TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 4, pp. 743–753, August 2016
DOI: 10.11650/tjm.20.2016.5805
This paper is available online at http://journal.tms.org.tw

Some Results on Local Cohomology Modules with Respect to a Pair of Ideals

Tran Tuan Nam* and Nguyen Minh Tri

Abstract. We study the finiteness of the sets $\operatorname{Ass}(H^d_{I,J}(M))$ and $\operatorname{Ass}(\operatorname{Hom}_R(R/I, H^d_{I,J}(M)))$ concerning Grothendieck's conjecture. We also show some properties of local cohomology modules $H^i_{I,J}(M)$ from the point of view of Serre subcategories.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and I, J are two ideals of R. In [6] Takahashi, Yoshino and Yoshizawa introduced the definition of local cohomology modules with respect to a pair of ideals (I, J) which is a generalization of the definition of local cohomology modules with respect to an ideal I of Grothendieck. Let M be an R-module, the (I, J)-torsion submodule $\Gamma_{I,J}(M)$ of M is

$$\Gamma_{I,J}(M) = \{x \in M \mid I^n x \subseteq Jx \text{ for some } n \gg 1\}.$$

Thus, there is a covariant functor $\Gamma_{I,J}$ from the category of R-modules to itself. For an integer i, the i-th local cohomology functor $H^i_{I,J}$ with respect to a pair of ideals (I, J) is the i-th right derived functor $R^i\Gamma_{I,J}$ of $\Gamma_{I,J}$. Note that if $J = 0$, then $H^i_{I,J}$ coincides with the ordinary local cohomology functor H^i_I of Grothendieck.

In [4] Grothendieck gave a conjecture that: For any ideal I of R and any finitely generated R-module M, the module $\operatorname{Hom}_R(R/I, H^i_I(M))$ is finitely generated, for all i. One year later, Hartshorne provided a counterexample to Grothendieck's conjecture. He defined an R-module M to be I-cofinite if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is finitely generated for all i and asked: For which rings R and ideals I are the modules $H^i_I(M)$ I-cofinite for all i and all finitely generated modules M?

The organization of the paper is as follows. In next section, we will be concerned with Grothendieck's conjecture. Denote $W(I, J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some positive integer } n\}$. An R-module M is said to be (I, J)-weakly cofinite if $\operatorname{Supp}_R(M) \subseteq W(I, J)$

Received January 19, 2015; Accepted March 3, 2016.
Communicated by Ching Hung Lam.
2010 *Mathematics Subject Classification.* 13D45.
Key words and phrases. Local cohomology, Weakly Laskerian, Weakly cofinite.
This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant no. 101.04-2015.22.
*Corresponding author.

and $\mathrm{Ext}^i_R(R/I, M)$ is weakly Laskerian for all $i \geq 0$. We prove in Theorem 2.5 that if d is a non-negative integer, M and $H^i_{I,J}(M)$ are weakly Laskerian R-modules for all $i < d$, then $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^d_{I,J}(M)))$ is a finite set. Next, we will see in Theorem 2.8 that $\mathrm{Ass}_R(H^d_{I,J}(M))$ is a finite set if $H^i_{I,J}(M)$ is a weakly Laskerian R-module for all $i < d$ provided there is an ideal $\mathfrak{a}$ of R such that $0 :_M \mathfrak{a} = 0 :_{\Gamma_{I,J}(M)} \mathfrak{a}$ and $W(I,J) \subseteq V(\mathfrak{a})$. This section is closed by Theorem 2.9 which shows that if $H^i_{I,J}(M)$ is (I,J)-weakly cofinite for all $i \neq d$, then $H^d_{I,J}(M)$ is also (I,J)-weakly cofinite.

The last section is devoted to studying some properties of $H^i_{I,J}(M)$ from the point of view of Serre subcategories. Theorem 3.1 says that if $\mathcal{S}$ is a Serre subcategory (of the category of R-modules) and if $H^i_{I,J}(M) \in \mathcal{S}$ for all $i < d$, then $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}$ for all $i < d$. In case $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module we prove that if $H^i_{I,J}(M) \in \mathcal{S}$ for all $i < d$, then $\mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M)) \in \mathcal{S}$ (Theorem 3.2).

