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On the Drinfeld Center of the Category of Comodules over a Co-quasitriangular Hopf Algebra

Haixing Zhu

Abstract. Let H be a co-quasitriangular Hopf algebra with bijective antipode. We prove that the Drinfeld center of the category of H-comodules is equivalent to the category of modules over some braided group. In particular, the equivalence holds not only for a finite dimensional H, but also for an infinite dimensional one.

1. Introduction

Braided tensor categories have played an important role in various areas such as conformal field theory, string theory and low-dimensional topology. Some important examples of braided tensor categories have been constructed by using the theory of Hopf algebras or quantum groups [4]. For example, there exists the structure of a braided tensor category in the (co-)representation category of (co-)quasitriangular Hopf algebras (for example, see [9]).

A categorical construction of a braided tensor category is the well-known Drinfeld's center construction, which was given by Drinfeld (unpublished), independently by Joyal and Street [8] and Majid [11]. In general, one can obtain a braided tensor category $\mathscr{Z}(\mathscr{C})$ from a tensor category $\mathscr{C}$, where $\mathscr{Z}(\mathscr{C})$ is often called the Drinfeld center of $\mathscr{C}$. Drinfeld centers are very important in the study of braided tensor categories [2, 5–7]. For example, the braided auto-equivalences of $\mathscr{Z}(\mathscr{C})$ were used to classify G-extensions of a given fusion category [7], and to compute the Brauer group $\mathrm{Br}(\mathscr{C})$ of a braided fusion category [2]. Thus it would be very interesting to study the Drinfeld center of a (braided) tensor category.

Denote by ${}_H\mathscr{M}$ the category of modules over a quasitriangular Hopf algebra (H, R). To investigate braided auto-equivalences of $\mathscr{Z}({}_H\mathscr{M})$, we redescribed $\mathscr{Z}({}_H\mathscr{M})$ in [19] by showing that $\mathscr{Z}({}_H\mathscr{M})$ is equivalent to the category ${}^{H_R}({}_H\mathscr{M})$ of comodules over some braided group H_R, namely, $\mathscr{Z}({}_H\mathscr{M}) \simeq {}^{H_R}({}_H\mathscr{M})$. Based on this key equivalence, it was verified that some special braided auto-equivalences can be induced by quantum-commutative Galois objects and then the Brauer group $\mathrm{Br}({}_H\mathscr{M})$ was characterized [3, 18].

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It is noticed that the notion of co-quasitriangular Hopf algebras was first introduced in [10] as the dual one of quasitriangular Hopf algebras. Therefore, it would be a natural idea to take an analogous way to investigate the Drinfeld center $\mathscr{Z}(\mathscr{M}^H)$ of the category $\mathscr{M}^H$ of comodules over a co-quasitriangular Hopf algebra (H,σ). An actual but important problem arises, namely, how to establish the relationship between the Drinfeld center and the category of comodules over some braided group. When H is finite dimensional, we have handled it by resorting to the quasitriangular Hopf algebra (H^*,R) (the dual of H) since the category of H-comodules is equivalent to the category of H^*-modules. Thus we have $\mathscr{Z}(\mathscr{M}^H)\simeq\mathscr{Z}({}_{H^*}\mathscr{M})\simeq{}^{H^*_R}({}_{H^*}\mathscr{M})$. However, when H is infinite dimensional, it becomes very difficult to obtain such an equivalence because the dual of an infinite-dimensional Hopf algebra is not a Hopf algebra. To avoid the difficulty, we turn to discuss the relationship between the Drinfeld center and the category of modules over some braided group H_σ. We show that there is a braided tensor equivalence between the Drinfeld center $\mathscr{Z}(\mathscr{M}^H)$ and the category $(\mathscr{M}^H)_{H_\sigma}$ of H_σ-modules, i.e.,

$$\mathscr{Z}(\mathscr{M}^H)\simeq(\mathscr{M}^H)_{H_\sigma}.$$

In particular, the result not only holds for a finite dimensional H, but also for an infinite dimensional one. Importantly, the equivalence indicates that the investigation of braided auto-equivalences of $\mathscr{Z}(\mathscr{M}^H)$ by Galois theory would be, in general, resorted to the study of Galois co-objects. This is different from the quasitriangular case.

The paper is organized as follows. In Section 2, we recall some necessary definitions such as a co-quasitriangular Hopf algebra, a Yetter-Drinfeld module and the Drinfeld center of a tensor category. In Section 3, we verify that there is some braided tensor equivalence between the Drinfeld center of the category of comodules and the category of modules over some braided group.

