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The Shock Reflection Phenomenon for Scalar Conservation Law with Dirac Measure Source Term

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Abstract. This paper is mainly concerned with the shock reflection phenomenon for convex scalar conservation law with Dirac measure source term. The Riemann solutions are constructed completely and then the impact of the strength of source term on the Riemann solutions is considered in detail. In order to illustrate the shock reflection phenomenon, the initial data with three pieces of constant states are considered and the interactions between a backward shock wave plus a stationary wave discontinuity and a rarefaction wave are displayed in all kinds of situations. Furthermore, the global solutions are constructed completely and the large time asymptotic states are obtained. In some certain situations, the shock reflection phenomenon is captured when a rarefaction wave interacts with a stationary wave discontinuity and then is divided into a transmitted rarefaction wave and a reflected shock wave at the critical point. In addition, it is shown that the Riemann solutions are stable with respect to the particular small perturbations of the initial data.

1. Introduction

The scalar conservation law with source term, which is usually called as the scalar balance law, may be written in the following form

$$u_t + f(u)_x = A(x, u), \tag{1.1}$$

where u is the state variable, f is the flux function and A is the source term. In particular, the source term in the form

$$A(x, u) = k'(x)b(u) \tag{1.2}$$

has been widely investigated such as in [5, 25] for the reason that it can be regarded as a model equation for the Saint-Venant equations. See [8, 10, 11, 14, 19, 20, 24, 31–33] and the

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references cited therein for the other source terms. Furthermore, if we take a step further to assume $b(u) = 1$ in the source term (1.2), then (1.1) becomes

$$u_t + f(u)_x = k'(x), \tag{1.3}$$

which is the scalar conservation law with spatially varying source term and is often served as the basis on the numerical scheme for (1.1). Let us notice that if the coefficient is given by the discontinuous one $k(x) = k_L H(-x) + k_R H(x)$, then (1.3) is reduced to the scalar conservation law with Dirac measure source term

$$u_t + f(u)_x = (k_R - k_L)\delta(x). \tag{1.4}$$

In this paper, we are concerned with the equation (1.4). For simplicity, a proper definition of even and convex for the flux function $f \in C^2$ is required. That is to say, both $f(0) = f'(0) = 0$ and $f''(u) > 0$ should be satisfied for any u. If $k(x)$ is allowed to be discontinuous, then it is more convenient to rewrite (1.3) as a system in the form

$$\begin{cases} k_t = 0, \\ u_t + (f(u) - k)_x = 0. \end{cases} \tag{1.5}$$

In fact, we restrict ourselves only to the situation that $k(x) = k_L H(-x) + k_R H(x)$, such that the Riemann problem for the system (1.5) is equivalent to that for the equation (1.4) when the same entropy condition is taken. The Riemann problem for equations of this type has been well established such as in [6, 9, 13, 17, 18]. The system (1.5), in particular when the coefficient $k(x)$ is discontinuous, has been extensively investigated as the detailed example for conservation laws with discontinuous flux [1–4, 7, 23, 26, 34].

It is easy to obtain that the characteristic eigenvalues of the system (1.5) are $\lambda_0 = 0$ and $\lambda_1 = f'(u)$ respectively, which can coincide at $u = 0$ and thus lead to the so-called resonance [15, 16, 19, 21, 22]. Usually, the total variation of the approximate solutions generated by Glimm scheme or front tracking is not uniformly bounded for the resonant systems, such that the convergence of these schemes is more complicated and does not follow the standard route. Compared with the strictly hyperbolic ones, the wave interaction problem for this nonlinear resonant systems leads to more rich wave phenomena and more complicated structures. When the interactions of elementary waves for the system (1.5) are studied, an interesting phenomenon is discovered in some particular situations, which is called as the so-called shock reflection phenomenon defined by:

Definition 1.1. The shock reflection phenomenon is called when a rarefaction wave interacts with a stationary wave discontinuity and then is divided into a transmitted rarefaction wave and a reflected shock wave at the reflection point where u achieves its critical value.

The purpose of the present paper is to investigate and analyze this interesting phenomenon for the system (1.5) in detail. In order to study the shock reflection phenomenon for the system (1.5), let us consider the following special initial data

$$u(x,0) = \begin{cases} u_L, & x < 0, \\ u_M, & 0 < x < x_0, \\ u_R, & x > x_0, \end{cases} \quad \text{and} \quad k(x) = \begin{cases} k_L, & x < 0, \\ k_R, & x > 0, \end{cases} \tag{1.6}$$

which enables us to investigate the wave interaction problem for the system (1.5). If x_0 is a sufficiently small positive number, then this special Cauchy problem (1.5) and (1.6) can be seen as the so-called perturbed Riemann problem. Here we discover that the solution to the perturbed Riemann problem (1.5) and (1.6) only depends on $A = k_R - k_L$. Without loss of generality, let us assume that $A > 0$ (namely $k_R > k_L$) in this paper for the reason that $A < 0$ (namely $k_R < k_L$) can be dealt with similarly.

In order to depict the shock reflection phenomenon for the system (1.5) for insight, we require that a backward shock wave plus a stationary wave discontinuity emits from the origin $(0, 0)$ and a rarefaction wave emits from the other initial point $(x_0, 0)$ whose wave back has the negative propagation speed. In order to meet the above requirements, we need to make the following assumptions:

H1. $u_M < 0$ and $u_M < u_R$, which enables us to obtain a rarefaction wave starting from the initial point $(x_0, 0)$ whose wave back has the negative propagation speed.

H2. $f(u_L) + A < f(u_M)$, which implies that a backward shock wave plus a stationary wave discontinuity emanates from the initial point $(0, 0)$.

With the above assumptions, we have the following theorem to describe the main result of this paper.

Theorem 1.2. *Assume that f is a smooth, even and convex function and $u_M < 0$, $u_M < u_R$ and $f(u_L) + A < f(u_M)$, then the shock reflection phenomenon cannot happen if and only if both $u_R < 0$ and $f(u_R) > A = k_R - k_L$ are satisfied, otherwise it happens. In addition, if the limit $x_0 \to 0$ is taken, then the solutions to the perturbed Riemann problem* (1.5) *and* (1.6) *converge to the corresponding Riemann solutions.*

To study the perturbed Riemann problem (1.5) and (1.6) is in essence to study various possible interactions of elementary waves. We are interested in the propagation of a rarefaction wave through a stationary wave discontinuity, which is the most interesting and difficult problem among all the interactions of elementary waves for the system (1.5). We will deal with the wave interaction problem in detail by employing the method of characteristics such that the global solutions to the perturbed Riemann problem (1.5) and

(1.6) can be constructed completely under our assumptions, even when the shock reflection phenomena appear in their solutions. The large time asymptotic states are obtained after all the complicated nonlinear interactions have been completed such that all the hyperbolic waves are expected to combine and cancel and eventually become noninteracting. Thus, we can point out that the Riemann solutions for the system (1.5) are stable with respect to the particular small perturbations of initial data (1.6) in all the possible situations under our assumptions, although the shock reflection phenomena may happen in the solutions to the perturbed Riemann problem (1.5) and (1.6). It was shown in [30] that the instability of Riemann solution with respect to the small perturbation of initial data may occur in some specific situations but the shock reflection phenomenon cannot be captured therein for the system (1.5).