2. Weakly Laskerian modules and cofinite modules

We begin by recalling the definition of weakly Laskerian modules ([2, 2.1]). An R-module M is said to be weakly Laskerian if the set of associated primes of any quotient module of M is finite.

Lemma 2.1. [2, 2.3]

(i) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Then M is weakly Laskerian if and only if L and N are both weakly Laskerian. Thus any subquotient of a weakly Laskerian module as well as any finite direct sum of weakly Laskerian modules is weakly Laskerian.*

(ii) *Let M and N be two R-modules. If M is weakly Laskerian and N is finitely generated, then $\mathrm{Ext}^i_R(N, M)$ and $\mathrm{Tor}^R_i(N, M)$ are weakly Laskerian for all $i \geq 0$.*

An R-module M is said (I,J)-cofinite if $\mathrm{Supp}_R(M) \subseteq W(I,J)$ and $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geq 0$ [7]. The following definition is an extension of the definitions of (I,J)-cofinite modules and I-weakly cofinite modules [3].

Definition 2.2. An R-module M is said to be (I,J)-weakly cofinite if $\mathrm{Supp}_R(M) \subseteq W(I,J)$ and $\mathrm{Ext}^i_R(R/I, M)$ is weakly Laskerian for all $i \geq 0$.

We have the following corollary.

Corollary 2.3. (i) *Every (I,J)-cofinite module is an (I,J)-weakly cofinite module.*

(ii) *If $\mathrm{Supp}_R(M) \subseteq W(I,J)$ and M is weakly Laskerian, then M is (I,J)-weakly cofinite.*

(iii) *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence. If two of the modules are* (I, J)*-weakly cofinite, then so is the third one.*

Proof. (i) Let M be an (I, J)-cofinite R-module. Then we have that $\mathrm{Supp}_R(M) \subseteq W(I, J)$ and $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geq 0$. Hence $\mathrm{Ext}^i_R(R/I, M)$ is weakly Laskerian for all $i \geq 0$.

(ii) It should be noted by Lemma 2.1(ii) that $\mathrm{Ext}^i_R(R/I, M)$ is weakly Laskerian for all $i \geq 0$.

(iii) From the short exact sequence, we obtain

$$\mathrm{Supp}_R(M) = \mathrm{Supp}_R(L) \cup \mathrm{Supp}_R(N)$$

and a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/I, L) \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, N) \to \cdots .$$

Therefore the conclusion follows from the definition of (I, J)-weakly cofinite modules. □

Proposition 2.4. *Let* M *be an* R*-module and* d *a non-negative integer such that* $H^i_{I,J}(M)$ *is* (I, J)*-weakly cofinite for all* $i \leq d$*. Then* $\mathrm{Ext}^i_R(R/I, M)$ *is weakly Laskerian for all* $i \leq d$*.*

Proof. We now proceed by induction on d. When $d = 0$, the short exact sequence

$$0 \to \Gamma_{I,J}(M) \to M \to M/\Gamma_{I,J}(M) \to 0$$

induces an exact sequence

$$0 \to \mathrm{Hom}_R(R/I, \Gamma_{I,J}(M)) \to \mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/I, M/\Gamma_{I,J}(M)).$$

Since $M/\Gamma_{I,J}(M)$ is (I, J)-torsion free, it is also I-torsion free and then

$$\mathrm{Hom}_R(R/I, M/\Gamma_{I,J}(M)) = 0.$$

It follows

$$\mathrm{Hom}_R(R/I, M) \cong \mathrm{Hom}_R(R/I, \Gamma_{I,J}(M)).$$

Hence $\mathrm{Hom}_R(R/I, M)$ is a weakly Laskerian R-module.