2. Preliminaries

Throughout this paper k is a fixed field. Unless otherwise stated, unadorned tensor products will be over k. For a coalgebra over k, the coproduct will be denoted by Δ. We adopt Sweedler's notation for the comultiplication in [15], e.g., $\Delta(a)=a_1\otimes a_2$.

We assume that the reader is familiar with the notion of a (braided) tensor category and the theory of Hopf algebras [9]. And we make free use of the notions of algebras, bialgebras and Hopf algebras in a braided tensor category [14]. Throughout this paper a tensor category always means strict one.

2.1. Braided tensor categories and Drinfeld centers

Definition 2.1. [9, Definition XIII.1.1] Let $(\mathscr{C},\otimes,I)$ be a tensor category. A *braiding* in $(\mathscr{C},\otimes,I)$ consists of a family of natural isomorphisms

$$C_{U,V}\colon U\otimes V\to V\otimes U$$

defined for all objects U, V of $\mathscr{C}$ such that

$$C_{U\otimes V,W} = (C_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes C_{V,W}),$$
$$C_{U,V\otimes W} = (\mathrm{id}_V \otimes C_{U,W})(C_{U,V} \otimes \mathrm{id}_W),$$

for all U, V and W of $\mathscr{C}$. A *braided tensor category* is a tensor category $(\mathscr{C}, \otimes, I)$ equipped with a braiding. In particular, if $C_{-,-}$ is a braiding, then for any object V we have

$$(\mathrm{id}_V \otimes C_{V,V})(C_{V,V} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes C_{V,V}) = (C_{V,V} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes C_{V,V})(\mathrm{id}_V \otimes C_{V,V}).$$

Namely, $C_{V,V}$ is a solution to the *quantum Yang-Baxter equation.*

The most fundamental example of a braided tensor category is the Drinfeld center of a tensor category.

Definition 2.2. Let $(\mathscr{C}, \otimes, I)$ be a tensor category. By [9, Sec. XIII.4] the right *Drinfeld center* $\mathscr{Z}_r(\mathscr{C})$ of the tensor category $\mathscr{C}$ is the category, whose objects are pairs $(U, \nu_{-,U})$, where U is an object of $\mathscr{C}$ and $\nu_{-,U}$ is a natural family of isomorphisms, called *half-braidings*:

$$\nu_{M,U} \colon M \otimes U \to U \otimes M, \quad \forall\, M \in \mathscr{C}$$

satisfying the Hexagon Axioms. Similarly, one can define the left Drinfeld center of $\mathscr{C}$.

2.2. Co-quasitriangular Hopf algebras

Definition 2.3. A *co-quasitriangular Hopf algebra* is a pair (H, σ), where H is a Hopf algebra, and a k-linear map $\sigma \colon H \otimes H \to k$ satisfies:

$$(ab, c) = \sigma(a, c_1)\sigma(b, c_2), \quad \sigma(a, bc) = \sigma(a_1, c)\sigma(a_2, b),$$
$$\sigma(a_1, b_1)a_2b_2 = b_1a_1\sigma(a_2, b_2), \qquad \sigma(a, 1) = \varepsilon(a) = \sigma(1, a)$$

for any $a, b, c \in H$, and there exists $\sigma^{-1} \colon H \otimes H \to k$ such that

$$\sigma(a_1, b_1)\sigma^{-1}(a_2, b_2) = \varepsilon(a)\varepsilon(b) = \sigma^{-1}(a_1, b_1)\sigma(a_2, b_2)$$

for all $a, b, c \in H$, where σ^{-1} is called the *inverse* of σ, see [9].

In the sequence, (H, σ) will always denote a co-quasitriangular Hopf algebra with a k-linear map σ.

2.3. The category of comodules over a Hopf algebra

Let H be a Hopf algebra. Denote by $\mathscr{M}^H$ the category of right H-comodules. Let M and N be two right H-comodules. Define the following H-comodule on $M \otimes N$:

$$\rho^R(m \otimes n) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]} n_{[1]}$$

for all $m \in M$ and $n \in N$. Then we have a tensor category $(\mathscr{M}^H, \otimes, k)$, where k is the unit object.

In particular, if H is co-quasitriangular, then $\mathscr{M}^H$ can be equipped with a braiding C:

$$C_{M,N}(m \otimes n) = n_{[0]} \otimes m_{[0]} \sigma(m_{[1]}, n_{[1]}), \quad \text{for all } m \in M \text{ and } n \in N,$$

where M and N are any two objects in $\mathscr{M}^H$.