In fact, the shock reflection phenomenon has also been observed when the interactions of elementary waves are considered for the following scalar conservation law with a flux function involving discontinuous coefficients

$$u_t + (k(x)g(u))_x = 0, \tag{1.7}$$

which has been well investigated both numerically by Seguin and Vovelle [27] and theoretically in our early works [28, 29]. It is noticed that the shock reflection phenomenon is named as the so-called bifurcation phenomenon in the above references. However, to our knowledge, the shock reflection phenomenon for the system (1.5) has not been paid attention on until now. In this paper, we point out that the shock reflection phenomenon also happens for the system (1.5) and has the similar properties with that for (1.7).

The paper is organized in the following way. In Section 2, the Riemann problem (1.5) and (2.1) are considered when $k_R > k_L$ for self-contained. Consequently, we consider that the strength A $(= k_R - k_L > 0)$ has the influence on the Riemann solutions. In Section 3, the perturbed Riemann problem (1.5) and (1.6) is investigated in all kinds of situations under our assumptions and the shock reflection phenomena are analyzed by studying the interactions of elementary waves, including stationary wave discontinuities, shock waves and rarefaction waves. Furthermore, the global solutions of the perturbed Riemann problem (1.5) and (1.6) are constructed completely. Finally, the conclusions are drawn in Section 4.

2. The Riemann problem

In this section, for a smooth, even and convex function $f(u)$, we can describe how to solve the Riemann problem for the system (1.5) with the following Riemann initial data

$$(k,u)(x,0) = \begin{cases} (k_L, u_L), & x < 0, \\ (k_R, u_R), & x > 0, \end{cases} \tag{2.1}$$

which has also been considered such as in [9, 13, 17] etc. In this paper, we present the exact solutions to the Riemann problem (1.5) and (2.1) by using the visualization method and then check the influence of the strength A $(= k_R - k_L)$ on the Riemann solutions. Without loss of generality, let us assume $k_L < k_R$ here. Otherwise, if $k_L > k_R$, then the Riemann solutions can also be constructed in the same way. First, the definition of a weak solution to the system (1.5) is given below.

Definition 2.1. A measurable function $u \in L^\infty(R \times R_+)$ is called as a weak solution to the system (1.5) if and only if the equality

$$\int_{R_+}\int_R (u\varphi_t + (f(u) - k(x))\varphi_x)\, dxdt + \int_R u(x,0)\varphi(x,0)\, dx = 0 \tag{2.2}$$

holds for any test function $\varphi \in C_c^\infty(R \times R_+)$, in which $k(x)$ may be discontinuous with respect to x.

The eigenvalues of the system (1.5) are $\lambda_0 = 0$ and $\lambda_1 = f'(u)$ and the corresponding right eigenvectors are $\overrightarrow{r_0} = (f'(u), 1)^T$ and $\overrightarrow{r_1} = (0, 1)^T$ respectively. It is worthy noticed that $\lambda_1 < \lambda_0$ for $f'(u) < 0$ and $\lambda_1 > \lambda_0$ for $f'(u) > 0$. It is easy to get $\nabla\lambda_0 \cdot \overrightarrow{r_0} = 0$ and $\nabla\lambda_1 \cdot \overrightarrow{r_1} = f''(u) > 0$ where $\nabla = \left(\frac{\partial}{\partial k}, \frac{\partial}{\partial u}\right)$. Thus, the system (1.5) is non-strictly hyperbolic, where the characteristic family of λ_0 is linearly degenerate and that of λ_1 is genuinely nonlinear. Accordingly, the waves associated with λ_0 are the stationary wave discontinuities denoted by J and those associated with λ_1 are the rarefaction waves denoted by R or the shock waves denoted by S which are determined by the initial data. The Riemann invariants can be chosen as $w = f(u) - k$ and $z = k$ respectively.

We start with the simpler situation that $k_L = k_R$, where the Riemann problem (1.5) and (2.1) is reduced to that for a scalar conservation law with a convex flux function. In this situation, we know that u_L and u_R are linked either by a rarefaction wave when $u_L < u_R$ or by a shock wave when $u_L > u_R$. Consequently, the more complicated situation that $k_L \neq k_R$ is taken into account. It is clear that $k(x)$ is equal to k_L in the region $\{x < 0, t \geq 0\}$ and k_R in the region $\{x > 0, t \geq 0\}$ respectively for the Riemann problem (1.5) and (2.1). Therefore, the Riemann problem (1.5) and (2.1) may be seen as two disconnected problems in $\{x < 0, t \geq 0\}$ and in $\{x > 0, t \geq 0\}$ respectively, in connection with some interface conditions at $J : \{x = 0, t \geq 0\}$. Let us introduce the notations $u^- = u(x = 0^-, t \geq 0)$ and $u^+ = u(x = 0^+, t \geq 0)$, then the Riemann solutions to (1.5) and (2.1) can be constructed provided that the values of u^- and u^+ are determined.

Let us first consider the problem in the region $\{x < 0, t \geq 0\}$ as

$$\begin{cases} u_t + (f(u) - k_L)_x = u_t + f(u)_x = 0, \\ u(x,0) = u_L \ (x < 0), \quad u(x = 0^-, t) = u^-, \end{cases} \tag{2.3}$$

which consists of a u-wave with non-positive speed. If $u_L < u^- < 0$, then one may arrive at a backward rarefaction wave $\overleftarrow{R}$ whose propagation speed can be calculated by $\tau = f'(u)$ with u varying from u_L to u^-. Otherwise, if $u_L > u^-$ and $f(u_L) < f(u^-)$, then one can obtain a backward shock wave $\overleftarrow{S}$ whose propagation speed is given by $\sigma = \frac{f(u_L)-f(u^-)}{u_L-u^-}$.

Let us turn to the problem in the region $\{x > 0, t \geq 0\}$ as

$$\begin{cases} u_t + (f(u) - k_R)_x = u_t + f(u)_x = 0, \\ u(x,0) = u_R \ (x > 0), \quad u(x = 0^+, t) = u^+, \end{cases} \tag{2.4}$$

which consists of a u-wave with non-negative speed. If $0 < u^+ < u_R$, then one can get a forward rarefaction wave $\overrightarrow{R}$ whose propagation speed can be calculated by $\tau = f'(u)$ with u varying from u^+ to u_R. Otherwise, if $u^+ > u_R$ and $f(u^+) > f(u_R)$, then one can obtain a forward shock wave $\overrightarrow{S}$ whose propagation speed can be expressed as $\sigma = \frac{f(u_R)-f(u^+)}{u_R-u^+}$.