Let $d > 0$. Note that $H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M))$ for all $i > 0$ by [6, 1.13(4)]. Let $\overline{M} = M/\Gamma_{I,J}(M)$ and $E(\overline{M})$ denote the injective hull of $\overline{M}$. From the short exact sequence

$$0 \to \overline{M} \to E(\overline{M}) \to E(\overline{M})/\overline{M} \to 0$$

we get

$$\mathrm{Ext}^i_R(R/I, E(\overline{M})/\overline{M}) \cong \mathrm{Ext}^{i+1}_R(R/I, \overline{M})$$

and

$$H^i_{I,J}(E(\overline{M})/\overline{M}) \cong H^{i+1}_{I,J}(\overline{M})$$

for all $i \geq 0$. It follows from the hypothesis that $H^i_{I,J}(E(\overline{M})/\overline{M})$ is (I,J)-weakly cofinite for all $i \leq d-1$. By the inductive hypothesis $\operatorname{Ext}^i_R(R/I, E(\overline{M})/\overline{M})$ is weakly Laskerian for all $i \leq d-1$ and then $\operatorname{Ext}^i_R(R/I, \overline{M})$ is also weakly Laskerian for all $i \leq d$. Now the short exact sequence

$$0 \to \Gamma_{I,J}(M) \to M \to \overline{M} \to 0$$

gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(R/I, \Gamma_{I,J}(M)) \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, \overline{M}) \to \cdots .$$

Since $\Gamma_{I,J}(M)$ is (I,J)-weakly cofinite, $\operatorname{Ext}^i_R(R/I, \Gamma_{I,J}(M))$ is weakly Laskerian for all $i \geq 0$. Finally, it follows from the long exact sequence that $\operatorname{Ext}^i_R(R/I, M)$ is also weakly Laskerian for all $i \leq d$. □

The following theorem answers the question concerning Grothendieck's conjecture: When is the set $\operatorname{Ass}_R(\operatorname{Hom}_R(R/I; H^d_{I,J}(M)))$ finite?

Theorem 2.5. *Let M be a weakly Laskerian R-module and d a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-weakly cofinite for all $i < d$. Then $\operatorname{Hom}_R(R/I, H^d_{I,J}(M))$ is also weakly Laskerian. In particular, the set $\operatorname{Ass}_R(\operatorname{Hom}_R(R/I, H^d_{I,J}(M)))$ is finite.*

Proof. Let us consider functors $F = \operatorname{Hom}_R(R/I, -)$ and $G = \Gamma_{I,J}(-)$. It is clear that

$$FG = \operatorname{Hom}_R(R/I, \Gamma_{I,J}(-)) = \operatorname{Hom}_R(R/I, -).$$

Then we have a Grothendieck spectral sequence by [5, 10.47]

$$E_2^{p,q} = \operatorname{Ext}^p_R(R/I, H^q_{I,J}(M)) \underset{p}{\Rightarrow} \operatorname{Ext}^{p+q}_R(R/I, M).$$

By the hypothesis $E_2^{p,q}$ is weakly Laskerian for all $p \geq 0$ and $0 \leq q < d$. Hence so is $E_\infty^{p,q}$ since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$. Now we have a filtration Φ of $H^d = \operatorname{Ext}^d_R(R/I, M)$

$$0 = \Phi^{d+1}H^d \subseteq \Phi^d H^d \subseteq \cdots \subseteq \Phi^1 H^d \subseteq \Phi^0 H^d = H^d$$

such that

$$E_\infty^{i,d-i} \cong \Phi^i H^d / \Phi^{i+1} H^d.$$

The short exact sequence

$$0 \to \Phi^1 H^d \to H^d \to E_\infty^{0,d} \to 0$$

implies that $E_{\infty}^{0,d}$ is weakly Laskerian since $H^d = \operatorname{Ext}_R^d(R/I, M)$ is weakly Laskerian. We consider homomorphisms of the spectral sequence

$$E_k^{-k,d+k-1} \overset{d^{-k,d+k-1}}{\longrightarrow} E_k^{0,d} \overset{d^{0,d}}{\longrightarrow} E_k^{k,d+1-k}.$$

