TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 5, pp. 1065–1077, October 2016
DOI: 10.11650/tjm.20.2016.5762
This paper is available online at http://journal.tms.org.tw

Double Perturbations for Impulsive Differential Equations in Banach Spaces

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Abstract. In this article, we are concerned with the existence of extremal solutions to the initial value problem of impulsive differential equations in ordered Banach spaces. The existence and uniqueness theorem for the solution of the associated linear impulsive differential equation is established. With the aid of this theorem, the existence of minimal and maximal solutions for the initial value problem of nonlinear impulsive differential equations is obtained under the situation that the nonlinear term and impulsive functions are not monotone increasing by using perturbation methods and monotone iterative technique. The results obtained in this paper improve and extend some related results in abstract differential equations. An example is also given to illustrate the feasibility of our abstract results.

1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change in their states at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in communications, control technology, impact mechanics, electrical engineering and medicine. Impulsive ordinary and partial differential equations are used to describe various models of real processes and phenomena studied in physics, chemistry, biology, population and dynamics, engineering and economics. The theory of impulsive differential equations has been emerging as an important area of investigation in the last few decades, see the monographs of Lakshmikantham, Bainov and Simeonov [11, pp. 2–97], Benchohra, Henderson and Ntouyas [2, pp. 11–36] and the papers of Chen and Li [3, 4], Guo and Liu [9], Li and Liu [12], Liu, Wu and Guo [13] and Lu [14], where numerous properties of their solutions are studied and detailed bibliographies are given.

The theory of differential equations in abstract spaces is a fascinating field with important applications to a number of areas in analysis and other branches of mathematics.

Received January 7, 2015; Accepted September 10, 2015.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification.* 34A37, 34K30.
Key words and phrases. Initial value problem, Impulsive differential equation, Monotone iterative technique, Perturbation method, Measure of noncompactness.
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For instance, numerous significant partial differential equations can be unified into ordinary differential equations in infinite dimensional Banach spaces. Therefore, the theory of differential equations in Banach spaces arouses wide interests of many researchers, and has been emerging as an important area of investigation in recent years, see [15, pp. 6–21] and [1, 5–8, 16]. Many famous mathematicians, such as H. Amann, H. Brezis, K. Deimling, Y. Du, D. Guo, V. Lakshmikantham, R. Martin, J. Sun et al. have made important contribution in this field.

In this paper, by utilizing perturbation technique for the nonlinear term and impulsive functions as well as the monotone iterative method based on the lower and upper solutions, we investigate the existence of extremal solutions to the initial value problem (IVP) of impulsive differential equations in an ordered Banach space E

$$\begin{cases} u'(t) = f(t, u(t)), & t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \ldots, m, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $f \in C(J \times E, E)$, $J = [0, a]$, $a > 0$ is a constant, $0 < t_1 < t_2 < \cdots < t_m < a$, $I_k \in C(E, E)$ is an impulsive function, $k = 1, 2, \ldots, m$, $u_0 \in E$, $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

It is well-known that the main difference between the finite and the infinite dimensional differential equations lies in the absence of compactness for the solution operator in the later case. So, in order to overcome this difficulty, some additional assumptions are usually needed, and also some new thoughts, skills and methods are employed to study the differential equations in abstract spaces. One of the most important methods to investigate the differential equations in Banach spaces is monotone iterative technique in the presence of lower and upper solutions. The monotone iterative method based on lower and upper solutions is an effective and flexible mechanism to seek solutions of differential equations in abstract spaces. It yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. Early on, Du and Lakshmikantham [7] built a monotone iterative method for the initial value problem of the ordinary differential equation in Banach space E

$$\begin{cases} u'(t) = f(t, u(t)), & t \in J, \\ u(0) = x_0, \end{cases} \tag{1.2}$$

they proved that if IVP (1.2) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and nonlinear term f satisfies the monotonicity condition

$$f(t, u_2) - f(t, u_1) \geq -M(u_2 - u_1), \quad \forall\, t \in J,\ v_0(t) \leq u_1 \leq u_2 \leq w_0(t) \tag{1.3}$$

with a positive constant M, and the noncompactness measure condition

$$\alpha(f(t,D)) \le L\alpha(D), \quad \forall\, t \in J, \text{ bounded } D \subset E, \tag{1.4}$$

where $L > 0$ is a constant, $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness in E, then IVP (1.2) has minimal and maximal solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.