2.4. Yetter-Drinfeld modules and the Drinfeld center

Definition 2.4. [16] Let H be a Hopf algebra. A right H-module M is called a right *Yetter-Drinfeld module* if (M, ρ) is a right H-comodule satisfying the following condition:

$$(m \cdot h)_{[0]} \otimes (m \cdot h)_{[1]} = m_{[0]} \cdot h_2 \otimes S(h_1) m_{[1]} h_3, \tag{2.1}$$

for all $h \in H$ and $m \in M$. Here $\rho(m) = m_{[0]} \otimes m_{[1]}$ for all $m \in M$.

Denote by $\mathscr{Y}\mathscr{D}_H^H$ the category of right Yetter-Drinfeld modules. A Yetter-Drinfeld morphism is both right H-linear and right H-colinear. If the antipode of H is bijective, then $\mathscr{Y}\mathscr{D}_H^H$ is a braided tensor category with the braiding given by

$$C_{M,N}(m \otimes n) = n_{[0]} \otimes m_{[0]} \cdot n_{[1]},$$

where $m \in M \in \mathscr{Y}\mathscr{D}_H^H$ and $n \in N \in \mathscr{Y}\mathscr{D}_H^H$. In particular, if (H, σ) is a co-quasitriangular Hopf algebra, then every right H-module M is automatically a right Yetter-Drinfeld module with the following right action:

$$m \cdot h = m_{[0]} \sigma(m_{[1]}, h), \quad \forall\, m \in M,\ h \in H.$$

It is clear that the category $\mathscr{M}^H$ is a braided tensor subcategory of $\mathscr{Y}\mathscr{D}_H^H$.

Lemma 2.5. [9, Theorem XIII. 5.1] *Let H be a Hopf algebra with bijective antipode. Then $\mathscr{Z}_r(\mathscr{M}^H)$ is equivalent to $\mathscr{Y}\mathscr{D}_H^H$ as a braided tensor category.*

3. The Drinfeld center and the category of modules over a braided group

In this section, let (H, σ) be a co-quasitriangular Hopf algebra. We investigate the relationship between Yetter-Drinfeld modules and modules over some braided group. We first recall Majid's transmutation theory in [13].

Lemma 3.1. [13, Theoremm 4.1] *Let (H, σ) be a co-quasitriangular Hopf algebra. Then there is a Hopf algebra H_σ in the braided tensor category $\mathscr{M}^H$, where $H_\sigma = H$ as a linear vector space and as an object in $\mathscr{M}^H$ by*

$$\rho(b) = b_2 \otimes S(b_1)b_3, \quad \text{for all } b \in H_\sigma. \tag{3.1}$$

The coalgebra structure and unit in H_σ coincide with those of H. The multiplication is defined by

$$a \star b = a_2 b_2 \sigma(S(a_1)a_3, S(b_1)),$$

for all $a, b \in H_\sigma$. The antipode of H_σ is given by

$$\overline{S}(a) = S_H(a_2)\sigma(S(a_1), a_5)\sigma(S^2(a_3), a_4).$$

A Hopf algebra in a braided tensor category is usually called a braided group, see [11, 13]. In the sequel, H_σ will always denote the Hopf algebra H_σ in $\mathscr{M}^H$, and be called *braided group.*

Many examples of braided groups can be found in [12, 13]. Here we give one simple example.

Example 3.2. Let k be a field with $\mathrm{ch}(k) \neq 2$. Let H_4 be the Sweedler 4-dimensional Hopf algebra over k. Namely, H_4 is generated by two elements g and h satisfying

$$g^2 = 1, \quad h^2 = 0, \quad gh + hg = 0.$$

The comultiplication, the counit and the antipode are given as follows:

$$\Delta(g) = g \otimes g, \quad \Delta(h) = 1 \otimes h + h \otimes g,$$
$$\varepsilon(g) = 1, \quad S(g) = g, \quad \varepsilon(h) = 0, \quad S(h) = gh.$$

It is known that the co-quasitriangular structure in the basis $\{1, g, h, gh\}$ is the bilinear form σ

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & \alpha & -\alpha \\ 0 & 0 & \alpha & \alpha \end{pmatrix}$$

Then the braided group H_{4_σ} has the following algebraic structure:

$$g \star h = gh = h \star g, \quad h \star h = \alpha(1-g), \quad g \star g = 1.$$

The braided group H_{4_σ} was discussed in [12].