Since $E_k^{-k,d+k-1} = 0$ for all $k \geq 2$, $\operatorname{Ker} d_k^{0,d} = E_{k+1}^{0,d}$ and $E_k^{k,d+1-k} = 0$ for all $k \geq d+2$. It follows that $E_{d+2}^{0,d} = E_{d+3}^{0,d} = \cdots = E_{\infty}^{0,d}$ and then $E_{d+2}^{0,d}$ is weakly Laskerian. The exact sequence

$$0 \longrightarrow E_{k+2}^{0,d} \longrightarrow E_{k+1}^{0,d} \overset{d^{0,d}}{\longrightarrow} E_{k+1}^{k+1,d-k}$$

yields that $E_{k+1}^{0,d}$ is weakly Laskerian for all $1 \leq k \leq d$. In particular,

$$E_2^{0,d} = \operatorname{Hom}_R(R/I, H_{I,J}^d(M))$$

is weakly Laskerian, which completes the proof. □

The following consequence is a stronger result than the one of Theorem 2.5.

Corollary 2.6. *Let M be a weakly Laskerian R-module and d a non-negative integer. If $H_{I,J}^i(M)$ is weakly Laskerian for all $i < d$, then* $\operatorname{Hom}_R(R/I, H_{I,J}^d(M)/N)$ *is also weakly Laskerian for any weakly Laskerian R-submodule N of $H_{I,J}^d(M)$. In particular, the set* $\operatorname{Ass}_R(\operatorname{Hom}_R(R/I, H_{I,J}^d(M)/N))$ *is finite.*

Proof. The short exact sequence

$$0 \to N \to H_{I,J}^d(M) \to H_{I,J}^d(M)/N \to 0$$

gives rise to an exact sequence

$$\operatorname{Hom}_R(R/I, H_{I,J}^d(M)) \to \operatorname{Hom}_R(R/I, H_{I,J}^d(M)/N) \to \operatorname{Ext}_R^1(R/I, N).$$

It follows from Lemma 2.1(ii) that $\operatorname{Ext}_R^1(R/I, N)$ is weakly Laskerian. Furthermore, $\operatorname{Hom}_R(R/I, H_{I,J}^d(M))$ is also weakly Laskerian by Theorem 2.5. Therefore $\operatorname{Hom}_R(R/I, H_{I,J}^d(M)/N)$ is weakly Laskerian. □

Note that finitely generated modules or modules that have finite support are weakly Laskerian modules. So we have an immediate consequence.

Corollary 2.7. *Let M be a finitely generated R-module and d a non-negative integer. If $H_{I,J}^i(M)$ is finitely generated or* $\operatorname{Supp}_R(H_{I,J}^i(M))$ *is a finite set for all $i < d$, then* $\operatorname{Ass}_R(\operatorname{Hom}_R(R/I, H_{I,J}^d(M)))$ *is finite.*

We now provide a finite result on the associated primes of $H_{I,J}^d(M)$.

Theorem 2.8. *Let M be a weakly Laskerian R-module and d a nonnegative integer. Suppose that there is an ideal $\mathfrak{a}$ of R such that $0 :_M \mathfrak{a} = 0 :_{\Gamma_{I,J}(M)} \mathfrak{a}$ and $W(I,J) \subseteq V(\mathfrak{a})$. If $H^i_{I,J}(M)$ is a weakly Laskerian R-module for all $i < d$, then $\operatorname{Ass}_R(H^d_{I,J}(M))$ is a finite set.*

Proof. Let us consider functors $F = \operatorname{Hom}_R(R/\mathfrak{a}, -)$ and $G = \Gamma_{I,J}(-)$. Then $FG = \operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{I,J}(-))$. From the hypothesis, there is an isomorphism $\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{I,J}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, M)$. By [5, 10.47] we have a Grothendieck spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_R(R/\mathfrak{a}, H^q_{I,J}(M)) \underset{p}{\Rightarrow} \operatorname{Ext}^{p+q}_R(R/\mathfrak{a}, M).$$