Later, the monotone iterative method has also been generalized to impulsive differential equations in Banach spaces, see [3, 4, 9, 12–14, 16] and the references therein. We mention the results of Guo and Liu [9], in which the authors obtained the existence of extremal solutions for IVP (1.1) under condition (1.3) for the nonlinear term f and monotonicity condition

$$I_k(u_1) \le I_k(u_2), \quad k = 1, 2, \ldots, m,\ \forall\, t \in J,\ v_0(t) \le u_1 \le u_2 \le w_0(t) \tag{1.5}$$

for impulsive function I_k. They demand also that the nonlinear term f and impulsive function I_k satisfy the noncompactness condition (1.4) the following condition

$$\alpha(I_k(U)) \le L_k\alpha(U), \quad \text{bounded } U \subset E, \tag{1.6}$$

where L_k are positive constants, and satisfy

$$2a(M+L) + \sum_{k=1}^{m} L_k < 1. \tag{1.7}$$

One can easily see that the inequality (1.7) is a strongly restricted condition, and it is not easy to satisfy the condition (1.7) in applications. Recently, Li and Liu [12] deleted the restriction condition (1.7) by adopting a method of piecewise argument, and then largely improved the results in [9].

We observed that in the previous papers, such as [3, 4, 9, 12–14, 16], in which the monotone iterative method is used to study the impulsive differential equations in abstract spaces, the authors all demand that the impulsive function I_k satisfies the monotonicity condition (1.5), i.e., the impulsive function I_k is monotone increasing on order interval. This condition is a strong assumption for the impulsive function I_k, and it is very difficult to be satisfied in many applications. For this reason, in this paper, we will improve and extend the above-mentioned results to the situation that the impulsive function I_k is not monotone increasing by utilizing a perturbation technique for I_k. We construct a monotone iterative method for the IVP (1.1), and obtain the existence of minimal and maximal solutions between lower and upper solutions by applying the perturbation technique for the nonlinear term f and impulsive function I_k and monotone iterative method. Compared with the earlier related existence results for impulsive differential equations, the major difference is that the impulsive function is not monotone increasing in ordered interval, which extended some related results to a large extent.

2. Preliminaries

In this section, we recall some basic facts on cone, partial order and linear impulsive differential equations, which are needed to prove our main results.

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order "$\leq$", whose positive cone $P = \{x \in E \mid x \geq \theta\}$ is normal with normal constant N. Set $\mathrm{PC}(J,E) = \{u\colon J \to E \mid u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$, then $\mathrm{PC}(J,E)$ is a Banach space with the norm $\|u\|_{\mathrm{PC}} = \sup_{t \in J} \|u(t)\|$. Evidently, $\mathrm{PC}(J,E)$ is also an ordered Banach space with the partial order "$\leq$" induced by the positive cone $K_{\mathrm{PC}} = \{u \in \mathrm{PC}(J,E) \mid u(t) \geq \theta, t \in J\}$. K_{PC} is also normal with the same normal constant N. For v, $w \in \mathrm{PC}(J,E)$ with $v \leq w$, we use $[v,w]$ to denote the order interval $\{u \in \mathrm{PC}(J,E) \mid v \leq u \leq w\}$ in $\mathrm{PC}(J,E)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in E \mid v(t) \leq u(t) \leq w(t), t \in J\}$ in E. For more definitions and details of the cone and partial order, we refer to the paper [1] and monograph [8, pp. 1–37].

Definition 2.1. An abstract function $u \in \mathrm{PC}(J,E) \cap C^1(J',E)$ is called a solution of IVP (1.1) if $u(t)$ satisfies all the equalities of (1.1), where $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$.