By [14] a right H-comodule M is called a *right H_σ-module* in the category $\mathscr{M}^H$ if $(M, \leftharpoonup)$ is a right H_σ-module such that $\leftharpoonup$ is right H-colinear, i.e.,

$$\rho(m \leftharpoonup b) = m_{[0]} \leftharpoonup b_{[0]} \otimes m_{[1]} b_{[1]}, \quad \forall\, b \in H_\sigma,\ m \in M, \tag{3.2}$$

where $\rho(b) = b_{[0]} \otimes b_{[1]} = b_2 \otimes S(b_1) b_3$ (see Lemma 3.1).

Similarly, one can define a left H_σ-module in the category $\mathscr{M}^H$. In the sequel, *a right H_σ-module in the category $\mathscr{M}^H$ will be called a right H_σ-module for short.*

Let $(M, \leftharpoonup)$ and $(N, \leftharpoonup)$ be two right H_σ-modules. It is not hard to see that the tensor product $M \otimes N$ is a right H_σ-module with the right coaction ρ and action $\leftharpoonup$:

$$\rho(m \otimes n) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]} n_{[1]},$$
$$(m \otimes n) \leftharpoonup a = m \leftharpoonup a_2 \otimes n_{[0]} \leftharpoonup a_4 \sigma(n_{[1]}, S(a_1) a_3),$$

for all $m \otimes n \in M \otimes N$ and $a \in H_\sigma$. Denote by $(\mathscr{M}^H)_{H_\sigma}$ the category of right H_σ-modules. Here a morphism in $(\mathscr{M}^H)_{H_\sigma}$ is both right H-colinear and right H_σ-linear.

By [14] the category $(\mathscr{M}^H)_{H_\sigma}$ is a tensor category with the unit object k. Now we discuss the relationship between right H_σ-modules and Yetter-Drinfeld modules.

Lemma 3.3. *Let (H, σ) be a co-quasitriangular Hopf algebra. If $(M, \leftharpoonup)$ is a right H_σ-module, then M is a right Yetter-Drinfeld module with the following H-module structure $\tilde{\cdot}$:*

$$m \,\tilde{\cdot}\, a = m_{[0]} \leftharpoonup a_2 \sigma(m_{[1]}, a_1),$$

for all $m \in M$ and $a \in H$.

Proof. We first verify that $(M, \tilde{\cdot})$ is a right H-module. For $a, b \in H$, we have:

$$\begin{aligned}
(m \,\tilde{\cdot}\, a) \,\tilde{\cdot}\, b &= [m_{[0]} \leftharpoonup a_2 \sigma(m_{[1]}, a_1)] \,\tilde{\cdot}\, b \\
&= [m_{[0]} \leftharpoonup a_2]_{[0]} \leftharpoonup b_2 \sigma([m_{[0]} \leftharpoonup a_2]_{[1]}, b_1) \sigma(m_{[1]}, a_1) \\
&= [m_{[0]} \leftharpoonup a_{2_{[0]}}] \leftharpoonup b_2 \sigma(m_{[1]} a_{2_{[1]}}, b_1) \sigma(m_{[2]}, a_1) \\
&= [m_{[0]} \leftharpoonup a_3] \leftharpoonup b_2 \sigma(m_{[1]} S(a_2) a_4, b_1) \sigma(m_{[2]}, a_1) \\
&= [m_{[0]} \leftharpoonup (a_3 \star b_2) \sigma(m_{[1]} S(a_2) a_4, b_1) \sigma(m_{[2]}, a_1) \\
&= [m_{[0]} \leftharpoonup (a_4 b_3) \sigma(S(a_3) a_5, S(b_2)) \sigma(m_{[1]} S(a_2) a_6, b_1) \sigma(m_{[2]}, a_1) \\
&= [m_{[0]} \leftharpoonup (a_4 b_4) \sigma(S(a_3) a_5, S(b_3)) \sigma(m_{[1]}, b_1) \sigma(S(a_2) a_6, b_2) \sigma(m_{[2]}, a_1) \\
&= [m_{[0]} \leftharpoonup (a_2 b_2) \sigma(m_{[1]}, b_1) \sigma(m_{[2]}, a_1) \\
&= [m_{[0]} \leftharpoonup (a_2 b_2) \sigma(m_{[1]}, a_1 b_1) = m \,\tilde{\cdot}\, (ab).
\end{aligned}$$

It is easy to see that $m \,\tilde{\cdot}\, 1 = m$.