We consider homomorphisms of the spectral sequence

$$E_k^{-k,d+k-1} \xrightarrow{d^{-k,d+k-1}} E_k^{0,d} \xrightarrow{d^{0,d}} E_k^{k,d+1-k}.$$

Since $E_k^{-k,d+k-1} = 0$ for all $k \geq 2$, $\operatorname{Ker} d_k^{0,d} = E_{k+1}^{0,d}$. There exists an exact sequence

$$0 \longrightarrow E_{k+1}^{0,d} \longrightarrow E_k^{0,d} \xrightarrow{d^{0,d}} E_k^{k,d+1-k}.$$

Hence

$$\operatorname{Ass}_R(E_k^{0,d}) \subseteq \operatorname{Ass}_R(E_{k+1}^{0,d}) \cup \operatorname{Ass}_R(E_k^{k,d+1-k}).$$

By iterating this for all $k = 2, \ldots, d+1$, we get

$$\operatorname{Ass}_R(E_2^{0,d}) \subseteq \left(\bigcup_{k=2}^{d+1} \operatorname{Ass}_R(E_k^{k,d+1-k}) \right) \cup \operatorname{Ass}_R(E_{d+2}^{0,d}).$$

It is clear that

$$E_{d+2}^{0,d} = E_{d+3}^{0,d} = \cdots = E_\infty^{0,d}.$$

Therefore

$$\operatorname{Ass}_R(E_2^{0,d}) \subseteq \left(\bigcup_{k=2}^{d+1} \operatorname{Ass}_R(E_k^{k,d+1-k}) \right) \cup \operatorname{Ass}_R(E_\infty^{0,d}).$$

For all $k = 2, \ldots, d+1$ as $H^{d+1-k}_{I,J}(M)$ is a weakly Laskerian R-module, so is $E_2^{k,d+1-k} = \operatorname{Ext}^k_R(R/\mathfrak{a}, H^{d+1-k}_{I,J}(M))$ by Lemma 2.1(ii). As $E_k^{k,d+1-k}$ is a subquotient of $E_2^{k,d+1-k}$, it follows from Lemma 2.1(i) that $E_k^{k,d+1-k}$ is a weakly Laskerian R-module. Thus $\bigcup_{k=2}^{d+1} \operatorname{Ass}_R(E_k^{k,d+1-k})$ is a finite set.

To prove the finiteness of $\operatorname{Ass}_R(E_2^{0,d})$, we show that the set $\operatorname{Ass}_R(E_\infty^{0,d})$ is finite. Indeed, there is a filtration Φ of $H^{p+q} = \operatorname{Ext}^{p+q}_R(R/\mathfrak{a}, M)$ with

$$0 = \Phi^{p+q+1} H^{p+q} \subseteq \Phi^{p+q} H^{p+q} \subseteq \cdots \subseteq \Phi^1 H^{p+q} \subseteq \Phi^0 H^{p+q} = \operatorname{Ext}^{p+q}_R(R/\mathfrak{a}, M)$$

and

$$E_\infty^{k,p+q-k} \cong \Phi^k H^{p+q} / \Phi^{k+1} H^{p+q}, \quad 0 \leq k \leq p+q.$$

It follows that $E_\infty^{p,q}$ is a weakly Laskerian R-module, so $\mathrm{Ass}_R(E_\infty^{p,q})$ is finite for all p, q. In particular, $\mathrm{Ass}_R(E_\infty^{0,d})$ is finite. It should be noted by [6, 1.7] that $\mathrm{Ass}_R(H^d_{I,J}(M)) \subseteq W(I,J)$. Therefore

$$\begin{aligned}\mathrm{Ass}_R(E_2^{0,d}) &= \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H^d_{I,J}(M)))\\ &= V(\mathfrak{a}) \cap \mathrm{Ass}_R(H^d_{I,J}(M))\\ &\supseteq W(I,J) \cap \mathrm{Ass}_R(H^d_{I,J}(M))\\ &= \mathrm{Ass}_R(H^d_{I,J}(M)),\end{aligned}$$

and the theorem follows. □

Theorem 2.9. *Let M be an R-module such that $\mathrm{Ext}^i_R(R/I, M)$ is weakly Laskerian for all i and d a non-negative integer. If $H^i_{I,J}(M)$ is (I,J)-weakly cofinite for all $i \neq d$, then $H^d_{I,J}(M)$ is also (I,J)-weakly cofinite.*