Definition 2.2. If a function $v_0 \in \mathrm{PC}(J,E) \cap C^1(J',E)$ satisfies that

$$\begin{cases} v_0'(t) \leq f(t, v_0(t)), & t \in J', \\ \Delta v_0|_{t=t_k} \leq I_k(v_0(t_k)), & k = 1, 2, \ldots, m, \\ v_0(0) \leq u_0, \end{cases} \tag{2.1}$$

we call it a lower solution of IVP (1.1); if all the inequalities of (2.1) are inverse, we call it an upper solution of IVP (1.1).

Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For any $D \subset \mathrm{PC}(J,E)$ and $t \in J$, set $D(t) = \{u(t) \mid u \in D\} \subset E$. If $D \subset \mathrm{PC}(J,E)$ is bounded, then $D(t)$ is bounded in E, and $\alpha(D(t)) \leq \alpha(D)$. For the details of the definition and properties of the measure of noncompactness, see [5, pp. 19–21].

The following lemma will be used in the proof of our main results.

Lemma 2.3. [10, Corollary 3.1] *Let E be a Banach space and let $D = \{u_n\} \subset \mathrm{PC}(J,E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integral on J, and*

$$\alpha\left(\left\{\int_J u_n(t)\,dt \,\middle|\, n \in \mathbb{N}\right\}\right) \leq 2\int_J \alpha(D(t))\,dt. \tag{2.2}$$

In order to study the nonlinear IVP (1.1), we consider the initial value problem (IVP) of linear impulsive differential equation in E

$$\begin{cases} u'(t) + Mu(t) = h(t), & t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} + C_k u(t_k) = y_k, & k = 1, 2, \dots, m, \\ u(0) = u_0, \end{cases} \tag{2.3}$$

where M is a positive constant, $h \in \mathrm{PC}(J, E)$, $C_k \in \mathbb{R}$, $y_k \in E$, $k = 1, 2, \dots, m$, $u_0 \in E$.

Lemma 2.4. *Let E be a Banach space. For any $h \in \mathrm{PC}(J, E)$, $C_k \in \mathbb{R}$, $y_k \in E$ and $u_0 \in E$, IVP (2.3) has a unique solution $u \in \mathrm{PC}(J, E) \cap C^1(J', E)$ given by*

$$\begin{aligned} u(t) = {} & \prod_{0 \le t_k < t} (1 - C_k) e^{-Mt} u_0 \\ & + \sum_{0 < t_k < t} \prod_{t_k \le t_i < t} (1 - C_i) \int_{t_{k-1}}^{t_k} e^{-M(t-s)} h(s)\, ds + \int_{t_p}^{t} e^{-M(t-s)} h(s)\, ds \\ & + \sum_{0 < t_k < t_p} \prod_{t_k < t_i < t} (1 - C_i) e^{-M(t-t_k)} y_k + e^{-M(t-t_p)} y_p, \quad t \in J, \end{aligned} \tag{2.4}$$

where $t_0 := 0$, $C_0 := 0$, $t_p < t$ ($p = 0, 1, \dots, m$) is the nearest impulsive point of t.

Proof. Let $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, where $t_0 = 0$, $t_{m+1} = a$. Let $y_0 = \theta$. If $u \in \mathrm{PC}(J, E) \cap C^1(J', E)$ is a solution of IVP (2.3), then the restriction of u on J_k satisfies the linear differential equation without impulse

$$\begin{cases} u'(t) + Mu(t) = h(t), & t_k < t \le t_{k+1}, \\ u(t_k^+) = (1 - C_k) u(t_k) + y_k, & k = 0, 1, \dots, m. \end{cases} \tag{2.5}$$

Hence, on J_k, $u(t)$ can be expressed by

$$u(t) = e^{-M(t-t_k)} [(1 - C_k) u(t_k) + y_k] + \int_{t_k}^{t} e^{-M(t-s)} h(s)\, ds. \tag{2.6}$$

Iterating successively in the above equality with $u(t_j)$, $j = k, k-1, \dots, 0$, we see that u satisfies (2.4).