Now we check that the compatible condition holds. In fact,

$$\begin{aligned}
\rho(m \,\tilde{\cdot}\, a) &= \rho(m_{[0]} \leftharpoonup a_2)\sigma(m_{[1]}, a_1)\\
&= m_{[0]} \leftharpoonup a_3 \otimes m_{[1]} S(a_2) a_4 \sigma(m_{[2]}, a_1)\\
&= m_{[0]} \leftharpoonup a_5 \sigma(m_{[1]}, a_4)\sigma(m_{[2]}, S(a_3)) \otimes m_{[3]} S(a_2) a_6 \sigma(m_{[4]}, a_1)\\
&= m_{[0]} \,\tilde{\cdot}\, a_4 \sigma(m_{[1]}, S(a_3)) \otimes m_{[2]} S(a_2) a_5 \sigma(m_{[3]}, a_1)\\
&= m_{[0]} \,\tilde{\cdot}\, a_4 \sigma(m_{[2]}, S(a_2)) \otimes S(a_3) m_{[1]} a_5 \sigma(m_{[3]}, a_1)\\
&= m_{[0]} \,\tilde{\cdot}\, a_2 \otimes S(a_1) m_{[1]} a_3,
\end{aligned}$$

for all $m \in M$ and $a \in H$. □

The following lemma says that the converse of Lemma 3.3 also holds.

Lemma 3.4. *Let (H, σ) be a co-quasitriangular Hopf algebra. If $(N, \cdot)$ is a right Yetter-Drinfeld module, then N is a right H_σ-module with the following structure $\widetilde{\leftharpoonup}$:*

$$n \,\widetilde{\leftharpoonup}\, a = n_{[0]} \cdot a_2 \sigma(n_{[1]}, S(a_1)),$$

where $\rho(n) = n_{[0]} \otimes n_{[1]}$ for all $n \in N$, and S is the antipode of H.

Proof. We first check that the action $\widetilde{\leftharpoonup}$ is H-colinear. For all $a \in H_\sigma$ and $n \in N$, we have

$$\begin{aligned}
\rho(n \,\widetilde{\leftharpoonup}\, a) &= \rho(n_{[0]} \cdot a_2)\sigma(n_{[1]}, S(a_1))\\
&= n_{[0]} \cdot a_3 \otimes S(a_2) n_{[1]} a_4 \sigma(n_{[2]}, S(a_1))\\
&= n_{[0]} \cdot a_3 \otimes n_{[2]} S(a_1) a_4 \sigma(n_{[1]}, S(a_2))\\
&= n_{[0]} \,\widetilde{\leftharpoonup}\, a_2 \otimes n_{[2]} S(a_1) a_3.
\end{aligned}$$

Now we show that $(N, \widetilde{\leftharpoonup})$ is a right H_σ-module. For any $n \in N$,

$$\begin{aligned}
n \,\widetilde{\leftharpoonup}\, (a \star b) &= n \,\widetilde{\leftharpoonup}\, (a_2 b_2)\sigma(S(a_1)a_3, S(b_1))\\
&= n_{[0]} \cdot (a_3 b_3)\sigma(n_{[1]}, S(a_2 b_2))\sigma(S(a_1)a_4, S(b_1))\\
&= n_{[0]} \cdot (a_3 b_3)\sigma(n_{[1]}, S(b_2)S(a_2))\sigma(S(a_1)a_4, S(b_1))\\
&= n_{[0]} \cdot (a_3 b_3)\sigma(n_{[1]}, S(a_2))\sigma(n_{[2]}, S(b_2)))\sigma(S(a_1)a_4, S(b_1))\\
&= n_{[0]} \cdot (a_3 b_2)\sigma(n_{[1]}, S(a_2))\sigma(n_{[2]} S(a_1) a_4, S(b_1))\\
&= (n_{[0]} \cdot a_3) \cdot b_2 \sigma(n_{[2]}, S(a_1))\sigma(S(a_2) n_{[1]} a_4, S(b_1))\\
&= (n_{[0]} \cdot a_2)_{[0]} \cdot b_2 \sigma(n_{[2]}, S(a_1))\sigma((n_{[0]} \cdot (a_2)_{[1]}, S(b_1))\\
&= (n_{[0]} \cdot a_2)_{[0]} \,\widetilde{\leftharpoonup}\, b \sigma(n_{[1]}, S(a_1)) = (n \,\widetilde{\leftharpoonup}\, a) \,\widetilde{\leftharpoonup}\, b.
\end{aligned}$$

Hence the associativity holds. It is clear that $n \,\widetilde{\leftharpoonup}\, 1 = n$.