Proof. We use induction on d. When $d = 0$, set $\overline{M} = M/\Gamma_{I,J}(M)$, then the short exact sequence

$$0 \to \Gamma_{I,J}(M) \to M \to \overline{M} \to 0$$

gives rise a long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/I, \Gamma_{I,J}(M)) \to \mathrm{Ext}^i_R(R/I, M) \to \mathrm{Ext}^i_R(R/I, \overline{M}) \to \cdots .$$

We have $H^i_{I,J}(\overline{M}) \cong H^i_{I,J}(M)$ for all $i > 0$ and $H^0_{I,J}(\overline{M}) = 0$. From the hypothesis, $H^i_{I,J}(\overline{M})$ is (I,J)-weakly cofinite for all $i \geq 0$. It follows from Proposition 2.4 that $\mathrm{Ext}^i_R(R/I, \overline{M})$ is weakly Laskerian for all $i \geq 0$. Therefore, considering the long exact sequence and the hypothesis gives that $\mathrm{Ext}^i_R(R/I, \Gamma_{I,J}(M))$ is weakly Laskerian. This implies that $H^0_{I,J}(M)$ is (I,J)-weakly cofinite.

Let $d > 0$. The short exact sequence

$$0 \to \overline{M} \to E(\overline{M}) \to E(\overline{M})/\overline{M} \to 0$$

yields

$$\mathrm{Ext}^i_R(R/I, E(\overline{M})/\overline{M}) \cong \mathrm{Ext}^{i+1}_R(R/I, \overline{M})$$

and

$$H^i_{I,J}(E(\overline{M})/\overline{M}) \cong H^{i+1}_{I,J}(\overline{M})$$

for all $i \geq 0$. Then $H^i_{I,J}(E(\overline{M})/\overline{M})$ is (I,J)-weakly cofinite for all $i \neq d-1$. Note that $\mathrm{Ext}^i_R(R/I, \Gamma_{I,J}(M))$ is weakly Laskerian and then $\mathrm{Ext}^i_R(R/I, E(\overline{M})/\overline{M})$ is weakly Laskerian for all $i \geq 0$. By the inductive hypothesis, $H^{d-1}_{I,J}(E(\overline{M})/\overline{M})$ is (I,J)-weakly cofinite. Therefore $H^d_{I,J}(M)$ is (I,J)-weakly cofinite. □

Combining Lemma 2.1(ii) with Theorem 2.9 we obtain the following consequence.

Corollary 2.10. *Let M be a weakly Laskerian R-module and d a non-negative integer. If $H^i_{I,J}(M)$ is (I,J)-weakly cofinite for all $i \neq d$, then $H^d_{I,J}(M)$ is also (I,J)-weakly cofinite.*

Corollary 2.11. *Let I be a principal ideal of R and M a weakly Laskerian module. Then $H^i_{I,J}(M)$ is (I,J)-weakly cofinite for all $i \geq 0$.*

Proof. It follows from [6, 4.11] that $H^i_{I,J}(M) = 0$ for all $i > 1$. Moreover, $H^0_{I,J}(M)$ is a weakly Laskerian R-module, since $H^0_{I,J}(M)$ is a submodule of M. This means that $H^i_{I,J}(M)$ is (I,J)-weakly cofinite for all $i \neq 1$. Now the conclusion follows from Theorem 2.9. □

3. On Serre subcategory

Recall that a class $\mathcal{S}$ of R-modules is a Serre subcategory of the category of R-modules if it is closed under taking submodules, quotients and extensions. Throughout this section, let $\mathcal{S}$ denote a given Serre subcategory of the category of R-modules.