Finally, we deal with the interface Rankine-Hugoniot relation

$$f(u^-) - k_L = f(u^+) - k_R, \tag{2.5}$$

namely

$$f(u^-) + A = f(u^+), \tag{2.6}$$

where $A = k_R - k_L > 0$. There exist infinitely many possible values for u^- and u^+ to obey (2.6), such that an additional entropy condition should be proposed in order to ensure the uniqueness of Riemann solution. In this paper, we choose the minimal jump entropy condition [12, 13, 17, 27] where the jump $|u^+ - u^-|$ is the smallest possible jump satisfying (2.6). For the convex flux function $f(u)$, this is equivalent to choosing the minimal possible flux across the jump. More specifically, the minimal jump entropy condition becomes

$$u^+ < u^- \leq 0 \quad \text{for} \quad u^- \leq 0 \quad \text{and} \quad 0 \leq u^- < u^+ \quad \text{for} \quad u^- \geq 0, \tag{2.7}$$

which implies that $u^- u^+ \geq 0$ such that only one characteristic leaves the jump.

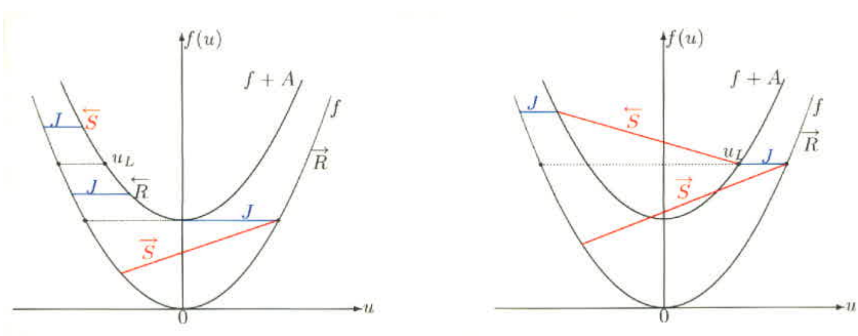

Figure 2.1: The phase plane $(u, f(u))$ where $A = k_R - k_L$, left for $u_L < 0$ and right for $u_L > 0$.

It can be concluded from the above discussions that the Riemann solutions to (1.5) and (2.1) just rely on the difference $A = k_R - k_L$ in k to the left and right. For the situation $A > 0$ (namely $k_R > k_L$), the construction of Riemann solutions is divided into the two parts in the light of u_L. If $u_L < 0$, then we can obtain four cases listed below (see the left in Figure 2.1).

(i) If $u_R < u_L < 0$ and $f(u_L) + A < f(u_R)$, then the Riemann solution is $\overleftarrow{S} + J$ which can be expressed as

$$(k,u)(x,t) = \begin{cases} (k_L, u_L), & x < \sigma t, \\ (k_L, u^-), & \sigma t < x < 0, \\ (k_R, u_R), & x > 0, \end{cases} \tag{2.8}$$

in which u^- (< 0) satisfies $f(u^-) + A = f(u_R)$, $u^+ = u_R$ and $\sigma = \frac{f(u_L)-f(u^-)}{u_L-u^-}$.

(ii) If $u_R < 0$, $u_L < 0$ and $A < f(u_R) < f(u_L) + A$, then the Riemann solution is $\overleftarrow{R} + J$ which can be represented as

$$(k,u)(x,t) = \begin{cases} (k_L, u_L), & x < f'(u_L)t, \\ (k_L, (f')^{-1}(\frac{x}{t})), & f'(u_L)t \leq x \leq f'(u^-)t, \\ (k_L, u^-), & f'(u^-)t < x < 0, \\ (k_R, u_R), & x > 0, \end{cases} \tag{2.9}$$

where u^- and u^+ are the same as those in case (i).

(iii) If $u_L < 0$ and $f(u_R) < A$, then the Riemann solution is $\overleftarrow{R} + J + \overrightarrow{S}$ which can be expressed as

$$(k,u)(x,t) = \begin{cases} (k_L, u_L), & x < f'(u_L)t, \\ (k_L, (f')^{-1}(\frac{x}{t})), & f'(u_L)t \leq x \leq 0, \\ (k_R, u^+), & 0 < x < \sigma t, \\ (k_R, u_R), & x > \sigma t, \end{cases} \tag{2.10}$$

where u^+ (> 0) satisfies $f(u^+) = A$, $u^- = 0$ and $\sigma = \frac{f(u_R)-f(u^+)}{u_R-u^+}$. It is remarkable that the propagation speed of the wave front of R is zero, which implies that the wave front of R coincides with J.

(iv) If $u_L < 0 < u_R$ and $f(u_R) > A$, then the Riemann solution is $\overleftarrow{R} + J + \overrightarrow{R}$ which is given by

$$(2.11)\qquad (k,u)(x,t)=\begin{cases}(k_L,u_L), & x<f'(u_L)t,\\ (k_L,(f')^{-1}(\frac{x}{t})), & f'(u_L)t\le x\le 0,\\ (k_R,u^+), & 0<x<f'(u^+)t,\\ (k_R,(f')^{-1}(\frac{x}{t})), & f'(u^+)t\le x\le f'(u_R)t,\\ (k_R,u_R), & x>f'(u_R)t,\end{cases}$$

where u^- and u^+ have the same representations as those in case (iii) and the wave front of the backward rarefaction wave $\overleftarrow{R}$ also coincides with J.

Otherwise, if $u_L>0$, then we can also get three cases listed below (see the right in Figure 2.1).

(v) If $u_R<0<u_L$ and $f(u_L)+A<f(u_R)$, then the Riemann solution is also $\overleftarrow{S}+J$ given by (2.8).

(vi) If $0<u_L$ and $f(u_L)+A>f(u_R)$, then the Riemann solution is $J+\overrightarrow{S}$ as follows:

$$(2.12)\qquad (k,u)(x,t)=\begin{cases}(k_L,u_L), & x<0,\\ (k_R,u^+), & 0<x<\sigma t,\\ (k_R,u_R), & x>\sigma t,\end{cases}$$

where u^+ (>0) satisfies $f(u_L)+A=f(u^+)$, $u^-=u_L$ and $\sigma=\frac{f(u_R)-f(u^+)}{u_R-u^+}$.