Therefore, $(N, \widetilde{\leftharpoonup})$ is a right module over the braided group H_σ. □

Now we formulate the main theorem in this paper.

Theorem 3.5. *Let (H,σ) be a co-quasitriangular Hopf algebra. Then there is a tensor equivalence $\mathcal{F}$ from the category $(\mathscr{M}^H)_{H_\sigma}$ of right H_σ-modules to the category $\mathscr{Y}\mathscr{D}^H_H$ of right Yetter-Drinfeld modules:*

$$\mathcal{F}\colon (\mathscr{M}^H)_{H_\sigma} \to \mathscr{Y}\mathscr{D}^H_H, \quad (M,\leftharpoonup) \mapsto (M,\tilde{\cdot}),$$

where $\tilde{\cdot}$ is defined in Lemma 3.3. The quasi-inverse of $\mathcal{F}$ is

$$\mathcal{G}\colon \mathscr{Y}\mathscr{D}^H_H \to (\mathscr{M}^H)_{H_\sigma}, \quad (N,\cdot) \mapsto (N,\widetilde{\leftharpoonup}),$$

where $\widetilde{\leftharpoonup}$ is defined in Lemma 3.4.

Proof. We first show that $\mathcal{G}\mathcal{F}(M,\leftharpoonup) = (M,\leftharpoonup)$ for any object M in $(\mathscr{M}^H)_{H_\sigma}$. It is enough to verify that $m \widetilde{\leftharpoonup} a = m \leftharpoonup a$ for all $a \in H_\sigma$ and $m \in M$. Indeed,

$$\begin{aligned} m \widetilde{\leftharpoonup} a &= m_{[0]} \tilde{\cdot}\, a_2 \sigma(m_{[1]}, S(a_1)) \\ &= m_{[0]} \leftharpoonup a_3 \sigma(m_{[1]}, a_2)\sigma(m_{[2]}, S(a_1)) = m \leftharpoonup a. \end{aligned}$$

Next we show that $\mathcal{F}\mathcal{G}(N,\cdot) = (N,\cdot)$ for any object of ${}^H_H\mathscr{Y}\mathscr{D}$. For all $n \in N$ and $h \in H$,

$$\begin{aligned} n \,\tilde{\cdot}\, b &= n_{[0]} \widetilde{\leftharpoonup} b_2 \sigma(n_{[1]}, b_1) \\ &= n_{[0]} \cdot b_3 \sigma(n_{[1]}, S(b_2))\sigma(n_{[2]}, b_1) = n \cdot b. \end{aligned}$$

Finally, we verify that the triple $(\mathcal{G}, \mathrm{Id}, \mathrm{Id})$ is a tensor functor. It is clear that $\mathcal{G}(k) = k$. Note that for any two Yetter-Drinfeld modules U and V, we have the right H_σ-module structure on $\mathcal{G}(U) \otimes \mathcal{G}(V)$:

$$(u \otimes v) \leftharpoonup a = u \leftharpoonup a_2 \otimes v_{[0]} \leftharpoonup a_4 \sigma(v_{[1]}, S(a_1)a_3),$$

where $u \in U$ and $v \in V$. Now we have

$$\begin{aligned} (u \otimes v) \widetilde{\leftharpoonup} a &= (u_{[0]} \otimes v_{[0]}) \cdot a_2 \sigma(u_{[1]}v_{[1]}, S(a_1)) \\ &= u_{[0]} \cdot a_2 \otimes v_{[0]} \cdot a_3 \sigma(u_{[1]}v_{[1]}, S(a_1)) \\ &= u_{[0]} \cdot a_3 \otimes v_{[0]} \cdot a_4 \sigma(u_{[1]}, S(a_2))\sigma(v_{[1]}, S(a_1)) \\ &= u_{[0]} \widetilde{\leftharpoonup} a_2 \otimes v_{[0]} \cdot a_3 \sigma(v_{[1]}, S(a_1)) \\ &= u_{[0]} \widetilde{\leftharpoonup} a_2 \otimes v_{[0]} \cdot a_5 \sigma(v_{[1]}, a_3 S(a_4))\sigma(v_{[2]}, S(a_1)) \\ &= u_{[0]} \widetilde{\leftharpoonup} a_2 \otimes v_{[0]} \widetilde{\leftharpoonup} a_4 \sigma(v_{[1]}, a_3)\sigma(v_{[2]}, S(a_1)) \\ &= u_{[0]} \widetilde{\leftharpoonup} a_2 \otimes v_{[0]} \widetilde{\leftharpoonup} a_4 \sigma(v_{[1]}, S(a_1)a_3). \end{aligned}$$