Theorem 3.1. *Let M be an R-module and d a non-negative integer. If $H^i_{I,J}(M) \in \mathcal{S}$ for all $i < d$, then $\operatorname{Ext}^i_R(R/I, M) \in \mathcal{S}$ for all $i < d$.*

Proof. We begin by considering functors $F = \operatorname{Hom}_R(R/I, -)$ and $G = \Gamma_{I,J}(-)$. It is clear that $FG = \operatorname{Hom}_R(R/I, \Gamma_{I,J}(-)) = \operatorname{Hom}_R(R/I, -)$. Then there is a Grothendieck spectral sequence by [5, 10.47]

$$E_2^{p,q} = \operatorname{Ext}^p_R(R/I, H^q_{I,J}(M)) \underset{p}{\Rightarrow} \operatorname{Ext}^{p+q}_R(R/I, M).$$

Since $H^i_{I,J}(M) \in \mathcal{S}$ for all $i < d$, $E_2^{p,q} \in \mathcal{S}$ for all $p \geq 0$, $0 \leq q < d$.

We consider homomorphisms of the spectral sequence for all $p \geq 0$, $0 \leq t < d$ and $i \geq 2$,

$$E_i^{p-i,t+i-1} \overset{d_i^{p-i,t+i-1}}{\longrightarrow} E_i^{p,t} \overset{d_i^{p,t}}{\longrightarrow} E_i^{p+i,t-i+1}.$$

Note that $E_i^{p,t} = \operatorname{Ker} d_{i-1}^{p,t} / \operatorname{Im} d_{i-1}^{p-i+1,t+i-2}$ and $E_i^{p,j} = 0$ for all $j < 0$. This implies

$$\operatorname{Ker} d_{t+2}^{p,t-p} \cong E_{t+2}^{p,t-p} \cong \cdots \cong E_\infty^{p,t-p}$$

for all $0 \leq p \leq t$. We now have a filtration Φ of $H^t = \operatorname{Ext}^t_R(R/I, M)$ such that

$$0 = \Phi^{t+1}H^t \subseteq \Phi^t H^t \subseteq \cdots \subseteq \Phi^1 H^t \subseteq \Phi^0 H^t = \operatorname{Ext}^t_R(R/I, M)$$

and

$$\Phi^i H^t / \Phi^{i+1} H^t \cong E_\infty^{i,t-i}$$

for all $0 \leq i \leq t$. Then there is a short exact sequence

$$0 \to \Phi^{i+1}H^t \to \Phi^i H^t \to E_\infty^{i,t-i} \to 0.$$

From the proof above we have $E_\infty^{i,t-i} \cong E_{t+2}^{i,t-i} \cong \operatorname{Ker} d_{t+2}^{i,t-i}$ a subquotient of $E_2^{i,t-i}$ and $E_2^{i,t-i} \in \mathcal{S}$ for all $0 \leq i \leq t$. It follows that $E_\infty^{i,t-i} \in \mathcal{S}$ for all $0 \leq i \leq t$. By induction on i we get $\Phi^i H^t \in \mathcal{S}$ for all $0 \leq i \leq t$. Finally $\operatorname{Ext}_R^t(R/I, M) \in \mathcal{S}$ for all $t < d$. □

Theorem 3.2. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$ and d a non-negative integer. If $H_{I,J}^i(M) \in \mathcal{S}$ for all $i < d$, then $\operatorname{Hom}_R(R/\mathfrak{m}, H_{I,J}^d(M)) \in \mathcal{S}$.*

Proof. The proof is by induction on d. When $d = 0$, since M is finitely generated, so is $H_{I,J}^0(M)$. Hence $\operatorname{Hom}_R(R/\mathfrak{m}, H_{I,J}^0(M))$ has finite length and then $\operatorname{Hom}_R(R/\mathfrak{m}, H_{I,J}^0(M)) \in \mathcal{S}$ by [1, 2.11].