Inversely, we can verify directly that the function $u \in \mathrm{PC}(J, E) \cap C^1(J', E)$ defined by (2.4) is a solution of IVP (2.3). □

3. Main results

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone P is regular, $f \in C(J \times E, E)$, $I_k \in C(E, E)$, $k = 1, 2, \dots, m$. Assume that the IVP (1.1) has a lower solution $v_0 \in \mathrm{PC}(J, E) \cap C^1(J', E)$ and an upper solution $w_0 \in \mathrm{PC}(J, E) \cap C^1(J', E)$ with $v_0 \le w_0$. Suppose that the following conditions are satisfied:*

(F_1) *There exists a constant* $M \geq 0$ *such that*

$$f(t, u_2) - f(t, u_1) \geq -M(u_2 - u_1),$$

for $\forall\, t \in J$, *and* $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$.

(I_1) *There exist constants* $0 \leq C_k < 1$ *such that*

$$I_k(u_2) - I_k(u_1) \geq -C_k(u_2 - u_1), \quad k = 1, 2, \dots, m,$$

for any $t \in J$, *and* $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$.

Then the IVP (1.1) *has minimal and maximal solutions* $\underline{u}$ *and* $\overline{u}$ *between* v_0 *and* w_0, *which can be obtained by a monotone iterative procedure starting from* v_0 *and* w_0, *respectively.*

Proof. It is easy to see that the solution of the IVP (1.1) is equivalent to the solution of the following initial value problem

$$\begin{cases} u'(t) + Mu(t) = f(t, u(t)) + Mu(t), & t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} + C_k u(t_k) = I_k(u(t_k)) + C_k u(t_k), & k = 1, 2, \dots, m, \\ u(0) = u_0. \end{cases} \tag{3.1}$$

Therefore, we define an operator $\Psi\colon \mathrm{PC}(J, E) \to \mathrm{PC}(J, E)$ by

$$\begin{aligned} (\Psi u)(t) = & \prod_{0 \leq t_k < t} (1 - C_k) e^{-Mt} u_0 \\ & + \sum_{0 < t_k < t} \prod_{t_k \leq t_i < t} (1 - C_i) \int_{t_{k-1}}^{t_k} e^{-M(t-s)} \left[f(s, u(s)) + Mu(s) \right] ds \\ & + \int_{t_p}^{t} e^{-M(t-s)} \left[f(s, u(s)) + Mu(s) \right] ds \\ & + \sum_{0 < t_k < t_p} \prod_{t_k < t_i < t} (1 - C_i) e^{-M(t-t_k)} \left[I_k(u(t_k)) + C_k u(t_k) \right] \\ & + e^{-M(t-t_p)} \left[I_p(u(t_p)) + C_p u(t_p) \right], \quad t \in J. \end{aligned} \tag{3.2}$$

It is clear that $Q\colon \mathrm{PC}(J, E) \to \mathrm{PC}(J, E)$ is continuous. By Lemma 2.4, we know that the solution of the IVP (1.1) is equivalent to the fixed point of operator Ψ defined by (3.2). From the definition of operator Ψ and the assumptions (F_1) and (I_1), it is easy to prove that Ψ is an increasing operator in $[v_0, w_0]$.

Next, we show that $v_0 \leq \Psi v_0$, $\Psi w_0 \leq w_0$. Let $h(t) = v_0'(t) + Mv_0(t)$, by Definition 2.2, we know that $h \in \mathrm{PC}(J, E)$ and $h(t) \leq f(t, v_0(t)) + Mv_0(t)$ for $t \in J'$. By Lemma 2.4 and