Hence $\mathcal{G}(U \otimes V) = \mathcal{G}(U) \otimes \mathcal{G}(V)$. The other axioms for a tensor functor are obvious. □

Note that the category of Yetter-Drinfeld modules is braided if H has bijective antipode. The equivalence $\mathcal{G}$ in Theorem 3.5 induces a braiding in the category of right H_σ-modules such that the equivalence becomes braided.

Corollary 3.6. *Let (H,σ) be a co-quasitriangular Hopf algebra with bijective antipode. Then the category of right H_σ-modules is a braided tensor category with a braiding $\widetilde{C}$ given by*

$$\widetilde{C}(u\otimes v)=v_{[0]}\otimes u_{[0]}\leftharpoonup v_{[2]}\sigma(u_{[1]},v_{[1]}),\quad \forall\, u\in U,\ v\in V,$$

where U and V are any two right H_σ-modules. Moreover, the functor $\mathcal{G}$ in Theorem 3.5 gives a braided tensor equivalence.

Proof. It is not hard to check that $\widetilde{C}$ satisfies the axioms of a braiding, and has the inverse C' defined as follows

$$C'(u\otimes v)=v_{[0]}\,\widetilde{\leftharpoonup}\, S^{-1}(u_{[1]})\otimes u_{[0]}\sigma(v_{[1]},S^{-1}(u_{[2]})),$$

for all $u\otimes v\in U\otimes V$, where S^{-1} denotes the inverse of the antipode S. □

Corollary 3.7. *Let (H,σ) be a co-quasitriangular Hopf algebra with bijective antipode. Then the Drinfeld center of the category of right H-comodules is equivalent to the category of right H_σ-modules with a braiding $\widetilde{C}$ in Corollary 3.6 as a braided tensor category.*

Proof. Following Corollary 3.6 and Lemma 2.5. □

Example 3.8. Let k be a field with $\mathrm{ch}(k)\neq 2$. Let H_4 be the Sweedler 4-dimensional Hopf algebra over k. In Example 3.2 the corresponding braided group H_{4_σ} was given. By Corollary 3.7 the Drinfeld center of the category of H_4-comodules is equivalent to the category of H_{4_σ}-modules. This equivalence played crucial role in the computation of the Brauer group of H_4 [17].

Example 3.9. By [1, Example 3.5], we have the infinite dimensional co-quasitriangular Hopf algebra H. Here H is generated as an algebra by the grouplike element g and the $(1,g)$-primitive element x with

$$gx=-xg,\quad x^2=0.$$

The co-quasitriangular strucutre σ of H is defined by

$$\sigma(g^ix^j,g^tx^s)=\delta_{s,0}\delta_{j,0}(-1)^{it}.$$

By Lemma 3.1, we have the braided group H_σ. By Corollary 3.7 the Drinfeld center of the category of H-comodules is equivalent to the category of H_σ-modules.

Example 3.10. Let R be a solution of the quantum Yang-Baxter equation. One can obtain a co-quasitriangular Hopf algebra $(H(R),\sigma)$ by the celebrated FRT construction (see [9]). By Corollary 3.7 the Drinfeld center of the category of $H(R)$-comodules is equivalent to the category of modules over the braided Hopf algebra $H(R)_\sigma$.

Remark 3.11. Let (H,σ) be a co-quasitriangular Hopf algebra with bijective antipode. The equivalence in Corollary 3.7 suggests that the construction of braided auto-equivalences of the Drinfeld center $\mathscr{Z}(\mathscr{M}^H)$ by braided Galois theory would be, in general, resorted to the study of some special Galois co-objects. This is different from the quasitriangular case.