Let $d > 0$. It follows from [6, 1.13(4)] that

$$H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$$

for all $i > 0$. Thus we can assume, by replacing M with $M/\Gamma_{I,J}(M)$, that M is (I,J)-torsion-free. Since $\Gamma_I(M) \subseteq \Gamma_{I,J}(M) = 0$, it follows that M is also I-torsion-free. Hence, there exists an element $x \in I$ which is non-zerodivisor on M. Set $\overline{M} = M/xM$, the short exact sequence

$$0 \to M \xrightarrow{\cdot x} M \to \overline{M} \to 0$$

gives rise to an exact sequence

$$H_{I,J}^{d-1}(M) \xrightarrow{\cdot x} H_{I,J}^{d-1}(M) \xrightarrow{f} H_{I,J}^{d-1}(\overline{M}) \xrightarrow{g} H_{I,J}^d(M) \xrightarrow{\cdot x} H_{I,J}^d(M).$$

As $H_{I,J}^i(M) \in \mathcal{S}$ for all $i < d$, $H_{I,J}^i(\overline{M}) \in \mathcal{S}$ for all $i < d-1$. Then $\operatorname{Hom}_R(R/\mathfrak{m}, H_{I,J}^{d-1}(\overline{M})) \in \mathcal{S}$ by the inductive hypothesis. Applying the functor $\operatorname{Hom}_R(R/\mathfrak{m}, -)$ to the short exact sequence

$$0 \to \operatorname{Im} f \to H_{I,J}^{d-1}(\overline{M}) \to \operatorname{Im} g \to 0$$

we get a long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Im} f) &\to \operatorname{Hom}_R(R/\mathfrak{m}, H_{I,J}^{d-1}(\overline{M})) \\ &\to \operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Im} g) \to \operatorname{Ext}_R^1(R/\mathfrak{m}, \operatorname{Im} f) \to \cdots . \end{aligned}$$

Note that $\operatorname{Ext}_R^1(R/\mathfrak{m}, \operatorname{Im} f) \in \mathcal{S}$, so $\operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Im} g) \in \mathcal{S}$. Now from the exact sequence

$$0 \to \operatorname{Im} g \to H_{I,J}^d(M) \xrightarrow{\cdot x} H_{I,J}^d(M)$$

we obtain an exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Im} g) \to \operatorname{Hom}_R(R/\mathfrak{m}, H_{I,J}^d(M)) \xrightarrow{\cdot x} \operatorname{Hom}_R(R/\mathfrak{m}, H_{I,J}^d(M)).$$

It is clear that

$$\mathrm{Im}(\mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M)) \xrightarrow{\cdot x} \mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M)) = 0.$$

Therefore

$$\mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Im}\, g) \cong \mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M))$$

and then $\mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M)) \in \mathcal{S}$. □

It should be mentioned that if M is a finitely generated module over a local ring $(R, \mathfrak{m})$ with $\mathrm{Supp}_R(M) \subseteq \{\mathfrak{m}\}$, then M is artinian. From Theorem 3.2 we obtain the following consequence.

Corollary 3.3. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$ and d a non-negative integer. If $H^i_{I,J}(M)$ is finitely generated for all $i < d$, then* $\mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M))$ *has finite length.*

Proof. It follows from Theorem 3.2 that $\mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M))$ is finitely generated. Moreover, $\mathrm{Supp}_R(\mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M))) \subseteq \{\mathfrak{m}\}$. Therefore $\mathrm{Hom}_R(R/\mathfrak{m}, H^d_{I,J}(M))$ is an artinian R-module and then it has finite length. □

Acknowledgments

The authors are deeply grateful to the referees for careful reading of the manuscript and for the helpful suggestions. The first author is partially supported by the Vietnam Institute for Advanced Study in Mathematics (VIASM), Hanoi, Vietnam.

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Tran Tuan Nam
Department of Mathematics-Informatics, Ho Chi Minh University of Pedagogy, Ho Chi Minh city, Vietnam
E-mail address: namtuantran@gmail.com

Nguyen Minh Tri
Department of Natural Science Education, Dong Nai University, Dong Nai, Vietnam
and
Faculty of Mathematics and Computer Science, University of Science, VNU-HCMC, Ho Chi Minh city, Vietnam
E-mail address: triminhng@gmail.com