Definition 2.2, we have

$$
\begin{aligned}
v_0(t) = {} & \prod_{0 \le t_k < t} (1 - C_k) e^{-Mt} v_0(0) \\
& + \sum_{0 < t_k < t} \prod_{t_k \le t_i < t} (1 - C_i) \int_{t_{k-1}}^{t_k} e^{-M(t-s)} h(s)\, ds + \int_{t_p}^{t} e^{-M(t-s)} h(s)\, ds \\
& + \sum_{0 < t_k < t_p} \prod_{t_k < t_i < t} (1 - C_i) e^{-M(t-t_k)} \left[\Delta v_0|_{t=t_k} + C_k v_0(t_k)\right] \\
& + e^{-M(t-t_p)} \left[\Delta v_0|_{t=t_p} + C_k v_0(t_p)\right] \\
\le {} & \prod_{0 \le t_k < t} (1 - C_k) e^{-Mt} u_0 \\
& + \sum_{0 < t_k < t} \prod_{t_k \le t_i < t} (1 - C_i) \int_{t_{k-1}}^{t_k} e^{-M(t-s)} \left[f(s, v_0(s)) + M v_0(s)\right] ds \\
& + \int_{t_p}^{t} e^{-M(t-s)} \left[f(s, v_0(s)) + M v_0(s)\right] ds \\
& + \sum_{0 < t_k < t_p} \prod_{t_k < t_i < t} (1 - C_i) e^{-M(t-t_k)} \left[I_k(v_0(t_k)) + C_k v_0(t_k)\right] \\
& + e^{-M(t-t_p)} \left[I_p(v_0(t_p)) + C_p v_0(t_p)\right] \\
= {} & (\Psi v_0)(t), \quad t \in J,
\end{aligned}
\tag{3.3}
$$

namely, $v_0 \le \Psi v_0$. Similarly, it can be show that $\Psi w_0 \le w_0$. Therefore, $\Psi\colon [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$
v_n = \Psi v_{n-1}, \quad w_n = \Psi w_{n-1}, \quad n = 1, 2, \ldots. \tag{3.4}
$$

Then from the monotonicity of Ψ, it follows that

$$
v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{3.5}
$$

From (3.5) we can easily see that $\{v_n(t)\}$ is an increasing sequence with upper bounded in E and $\{w_n(t)\}$ is an decreasing sequence with lower bounded in E, combining this with the fact that the positive cone P of E is regular, we know that the sequences $\{v_n\}$ and $\{w_n\}$ are convergent in J. Let

$$
\lim_{n \to \infty} v_n(t) = \underline{u}(t), \quad \lim_{n \to \infty} w_n(t) = \overline{u}(t), \quad t \in J. \tag{3.6}
$$

Evidently, $\{v_n(t)\} \subset \mathrm{PC}(J, E)$, so $\underline{u}(t)$ is bounded integrable in every J_k, $0 \le k \le m$.

Since for any $t \in J$,

$$
\begin{aligned}
v_n(t) = (\Psi v_{n-1})(t) = &\prod_{0\le t_k<t}(1-C_k)e^{-Mt}u_0 \\
&+ \sum_{0<t_k<t}\prod_{t_k\le t_i<t}(1-C_i)\int_{t_{k-1}}^{t_k} e^{-M(t-s)}\left[f(s,v_{n-1}(s)) + Mv_{n-1}(s)\right]ds \\
&+ \int_{t_p}^{t} e^{-M(t-s)}\left[f(s,v_{n-1}(s)) + Mv_{n-1}(s)\right]ds \\
&+ \sum_{0<t_k<t_p}\prod_{t_k<t_i<t}(1-C_i)e^{-M(t-t_k)}\left[I_k(v_{n-1}(t_k)) + C_k v_{n-1}(t_k)\right] \\
&+ e^{-M(t-t_p)}\left[I_p(v_{n-1}(t_p)) + C_p v_{n-1}(t_p)\right],
\end{aligned} \tag{3.7}
$$

letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we know that $\underline{u}(t) = \Psi\underline{u}(t)$ and $\underline{u}(t) \in \mathrm{PC}(J,E)$. Similarly, $\overline{u}(t) = \Psi\overline{u}(t)$ and $\overline{u}(t) \in \mathrm{PC}(J,E)$. Combining this with monotonicity (3.5), we see that $v_0(t) \le \underline{u}(t) \le \overline{u}(t) \le w_0(t)$, $t \in J$.