Next we will end this paper by investigating the commutative and cocommutative case. In the sequel, (H,σ) always denotes a co-quasitriangular Hopf algebra such that H is commutative and cocommutative. We first consider the braided group H_σ in Lemma 3.1. Note that H is commutative and cocommutative. We have for all $a,b\in H_\sigma$

$$\begin{aligned}\rho(b)&=b_{[0]}\otimes b_{[1]}=b_2\otimes S(b_1)b_3=b\otimes 1,\\ a\star b&=a_2b_2\sigma(S(a_1)a_3,S(b_1))=ab_2\sigma(1,S(b_1))=ab.\end{aligned}$$

So the right H-coaction ρ of H_σ is trivial and the multiplication of H_σ coincides that with H. Similarly, one can get $\overline{S}=S_H$. Thus $H=H_\sigma$.

Lemma 3.12. *Let (H,σ) be a co-quasitriangular Hopf algebra such that H is commutative and cocommutative. The category $\mathscr{YD}_H^H$ and $(\mathscr{M}^H)_{H_\sigma}$ are the same.*

Proof. We first have that the compatible condition (1) of a Yetter-Drinfeld module $(M,\rho,\cdot)$ reduces to be $\rho(m\cdot h)=m_{[0]}\cdot h\otimes m_{[1]}$ for all $h\in H$ and $m\in M$ since

$$\rho(m\cdot h)=m_{[0]}\cdot h_2\otimes S(h_1)m_{[1]}h_3=m_{[0]}\cdot h_2\otimes S(h_1)h_3m_{[1]}=m_{[0]}\cdot h\otimes m_{[1]}.$$

At the same time, the compatible condition (3) of a right H_σ-module $(N,\rho,\leftharpoonup)$ becomes to be

$$\begin{aligned}\rho(m\leftharpoonup b)&=m_{[0]}\leftharpoonup b_{[0]}\otimes m_{[1]}b_{[1]}\\ &=m_{[0]}\leftharpoonup b_2\otimes m_{[1]}S(b_1)b_3\\ &=m_{[0]}\leftharpoonup b\otimes m_{[1]}\end{aligned}$$

for all $b\in H_\sigma$ and $n\in N$. Note that $H=H_\sigma$. Hence, these two compatible conditions are the same. Therefore, we have the following identity functor:

$$\mathrm{Id}:\mathscr{YD}_H^H=(\mathscr{M}^H)_{H_\sigma},\quad (M,\rho,\cdot)\longleftrightarrow(M,\rho,\cdot).\qquad\square$$

Corollary 3.13. *Let (H,σ) be a co-quasitriangular Hopf algebra with bijective antipode such that H is commutative and cocommutative. Then the functor $\mathcal{G}$ in Theorem 3.5 is a braided auto-equivalence of the category $\mathscr{YD}_H^H$ of Yetter-Drinfeld modules. In particular, if σ is not trivial (i.e., $\varepsilon\otimes\varepsilon$), then the functor $\mathcal{G}$ is, in general, not trivial (i.e., the identity functor).*

Proof. By Lemma 3.12, $\mathscr{YD}_H^H = (\mathscr{M}^H)_{H_\sigma}$. It follows from Corollary 3.6 that the functor $\mathcal{G}$ is a braided auto-equivalence of the category $\mathscr{YD}_H^H$ of Yetter-Drinfeld modules. Assume that σ is not trivial. It is not hard to see that the induced action $\widetilde{\rightharpoonup}$ in Lemma 3.4 is, in general, not trivial. Thus the functor $\mathcal{G}$ is, in general, not the identity functor. □

Example 3.14. Let k be a field. Let G be an Abelian group. A function $\alpha\colon G\otimes G\to k^*$ is a bicharacter of G if

$$\alpha(xy,z)=\alpha(x,z)\alpha(y,z),\quad \alpha(x,yz)=\alpha(x,y)\alpha(x,z)$$

for all $x,y,z\in G$. It is easy to see that $\alpha(x,1)=1=(1,x)$. The group algebra kG is a Hopf algebra with the following comultiplication and antipode:

$$\Delta(g)=g\otimes g,\quad S(g)=g^{-1},\quad \forall\, g\in G.$$

Assume that α is a bicharacter of G. The group algebra kG with the linear extension of α to kG is co-quasitriangular. Note that kG is commutative and cocommutative. By Corollary 3.13, a bicharacter α of G induces an auto-equivalence $\mathcal{G}$ of the category $\mathscr{YD}_{kG}^{kG}$ of Yetter-Drinfeld modules. If α is not trivial, the induced functor $\mathcal{G}$ is, in general, not the identity functor.

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Haixing Zhu
College of Economics and Management, Nanjing Forestry University, Nanjing 210037, P. R. China
E-mail address: zhuhaixing@163.com, haixing.zhu@njfu.edu.cn