(vii) If $0<u_L<u_R$ and $f(u_L)+A<f(u_R)$, then the Riemann solution is $J+\overrightarrow{R}$ as follows:

$$(2.13)\qquad (k,u)(x,t)=\begin{cases}(k_L,u_L), & x<0,\\ (k_R,u^+), & 0<x<f'(u^+)t,\\ (k_R,(f')^{-1}(\frac{x}{t})), & f'(u^+)t\le x\le f'(u_R)t,\\ (k_R,u_R), & x>f'(u_R)t,\end{cases}$$

in which u^- and u^+ can be calculated like as those in case (vi).

One can see that the Riemann solutions to (1.5) and (2.1) only depend on the difference $A=k_R-k_L$, such that we can only regard u as the unknown function and the above Riemann solutions to (1.5) and (2.1) is just the corresponding Riemann solutions for u to the following Riemann problem

$$(2.14)\qquad u_t+f(u)_x=A\delta(x),$$

with the initial data

$$(2.15)\qquad u(x,0)=\begin{cases}u_L, & x<0,\\ u_R, & x>0.\end{cases}$$

It is interesting to consider that the strength A of Dirac measure has the influence on the Riemann solutions to (2.14) and (2.15). Our discussion should also be divided into the following six cases according to the values of u_L and u_R.

(1) If $u_R < u_L < 0$, then the Riemann solution is only the stationary wave discontinuity J connecting u_L and u_R directly when $A = f(u_R) - f(u_L)$. If we lift A such that $f(u_R) \geq A > f(u_R) - f(u_L)$, then it turns out to be $\overleftarrow{R} + J$. Furthermore, if we lift A much larger such that $A > f(u_R)$, then it becomes $\overleftarrow{R} + J + \overrightarrow{S}$. On the other hand, if we drop A such that $f(u_R) - f(u_L) > A > 0$, then it becomes $\overleftarrow{S} + J$.

(2) If $u_L < u_R < 0$, then the Riemann solution is $\overleftarrow{R} + J$ for $0 < A \leq f(u_R)$ and then becomes $\overleftarrow{R} + J + \overrightarrow{S}$ if A is added up to satisfy $A > f(u_R)$.

(3) For $u_L < 0 < u_R$, the Riemann solution is $\overleftarrow{R} + J$ when $A = f(u_R)$. If we increase A such that $A > f(u_R)$, then it turns out to be $\overleftarrow{R} + J + \overrightarrow{S}$. Otherwise, if we drop A such that $0 < A < f(u_R)$, then it becomes $\overleftarrow{R} + J + \overrightarrow{R}$.

(4) For $u_R < 0 < u_L$, if $f(u_L) > f(u_R)$, then the Riemann solution is still $J + \overrightarrow{S}$ for all the $A > 0$. Otherwise, when $f(u_L) < f(u_R)$, if we take $A = f(u_R) - f(u_L)$, then it is the composite wave SJ with the zero propagation speed, for the reason that S coincides with J on the line $x = 0$. If we lift A such that $A > f(u_R) - f(u_L)$, then it is transformed into $J + \overrightarrow{S}$. If we drop A such that $0 < A < f(u_R) - f(u_L)$, then it becomes $\overleftarrow{S} + J$.

(5) If $0 < u_R < u_L$, then the Riemann solution is always $J + \overrightarrow{S}$ for all the $A > 0$.

(6) If $0 < u_L < u_R$, then the Riemann solution is also only J connecting u_L and u_R directly when $A = f(u_R) - k(u_L)$. If we lift A such that $A > f(u_R) - f(u_L)$, then it is changed into $J + \overrightarrow{S}$. Otherwise, if we drop A such that $f(u_R) - f(u_L) > A > 0$, then it is turned into $J + \overrightarrow{R}$.

3. The perturbed Riemann problem and shock reflection phenomenon

The main purpose of this section is dedicated to the shock reflection phenomenon for the system (1.5) when f is a smooth, even and convex function. In order to deal with it, we consider the perturbed Riemann problem (1.5) and (1.6) under the assumptions $u_M < 0$, $u_M < u_R$ and $f(u_L) + A < f(u_M)$, where the Riemann solution is $\overleftarrow{S} + J$ starting from the initial point $(0, 0)$ and the Riemann solution is the rarefaction wave R starting from the initial point $(x_0, 0)$ whose wave back has the negative propagation speed. Thus, the rarefaction wave R will meet the stationary wave discontinuity J in finite time and the interaction will happen, in which the shock reflection phenomenon may be captured.

During the construction of global solutions to the perturbed Riemann problem (1.5) and (1.6), wave interactions should be dealt with by employing the method of characteristics.

Let us denote the state between S and J with (k_L, u_1) where u_1 (< 0) is determined uniquely by

$$f(u_1) + A = f(u_M). \tag{3.1}$$

For sufficiently small t before interaction, the solution of the perturbed Riemann problem (1.5) and (1.6) may be represented by the juxtaposition of Riemann solutions at $(0, 0)$ and $(x_0, 0)$. That is to say, the solution of the perturbed Riemann problem (1.5) and (1.6) for sufficiently small t may be expressed briefly as:

$$(k_L, u_L) + S_1 + (k_L, u_1) + J + (k_R, u_M) + R_1 + (k_R, u_R).$$

More precisely, before the interaction happens, we have the solution in the form

$$(k,u)(x,t) = \begin{cases} (k_L, u_L), & x < \left(\frac{f(u_L)-f(u_1)}{u_L-u_1}\right) t, \\ (k_L, u_1), & \left(\frac{f(u_L)-f(u_1)}{u_L-u_1}\right) t < x < 0, \\ (k_R, u_M), & 0 < x < x_0 + f'(u_M)t, \\ (k_R, (f')^{-1}(\frac{x-x_0}{t})), & f'(u_M)t \le x - x_0 \le f'(u_R)t, \\ (k_R, u_R), & x - x_0 > f'(u_R)t. \end{cases} \tag{3.2}$$

Our discussion should be divided into two parts according to $u_L < 0$ or not. We first draw our attention on the situation $u_L < 0$ and have the following lemma to depict all the situations along with the variation of u_R.

Lemma 3.1. *Assume that $u_L < 0$, $u_M < 0$, $u_M < u_R$ and $f(u_L) + A < f(u_M)$, then the results about the perturbed Riemann problem* (1.5) *and* (1.6) *can be summarized as follows:*

(i) *If $u_R < 0$ and $f(u_L) + A < f(u_R) < f(u_M)$, then there is no shock reflection phenomenon and the large time asymptotic state is $\overleftarrow{S} + J$.*

(ii) *If $u_R < 0$ and $A < f(u_R) < f(u_L) + A$, then there is also no shock reflection phenomenon and the large time asymptotic state is $\overleftarrow{R} + J$.*

(iii) *If $f(u_R) < A$, then the shock reflection phenomenon occurs and the large time asymptotic state is $\overleftarrow{R} + J + \overrightarrow{S}$.*

(iv) *If $u_R > 0$ and $f(u_R) > A$, then the shock reflection phenomenon also occurs and the large time asymptotic state is $\overleftarrow{R} + J + \overrightarrow{R}$.*

Proof. We can first fix the relative positions of u_L and u_M from the assumption and then change u_R in view of the fixed u_L and u_M. In what follows, our discussion can be carried out according to the different positions of u_R.