Next, we show that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of Ψ in $[v_0, w_0]$, respectively. In fact, for any $u \in [v_0,w_0]$, $\Psi u = u$, we have $v_0 \le u \le w_0$, and $v_1 = \Psi v_0 \le \Psi u = u \le \Psi w_0 = w_1$. Continuing such a progress, we get $v_n \le u \le w_n$. Letting $n \to \infty$, we get $\underline{u} \le u \le \overline{u}$. Therefore, $\underline{u}$ and $\overline{u}$ are minimal and maximal solutions of IVP (1.1) in $[v_0, w_0]$, and $\underline{u}$ and $\overline{u}$ can be obtained by the iterative scheme defined by (3.4) starting from v_0 and w_0, respectively. □

Theorem 3.2. *Let E be an ordered and weakly sequentially complete Banach space, whose positive cone P is normal, $f \in C(J \times E, E)$, $I_k \in C(E,E)$, $k = 1,2,\ldots,m$. Assume that the IVP* (1.1) *has a lower solution $v_0 \in \mathrm{PC}(J,E) \cap C^1(J',E)$ and an upper solution $w_0 \in \mathrm{PC}(J,E) \cap C^1(J',E)$ with $v_0 \le w_0$, and the conditions* (F_1) *and* (I_1) *are satisfied. Then the IVP* (1.1) *has minimal and maximal solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. From the proof of Theorem 3.1, we know that the operator $\Psi\colon [v_0,w_0] \to [v_0,w_0]$ is continuous. Furthermore, if the assumptions (F_1) and (I_1) are satisfied, then the iterative sequences $\{v_n\}$ and $\{w_n\}$ defined by (3.4) satisfy the monotonicity (3.5). Therefore, for any $t \in J$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are monotone and order-bounded sequences in E. Noticing that E is a weakly sequentially complete Banach space, combining this fact with [6, Theorem 2.2], we know that $\{v_n(t)\}$ and $\{w_n(t)\}$ are precompact in E. Combining this with the monotonicity (3.5), it follows that $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly convergent in E. Similar with the proof of Theorem 3.1, we know that IVP (1.1) has minimal and maximal solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative scheme defined by (3.4) starting from v_0 and w_0, respectively. □

If E is a general ordered Banach space, whose positive cone P is normal, we have the following result.

Theorem 3.3. *Let E be an ordered Banach space, whose positive cone P is normal, $f \in C(J \times E, E)$, $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$. Assume that the IVP* (1.1) *has a lower solution $v_0 \in \mathrm{PC}(J, E) \cap C^1(J', E)$ and an upper solution $w_0 \in \mathrm{PC}(J, E) \cap C^1(J', E)$ with $v_0 \leq w_0$. Suppose that the assumptions* (F_1), (I_1) *and the following conditions are satisfied:*

(F_2) *There exists a constant $L > 0$ such that*

$$\alpha\left(\{f(t, u_n) + M u_n\}\right) \leq L\alpha(\{u_n\}), \quad t \in J,$$

for any equicontinuous and increasing or decreasing monotonic sequences $\{u_n\} \subset [v_0(t), w_0(t)]$,

(I_2) *There exist constants $0 < L_k < 1$ $(k = 1, 2, \ldots, m)$ such that*

$$\alpha\left(\{I_k(u_n) + C_k u_n\}\right) \leq L_k\alpha(\{u_n\}), \quad t \in J,$$

for any equicontinuous and increasing or decreasing monotonic sequences $\{u_n\} \subset [v_0(t), w_0(t)]$,

(S) $\left[\frac{2L}{M}\left(\sum_{k=1}^{m}\prod_{i=k}^{m}(1 - C_i) + 1\right) + \sum_{k=1}^{m-1}\prod_{i=k+1}^{m} L_k(1 - C_i) + L_m\right] < 1.$

Then the IVP (1.1) *has minimal and maximal solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. From the proof of Theorem 3.1, we know that the operator $\Psi\colon [v_0, w_0] \to [v_0, w_0]$ is continuous. Furthermore, if the assumptions (F_1) and (I_1) are satisfied, then the iterative sequences $\{v_n\}$ and $\{w_n\}$ defined by (3.4) satisfy the monotonicity (3.5). Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J.

