j}{m}\right)^n \left(a_m\frac{\Gamma(n,2\pi m/N)}{(n-1)!} + (-1)^\delta a_{-m}\frac{\Gamma(n,-2\pi m/N)}{(n-1)!}\right) \\
&\to \begin{cases} 0, & \text{if } m > m_j, \\ a_{m_j}(j) + (-1)^\delta a_{-m_j}(j), & \text{if } m = m_j, \end{cases} \qquad \text{as } n\to\infty
\end{aligned} \tag{4.2}$$

since $\lim_{n\to\infty}\frac{\Gamma(n,x)}{(n-1)!} = 1$. Note that if there are more negative terms with $m \leq -2m_j$, the contribution is zero by the first part of the proof. Thus, it remains to show that (4.2) tends to 0 for $0 < m < m_j$. This can be easily seen from the following estimate

$$\Gamma(n,x) - \Gamma(n,-x) = \int_x^{-x} e^{-t}t^{n-1}\,dt \ll \begin{cases} x^n, & \text{if } |x| > 1, \\ 1, & \text{if } |x| \leq 1. \end{cases} \qquad \square$$

Following two corollaries follows immediately from Theorem 1.2.

Corollary 4.2. *For $F(z) \in S^!_{k,\rho}$, we have*

$$\lim_{n\to\infty} \|L^*(F,n)\| = \infty.$$

Corollary 4.3. *For $F(z) \in S^!_{k,\rho}$, there are at most finitely many values n such that at least one of components of $L(F^*,n)$ vanishes.*

Now we give a proof of Theorem 1.3.

Proof of Theorem 1.3. We let

$$\begin{aligned}
\mathcal{G}(z) := &-(2i)^{1-k}R.\int_{-\overline{z}}^{i\infty} \xi_{2-k}(\mathcal{F})^c(w)(z+w)^{k-2}\,dw \\
&+ c_k^{-1}R.\int_z^{i\infty} D^{k-1}(\mathcal{F})(w)(z-w)^{k-2}\,dw,
\end{aligned} \tag{4.3}$$

where $g^c(z) := \sum_{j=1}^{p}\sum_{n\geq -M_j}\overline{a_n(j)}q^{n/N_j}\boldsymbol{e}_j$ for $g(z) = \sum_{j=1}^{p}\sum_{n\geq -M_j}a_n(j)q^{n/N_j}\boldsymbol{e}_j$. We first show that $\mathcal{G}(z)$ and $\mathcal{F}(z)$ are identical up to a constant term. Since $\xi_{2-k}(\mathcal{G}) = \xi_{2-k}(\mathcal{F})$ and $D^{k-1}(\mathcal{G}) = D^{k-1}(\mathcal{F})$, we see that

$$\mathcal{G}(z) = \mathcal{F}(z) + p(z),$$

where $p(z)$ is a vector-valued polynomial of degree at most $k-2$. Because $\mathcal{G}(z)$ and $\mathcal{F}(z)$ are both invariant under $|_{2-k,\rho}T$, we have that $p_j(z) = a_j$ is a constant for each $1 \leq j \leq p$. Let $\mathcal{F}_1(z)$ (respectively $\mathcal{F}_2(z)$) be the first (respectively second) summand of (4.3). Then we observe that $\mathcal{F}_1(z)$ (respectively $\mathcal{F}_2(z)$) is the same as $\mathcal{G}^-(z)$ (respectively $\mathcal{G}^+(z)$) up to constant.

From the transformation property of $\mathcal{F}(z)$, we derive that

$$\begin{aligned}
0 &= (\mathcal{F}|_{2-k,\rho}(1-S))(z)\\
&= (\mathcal{F}_1|_{2-k,\rho}(1-S))(z) + (\mathcal{F}_2|_{2-k,\rho}(1-S))(z) - (p|_{2-k,\rho}(1-S))(z),
\end{aligned}$$

where $(p|_{2-k,\rho}(1-S))(z)$ is a constant multiple of $1 - z^{k-2}$. A calculation shows that

$$(\mathcal{F}_1|_{2-k,\rho}S)(z) = -(2i)^{1-k}R.\int_{-\overline{z}}^{0}\xi_{2-k}(\mathcal{F})^c(w)(z+w)^{k-2}\,dw,$$

and

$$(\mathcal{F}_2|_{2-k,\rho}S)(z) = c_k^{-1}R.\int_{z}^{0}D^{k-1}(\mathcal{F})(w)(z-w)^{k-2}\,dw.$$

Therefore we have arrived at

$$\begin{aligned}
&(2i)^{1-k}R.\int_{0}^{i\infty}\xi_{2-k}(\mathcal{F})^c(w)(z+w)^{k-2}\,dw\\
&\equiv c_k^{-1}R.\int_{0}^{i\infty}D^{k-1}(\mathcal{F})(w)(z-w)^{k-2}\,dw \pmod{1-z^{k-2}}.
\end{aligned}$$

If we insert the binomial expansion, then we obtain

$$\begin{aligned}
&c_k^{-1}\sum_{n=0}^{k-2} i^{1-n}\binom{k-2}{n}L^*(D^{k-1}(\mathcal{F}), n+1)z^{k-2-n}\\
&\equiv (2i)^{1-k}\sum_{n=0}^{k-2} i^{1-n}\binom{k-2}{n}L^*(\xi_{2-k}(\mathcal{F})^c, n+1)(-z)^{k-2-n} \pmod{1-z^{k-2}}.
\end{aligned}$$

Then the third claim follows by comparing coefficients and using the fact that for integers $0 \leq n \leq k-2$ we have

$$L^*(f^c, n+1) = \overline{L^*(f, n+1)}. \tag{4.4}$$

The second claim comes from the following representation of the period polynomial of $F(z) \in S^!_{k,\rho}$ in terms of its special L-values

$$r(F;z) = \sum_{n=0}^{k-2} i^{1-n} \binom{k-2}{n} L^*(F, n+1) z^{k-2-n}.$$

For the first claim note that by the definition of mock period polynomial we have

$$\mathbb{P}(\mathcal{F}^+, S; z) = -\frac{(4\pi)^{k-1}}{(k-2)!} (\mathcal{F}^- - \mathcal{F}^-|_{2-k,\rho} S)(z).$$

From this we obtain that

$$\mathbb{P}(\mathcal{F}^+, S; z) = -\frac{(2\pi i)^{k-1}}{(k-2)!} \sum_{n=0}^{k-2} \binom{k-2}{n} i^{n+1} L^*(\xi_{2-k}(\mathcal{F})^c, n+1) z^{k-2-n}.$$

Using (4.4) we deduce that

$$\begin{aligned}\overline{\mathbb{P}(\mathcal{F}^+, S; \overline{z})} &= \frac{(2\pi i)^{k-1}}{(k-2)!} \sum_{n=0}^{k-2} \binom{k-2}{n} i^{-n-1} L^*(\xi_{2-k}(\mathcal{F}), n+1) z^{k-2-n} \\ &= \sum_{n=0}^{k-2} \frac{L(\xi_{2-k}(\mathcal{F}), n+1)}{(k-2-n)!} (2\pi i z)^{k-2-n},\end{aligned}$$

which concludes the first claim. □

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