(i) If $u_R < 0$ and $f(u_L) + A < f(u_R) < f(u_M)$, then it is clear that the rarefaction wave R_1 arrives at the stationary wave discontinuity J in finite time t_1 which is given by

$$t_1 = -\frac{x_0}{f'(u_M)}. \tag{3.3}$$

Consequently, R_1 begins to penetrate J and is denoted with R_2 after penetration. The propagation speeds of the matched characteristic lines in R_1 and R_2 can be calculated respectively by

$$\tau_1(u^+) = f'(u^+) \quad \text{and} \quad \tau_2(u^-) = f'(u^-), \tag{3.4}$$

in which the state u^+ in R_1 becomes the corresponding one u^- in R_2 when across J which should obey the interface Rankine-Hugoniot relation

$$f(u^-) + A = f(u^+), \tag{3.5}$$

with $u_M \le u^+ \le u_R$ and $u_1 \le u^- \le u_2$, where u_1 (< 0) is given by (3.1) and u_2 (< 0) is given by

$$f(u_2) + A = f(u_R). \tag{3.6}$$

For the matched characteristic lines in R_1 and R_2, it follows from (3.5) that $u^+ < u^- < 0$, such that we have $\tau_1(u^+) < \tau_2(u^-) < 0$ for the reason that $f'(u)$ is an increasing function with respect to u, which implies that the rarefaction wave decelerates backwards when it passes through J.

The propagation speeds of the wave back of the rarefaction wave R_2 and the shock wave S_1 can be calculated respectively by

$$\tau_2(u_1) = f'(u_1) \quad \text{and} \quad \sigma_1 = \frac{f(u_L) - f(u_1)}{u_L - u_1}. \tag{3.7}$$

Noticing that $u_1 < u_L < 0$, in view of the mean value theorem, there exists one and only one $\overline{u} \in (u_1, u_L)$ such that $\sigma_1 = f'(\overline{u})$. Remembering that $f'(u)$ is an increasing function with respect to u, it is shown that

$$\tau_2(u_1) < \sigma_1 < 0, \tag{3.8}$$

which implies that the rarefaction wave R_2 overtakes the shock wave S_1 from behind in finite time and the intersection point (x_2, t_2) is determined by

$$x_2 = \sigma_1 t_2 = \tau_2(u_1)(t_2 - t_1). \tag{3.9}$$

Consequently, S_1 begins to penetrate R_2 after the time t_2. During the process of penetration, we denote it with S_2 whose propagation speed is

$$\sigma_2(u^-) = \frac{dx}{dt} = \frac{f(u_L) - f(u^-)}{u_L - u^-}, \tag{3.10}$$

where u^- varies from u_1 to u_2. The curve of S_2 is determined by (3.10) together with

$$x = f'(u^-)(t - \bar{t}), \tag{3.11}$$

in which $\bar{t}$ is the time that the characteristic line with the corresponding state u^+ in R_1 arrives at the line $x = 0$ and can be calculated by

$$\bar{t} = -\frac{x_0}{f'(u^+)}, \tag{3.12}$$

where u^+ varies from u_M to u_R. It follows from (3.10) that

$$\frac{d^2x}{dt^2} = \frac{f'(u^-) - \frac{f(u^-)-f(u_L)}{u^- - u_L}}{u^- - u_L} \cdot \frac{du^-}{dt}. \tag{3.13}$$

By employing the mean value theorem again, there exists one and only one $\widetilde{u} \in (u^-, u_L)$ such that we have

$$\frac{f(u^-) - f(u_L)}{u^- - u_L} = f'(\widetilde{u}) > f'(u^-) \tag{3.14}$$

for every $u^- \in [u_1, u_2]$. Thus, one can infer that $\frac{d^2x}{dt^2} > 0$, which implies that the shock wave S_2 decelerates backwards when it penetrates the rarefaction wave R_2.

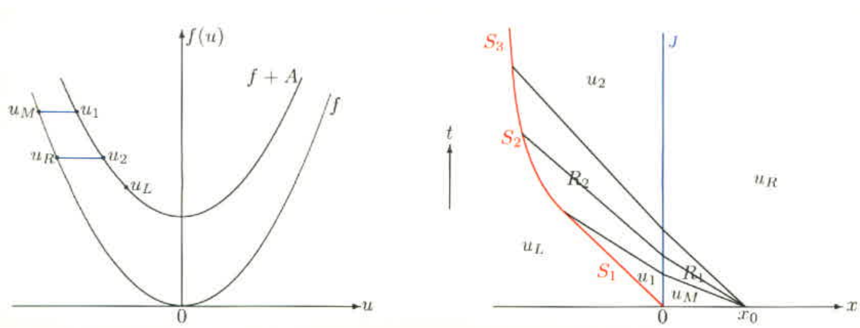

Figure 3.1: $f(u_L) + A < f(u_R) < f(u_M)$ and $u_M < u_R < u_L < 0$.

It is easy to obtain $u_2 < u_L < 0$ from $f(u_L) + A < f(u_R)$, which enables us to see that the shock wave S_2 has the ability to penetrate the whole rarefaction wave R_2 completely in finite time and is denoted with S_3 after penetration whose propagation speed is

$$\sigma_3 = \frac{f(u_L) - f(u_2)}{u_L - u_2}. \tag{3.15}$$

We can draw Figure 3.1 to illustrate this case and it is clear to see from Figure 3.1 that the large time asymptotic state is $\overleftarrow{S} + J$.

(ii) If $u_R < 0$ and $A < f(u_R) < f(u_L) + A$, then the situation is similar to that in case (i). The difference lies in that $u_L < u_2 < 0$ which can be derived from $A < f(u_R) < f(u_L) + A$. Thus, the shock wave S_2 has no ability to penetrate R_2 thoroughly and ultimately takes the characteristic line with the state $u^- = u_L$ in R_2 as its asymptote. We can also draw Figure 3.2 to illustrate it whose large time asymptotic state is $\overleftarrow{R} + J$.