For convenience, let $B = \{v_n \mid n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$. Since $B = \Psi(B_0)$, by the definition of operator Ψ and the boundedness of B_0, we can easily prove that B is equicontinuous in every interval J_k, $k = 0, 1, 2, \ldots, m$. From $B_0 = B \cup \{v_0\}$ it follows that $\alpha(B_0(t)) = \alpha(B(t))$ for every $t \in J$.

Now, we are in the position to prove $\alpha(B(t)) = 0$ in J. By the assumptions (F_2), (I_2), (3.2) and Lemma 2.3, we know that for any $t \in J$,

$$
\begin{aligned}
\alpha(B(t)) &= \alpha(\Psi(B_0(t))) \\
&\leq \alpha\left(\sum_{0<t_k<t}\prod_{t_k\leq t_i<t}(1-C_i)\int_{t_{k-1}}^{t_k} e^{-M(t-s)}\left[f(s,v_{n-1}(s))+Mv_{n-1}(s)\right]ds\right) \\
&\quad + \alpha\left(\int_{t_p}^{t} e^{-M(t-s)}\left[f(s,v_{n-1}(s))+Mv_{n-1}(s)\right]ds\right) \\
&\quad + \alpha\left(\sum_{0<t_k<t_p}\prod_{t_k<t_i<t}(1-C_i)e^{-M(t-t_k)}\left[I_k(v_{n-1}(t_k))+C_kv_{n-1}(t_k)\right]\right) \\
&\quad + \alpha\left(e^{-M(t-t_p)}\left[I_p(v_{n-1}(t_p))+C_pv_{n-1}(t_p)\right]\right) \\
&\leq 2\sum_{0<t_k<t}\prod_{t_k\leq t_i<t}(1-C_i)\int_{t_{k-1}}^{t_k} e^{-M(t-s)}\alpha(\{f(s,v_{n-1}(s))+Mv_{n-1}(s)\})\,ds \\
&\quad + 2\int_{t_p}^{t} e^{-M(t-s)}\alpha(\{f(s,v_{n-1}(s))+Mv_{n-1}(s)\})\,ds \\
&\quad + \sum_{0<t_k<t_p}\prod_{t_k<t_i<t}(1-C_i)e^{-M(t-t_k)}\alpha(\{I_k(v_{n-1}(t_k))+C_kv_{n-1}(t_k)\}) \\
&\quad + e^{-M(t-t_p)}\alpha(\{I_p(v_{n-1}(t_p))+C_pv_{n-1}(t_p)\}) \\
&\leq 2L\sum_{0<t_k<t}\prod_{t_k\leq t_i<t}(1-C_i)\int_{t_{k-1}}^{t_k} e^{-M(t-s)}\alpha(B_0(s))\,ds \\
&\quad + 2L\int_{t_p}^{t} e^{-M(t-s)}\alpha(B_0(s))\,ds \\
&\quad + \sum_{0<t_k<t_p}\prod_{t_k<t_i<t}(1-C_i)e^{-M(t-t_k)}L_k\alpha(B_0(t_k)) \\
&\quad + e^{-M(t-t_p)}L_p\alpha(B_0(t_p)) \\
&\leq \left[\frac{2L}{M}\left(\sum_{k=1}^{m}\prod_{i=k}^{m}(1-C_i)+1\right)+\sum_{k=1}^{m-1}\prod_{i=k+1}^{m}L_k(1-C_i)+L_m\right]\sup_{t\in J}\alpha(B(t)).
\end{aligned}
\tag{3.8}
$$

From (3.8) and the assumption (S) we know that $\alpha(B(t)) = 0$ for every $t \in J$. This means that the sequence $\{v_n(t)\}$ is uniformly convergent in J. Similarly, we can prove that the sequence $\{v_n(t)\}$ is uniformly convergent in J. By using a completely similar method with the proof of Theorem 3.1, we can prove that the IVP (1.1) has minimal and maximal solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure defined by (3.4) starting from v_0 and w_0, respectively. □

4. An example

In this section, we give an example to illustrate the applicability of our abstract results.