(iii) If $f(u_R) < A$, then one can capture the shock reflection phenomenon in the solution to the perturbed Riemann problem (1.5) and (1.6). In fact, when the state in R_2 arrives at $u^- = 0$, the corresponding one in R_1 is $u^+ = -u_3$, in which u_3 (> 0) is determined by

$$f(u_3) = A. \tag{3.16}$$

It should be emphasized that the propagation speed of the characteristic line supported on the state $u^- = 0$ in R_2 is zero for the reason that $\tau_2(0) = f'(0) = 0$, which is identical with J. At the same time, a reflected shock wave S_3 is generated when the state in R_1 achieves the critical one $u^+ = -u_3$ and the critical point (x_3, t_3) can be given by

$$(x_3, t_3) = \left(0, \frac{-x_0}{f'(-u_3)}\right). \tag{3.17}$$

The curve of the reflected shock wave S_3 can be derived from

$$\begin{cases} \sigma_3 = \frac{dx}{dt} = \frac{f(u^+) - f(u_3)}{u^+ - u_3}, \\ x - x_0 = f'(u^+)t, \end{cases} \tag{3.18}$$

with the initial condition (3.17) and u^+ varying from $-u_3$ to u_R. It is easy to see that S_3 is initially tangent to the line of the stationary wave discontinuity J, but it immediately has the positive speed. Hence, the shock reflection phenomenon can be captured, namely the rarefaction wave R_1 is separated into two waves: a transmitted rarefaction wave R_2 and a reflected shock wave S_3 after the time t_3.

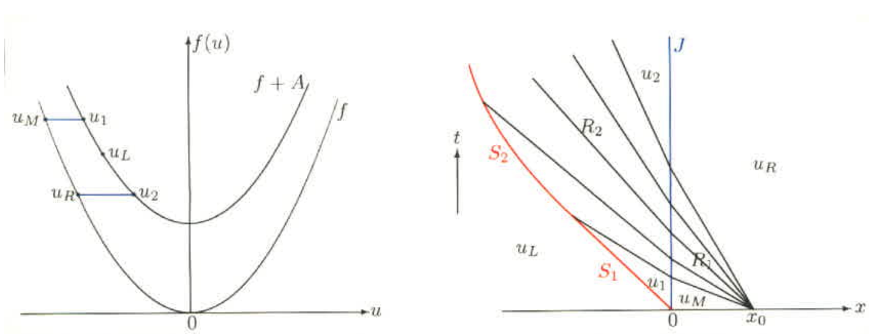

Figure 3.2: $A < f(u_R) < f(u_L) + A < f(u_M)$ with $u_M < u_R < 0$ and $u_L < 0$.

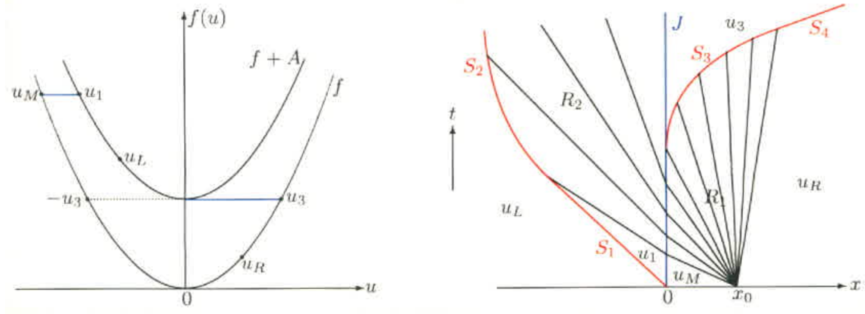

Figure 3.3: $f(u_R) < A$ and $f(u_L) + A < f(u_M)$ with $u_M < u_L < 0$.

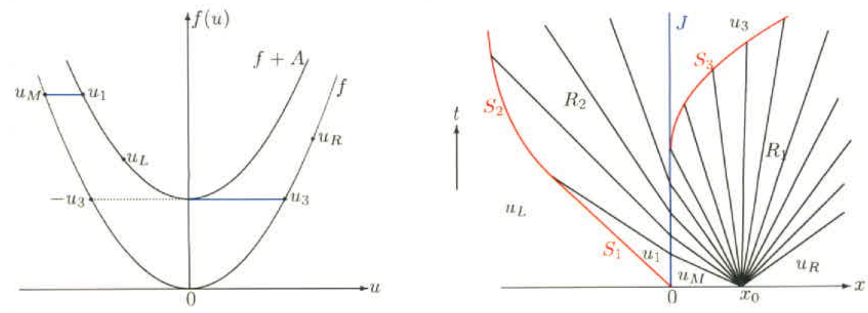

Figure 3.4: $f(u_R) > A$ and $f(u_L) + A < f(u_M)$ with $u_M < u_L < 0 < u_R$.

It can be obtained from (3.18) that

$$\frac{d^2x}{dt^2} = \frac{f'(u^+) - \frac{f(u^+)-f(u_3)}{u^+-u_3}}{u^+ - u_3} \cdot \frac{du^+}{dt}. \tag{3.19}$$

Like as before, for every $u^+ \in [-u_3, u_R]$, one can also obtain $\frac{d^2x}{dt^2} > 0$ from (3.19), which implies that the shock wave S_3 accelerates forwards when it penetrates R_1. For $u_R < u_3$, S_3 can penetrate the whole R_1 thoroughly in finite time and is denoted with S_4 after penetration whose propagation speed is given by

$$\sigma_4 = \frac{f(u_R) - f(u_3)}{u_R - u_3}. \tag{3.20}$$

On the other hand, for $u_L < 0$, S_2 cannot cross the total R_2 and at last also has the characteristic line with the state $u^- = u_L$ in R_2 as its asymptote. Let us draw Figure 3.3 to illustrate this case whose large time asymptotic state is $\overleftarrow{R} + J + \overrightarrow{S}$.

(iv) If $u_R > 0$ and $f(u_R) > A$, then the situation is similar to that in case (iii). The difference lies in that $0 < u_3 < u_R$ which can be derived from $u_R > 0$ and $f(u_R) > A$. Thus, S_3 cannot penetrate R_1 thoroughly and has the characteristic line with the state

$u^+ = u_3$ in R_1 as its asymptote in the end. Let us draw Figure 3.4 to explain the difference whose large time asymptotic state is $\overleftarrow{R} + J + \overrightarrow{R}$. □

We are now in a position to pay our attention on the situation $u_L > 0$ and also have the similar result as Lemma 3.1 to describe it.

Lemma 3.2. *Assume that $u_L > 0$, $u_M < 0$, $u_M < u_R$ and $f(u_L) + A < f(u_M)$, then the results about the perturbed Riemann problem* (1.5) *and* (1.6) *can be summarized as follows:*

(i) *If $u_R < 0$ and $f(u_L) + A < f(u_R) < f(u_M)$, then there is no shock reflection phenomenon and the large time asymptotic state is $\overleftarrow{S} + J$.*

(ii) *If $u_R < 0$ and $A < f(u_R) < f(u_L) + A$, then there is also no shock reflection phenomenon and the large time asymptotic state is $J + \overrightarrow{S}$.*

(iii) *If $f(u_R) < A$ or $A < f(u_R) < f(u_L) + A$ with $u_R > 0$, then the shock reflection phenomenon occurs and the large time asymptotic state is also $J + \overrightarrow{S}$.*

(iv) *If $u_R > 0$ and $f(u_R) > f(u_L) + A$, then the shock reflection phenomenon also occurs and the large time asymptotic state is $J + \overrightarrow{R}$.*

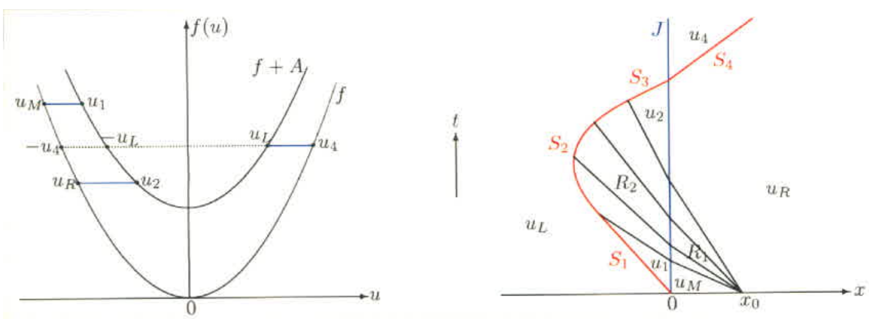

Figure 3.5: $A < f(u_R) < f(u_L) + A < f(u_M)$ with $u_M < u_R < 0 < u_L$.

Proof. The process of proof is analogous to that in Lemma 3.1. In fact, we also change u_R in view of the fixed u_L and u_M and our discussion can be carried out according to the different positions of u_R.

(i) If $u_R < 0$ and $f(u_L) + A < f(u_R) < f(u_M)$, then the situation is similar to that in Case (i) in Lemma 3.1 and we omit the detail.

(ii) If $u_R < 0$ and $A < f(u_R) < f(u_L) + A$, then the shock wave S_2 has the ability to penetrate the whole rarefaction wave R_2 completely in finite time for the reason that $u_2 < 0 < u_L$. During the process of penetration, it follows from (3.10) that the shock wave

S_2 changes its direction and has a positive propagation speed when it passes through the characteristic line with the state $u^- = -u_L$ in R_2 due to the fact that $u_1 < -u_L < u_2 < 0$. After penetration, it is denoted with S_3 whose propagation speed satisfies

$$\sigma_3 = \frac{f(u_L) - f(u_2)}{u_L - u_2} = \frac{f(u_L) + A - f(u_R)}{u_L - u_2} > 0, \tag{3.21}$$

which implies that the shock wave S_3 turns back to the line $x = 0$ of J in finite time and then passes through it. Consequently, it is denoted with S_4 whose propagation speed is

$$\sigma_4 = \frac{f(u_R) - f(u_4)}{u_R - u_4}, \tag{3.22}$$

in which u_4 (> 0) is given by

$$f(u_L) + A = f(u_4). \tag{3.23}$$

Noticing that $u_4 - u_R > u_L - u_2 > 0$ and $f(u_R) < f(u_L) + A$, it follows from (3.21)–(3.23) that

$$\sigma_4 = \frac{f(u_L) + A - f(u_R)}{u_4 - u_R} < \frac{f(u_L) + A - f(u_R)}{u_L - u_2} = \sigma_3, \tag{3.24}$$

which implies that the shock wave decelerates when it passes through the stationary wave discontinuity J. Let us draw Figure 3.5 to illustrate this case whose large time asymptotic state is $J + \overrightarrow{S}$.

(iii) If $f(u_R) < A$, then it follows from (3.10) that the shock wave S_2 changes its movement direction when it passes through the characteristic line with the state $u^- = -u_L$ in R_2. Consequently, it speeds up forwards and continues to penetrate R_2 until it arrives at J in finite time for the reason that the wave front of R_2 is identical with J. The shock wave is denoted with S_4 after penetration.

The propagation speeds of the shock waves before penetration and after penetration can be calculated respectively by

$$\sigma_2(0) = \frac{f(u_L) - f(0)}{u_L - 0} = \frac{f(u_L)}{u_L} \quad \text{and} \quad \sigma_4 = \frac{f(u_4) - f(u_3)}{u_4 - u_3}, \tag{3.25}$$

in which u_3 and u_4 are given by (3.16) and (3.23), respectively. We shall show that $0 < \sigma_2(0) < \sigma_4$, namely the shock wave accelerates forwards when it passes through J. In fact, it follows from (3.16) and (3.23) that

$$f(u_L) - f(0) = f(u_4) - f(u_3) > 0. \tag{3.26}$$

Thus, the assertion is true provided that one can prove that $u_L > u_4 - u_3 > 0$. Actually, if $u_L \le u_3$, then there exist one and only one $\xi_1 \in (0, u_L)$ and $\xi_2 \in (u_3, u_4)$ such that

$$f'(\xi_1)(u_L - 0) = f(u_L) - f(0) = f(u_4) - f(u_3) = f'(\xi_2)(u_4 - u_3), \tag{3.27}$$

which enables us to obtain $u_L > u_4 - u_3 > 0$ for $0 < f'(\xi_1) < f'(\xi_2)$. Otherwise, if $u_L < u_3$, then there exist one and only one $\eta_1 \in (0, u_3)$ and $\eta_2 \in (u_L, u_4)$ such that

$$f'(\eta_1)(u_3 - 0) = f(u_3) - f(0) = f(u_4) - f(u_L) = f'(\eta_2)(u_4 - u_L), \tag{3.28}$$

which also enables us to achieve $u_L > u_4 - u_3 > 0$ for $0 < f'(\eta_1) < f'(\eta_2)$.

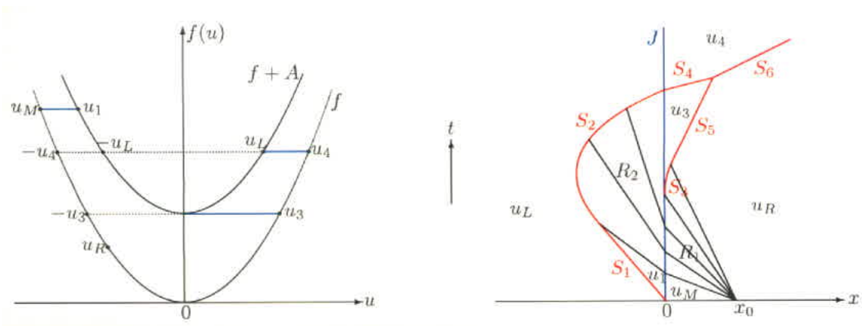

Figure 3.6: $f(u_R) < A < f(u_L) + A < f(u_M)$ with $u_M < 0 < u_L$.