Example 4.1. Consider the following initial value problem of infinite system for nonlinear impulsive differential equations

$$\begin{cases} u_n'(t) = \sin^2 u_n(t), & t \in J,\ t \neq t_k, \\ \Delta u_n|_{t=t_k} = e^{u_n(t_k)} - \ln 3u_n(t_k), & k = 1, 2, \ldots, m, \\ u_n(0) = 0, \end{cases} \tag{4.1}$$

where $n = 1, 2, \ldots$, $J = [0, a]$, $a > 0$ is a constant.

Let

$$E = \ell^2 = \left\{ u = (u_1, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n|^2 < \infty \right\} \tag{4.2}$$

with the norm

$$\|u\| = \left(\sum_{n=1}^{\infty} |u_n|^2 \right)^{\frac{1}{2}}, \tag{4.3}$$

and let

$$P = \left\{ u = (u_1, \ldots, u_n, \ldots) \in \ell^2 \mid u_n \geq 0, n = 1, 2, \ldots \right\}. \tag{4.4}$$

Then E is a weakly sequentially complete Banach space and P is a normal cone in E. Denote $u(t) = (u_1(t), \ldots, u_n(t), \ldots)$, $f(t, u(t)) = (\sin^2 u_1(t), \ldots, \sin^2 u_n(t), \ldots)$ and $I_k(u(t_k)) = (e^{u_1(t_k)} - \ln 3u_1(t_k), \ldots, e^{u_n(t_k)} - \ln 3u_n(t_k), \ldots)$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots, m$. Then the infinite system (4.1) can be transformed into the form of (1.1) in E.

Theorem 4.2. *Assume that there exist a series of positive functions $w_n(t) \in \mathrm{PC}(J) \cap C^1(J)$, $n = 1, 2, \ldots$, such that*

$$\begin{cases} w_n'(t) \geq \sin^2 w_n(t), & t \in J,\ t \neq t_k, \\ \Delta w_n|_{t=t_k} \geq e^{w_n(t_k)} - \ln 3w_n(t_k), & k = 1, 2, \ldots, m. \end{cases} \tag{4.5}$$

Then the infinite system (4.1) *has minimal and maximal solutions between $\theta = (0, 0, \ldots, 0, \ldots)$ and $w(t) = (w_1(t), \ldots, w_n(t), \ldots)$, which can be obtained by a monotone iterative procedure starting from θ and $w(t)$, respectively.*

Proof. By the inequality (4.5) one can easily see that $v_0 = \theta$ and $w_0 = w(t)$ are lower and upper solutions of the infinite system (4.1), respectively. From the definitions of the nonlinear term f and impulsive function I_k, it is easy to verify that assumptions (F_1) and (I_1) are satisfied with the constants $M = 1$ and $\ln 3 - 1 < C_k < 1$. Therefore, our conclusion follows from Theorem 3.1. □

Remark 4.3. The problem (4.1) can not be solved by the results in [3,4,9,12–14,16] because the impulsive function defined in (4.1) is not monotone increasing on any ordered interval.

Acknowledgments

The authors would like to express sincere thanks to the anonymous referee for his/her carefully reading the manuscript and valuable comments and suggestions. This research was supported by NNSF of China (No. 11501455 and No. 11261053). The Science Research Project for Colleges and Universities of Gansu Province (No. 2015A-003 and No. 2015A-213) and Project of NWNU-LKQN-14-3.

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