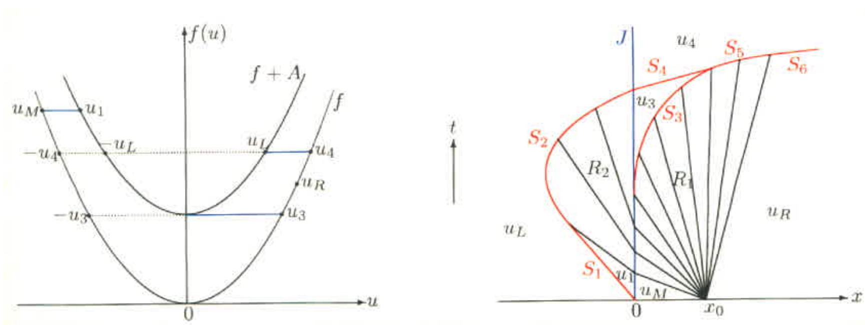

Figure 3.7: $f(u_R) > A$ and $f(u_L) + A < f(u_M)$ with $u_M < 0$, $u_L > 0$ and $u_R > 0$.

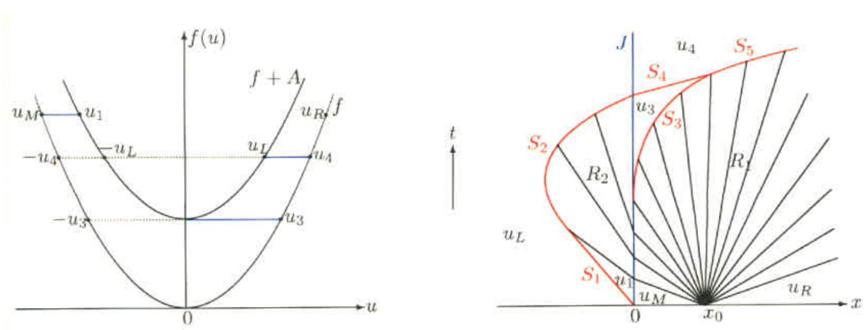

Figure 3.8: $f(u_L) + A < f(u_R)$ and $f(u_L) + A < f(u_M)$ with $u_M < 0$, $u_R > u_L > 0$.

Let us turn to the reflected shock wave S_3. If the state u_R is near $-u_3$, then S_3 has the ability to penetrate the entire R_1 completely before S_4 keeps up with it. It is denoted with S_5 after penetration whose propagation speed is

$$\sigma_5 = \frac{f(u_R) - f(u_3)}{u_R - u_3}. \tag{3.29}$$

It is easy to obtain $0 < \sigma_5 < \sigma_4$ from $0 < u_3 < u_R < u_4$, which implies that the shock wave S_4 keeps up with the shock wave S_5 in finite time. Finally, they coalesce into a new shock wave S_6 whose propagation speed is

$$\sigma_6 = \frac{f(u_R) - f(u_4)}{u_R - u_4}. \tag{3.30}$$

On the other hand, if the state u_R is near u_3, then S_3 has no ability to cross R_1 fully before the shock wave S_4 catches up with it. In this situation, they coalesce into a new shock wave who continues to penetrate the rarefaction wave R_1 and is able to penetrate it completely for $u_4 > u_R$. In fact, the situation also happens when both $A < f(u_R) < f(u_L) + A$ with $u_R > 0$ are satisfied. Let us draw Figures 3.6–3.7 to illustrate the difference and the large time asymptotic state is $J + \overrightarrow{S}$ for both of them.

(iv) If $u_R > 0$ and $f(u_R) > f(u_L) + A$, then the situation is similar to that illustrated in Figure 3.7. The difference lies in that the shock wave has no ability to penetrate R_1 completely and finally has the characteristic line with the state $u^+ = u_4$ in R_1 as its asymptote for $u_4 < u_R$. Let us draw Figure 3.8 to explain the situation whose large time asymptotic state is $J + \overrightarrow{R}$. □

One can observe clearly that the limits as $x_0 \to 0$ of the solutions to the perturbed Riemann problem (1.5) and (1.6) are in accordance with the corresponding Riemann solutions to (1.5) and (2.1), which can also be seen from the large time asymptotic states in all kinds of situations. Thus, in all the cases studied above, the Riemann solutions for the system (1.5) are stable with respect to the particular small perturbation (1.6) of Riemann initial data if x_0 is regarded as the perturbation parameter. It can be concluded from the above discussions that the shock reflection phenomenon occurs if and only if $u_R > -u_3$. Thus, Theorem 1.2 can be achieved by pasting Lemmas 3.1 and 3.2 together.

4. Conclusions

So far, the structures of global solutions and the large time asymptotic states for the perturbed Riemann problem (1.5) and (1.6) are obtained and illustrated for all the situations under our suitable assumptions, where the shock reflection phenomena appear in their solutions provided that $u_R > -u_3$. In fact, the shock reflection phenomena for the system (1.5) can be observed and analyzed well by using the perturbed Riemann problem here, which enables us to describe the critical state, the reflected shock wave and the transmitted rarefaction wave in detail.

If we take the assumption $f(u_L) + A > f(u_M)$ and $u_L < 0$ instead of the assumption $f(u_L)+A < f(u_M)$, then the shock reflection phenomenon can also be captured. But here the Riemann solution at the point $(0,0)$ is $\overleftarrow{R} + J$, namely a backward rarefaction wave plus a stationary wave discontinuity. The situation is relatively simpler compared with the situation in this paper for the reason that the wave front of this backward rarefaction wave is parallel to the wave back of this transmitted rarefaction wave and thus no interaction happens in the region $\{x < 0, t \geq 0\}$.

In addition, if we consider $A < 0$ (namely $k_R < k_L$) instead of $A > 0$ (namely $k_R > k_L$), then the shock reflection phenomenon can also be observed if we consider the perturbed Riemann problem for the system (1.5) with the following initial data

$$u(x,0) = \begin{cases} u_L, & x < x_0, \\ u_M, & x_0 < x < 0, \\ u_R, & x > 0, \end{cases} \quad \text{and} \quad k(x) = \begin{cases} k_L, & x < 0, \\ k_R, & x > 0, \end{cases} \tag{4.1}$$

under the similar assumptions. The Riemann problem (1.5) and (2.1) should be resolved for $A < 0$ (namely $k_R < k_L$) and the perturbed Riemann problem is axisymmetric compared with here.

The perturbed Riemann problem (1.5) and (1.6) is investigated through analyzing the phase plane which enables us to construct the global solutions completely. Thus, it is clear to see that the results in this paper can be generalized to the general convex (or concave) flux function $f(u)$ satisfying $f''(u) > 0$ (or $f''(u) < 0$) for all the u. Here we take the assumption that f is even in order to simplify our calculation which is not essential.

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