TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 2, pp. 431–447, April 2016
DOI: 10.11650/tjm.20.2016.5665
This paper is available online at http://journal.tms.org.tw

On Nonhomogeneous Elliptic Equations with Critical Sobolev Exponent and Prescribed Singularities

Mohammed Bouchekif and Sofiane Messirdi*

Abstract. In this paper we consider a class of nonhomogeneous elliptic equations involving multi-polar Hardy type potentials and a Sobolev critical nonlinearity in an open domain of $\mathbb{R}^N$, $N \geq 3$. By Ekeland's Variational Principle and the Mountain Pass Lemma, we prove the existence of multiple solutions under sufficient conditions on the data and the considered parameters.

1. Introduction

In this paper we study the existence of multiple solutions to the following problem:

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u - \displaystyle\sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u = |u|^{2^*-2} u + \sum_{i=1}^{k} \frac{\lambda_i}{|x-a_i|^{2-\alpha_i}} u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, $k \in \mathbb{N}^*$; for $i = 1, 2, \ldots, k$, $a_i \in \Omega$ with a_i different from a_j for $i \neq j$, λ_i and μ_i are nonnegative parameters and α_i are positive constants; f is a given bounded measurable function. Here $2^* = \frac{2N}{N-2}$ denotes the critical Sobolev exponent.

This model problem has a loss of compactness phenomena, since the nonlinearity has a critical growth imposed by the critical Sobolev exponent and / or the presence of singular potentials. In these situations, the classical methods fail to be applied directly which make the study more harder.

This class of elliptic equations contains singular potentials which arise in many fields, such as quantum mechanics, nuclear physics, molecular physics and quantum cosmology, for more details we refer to [6] or [7].

We start by giving a brief historic.

Received December 16, 2014, accepted September 24, 2015.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification.* 35J20, 35J50, 35B33.
Key words and phrases. Critical Sobolev exponent, Palais-Smale condition, Ekeland's variational principle, Multi-singular potentials, Concentration compactness principle, Hardy inequality.
*Corresponding author.

The regular case i.e., $\lambda_i = \mu_i = 0$ for $i = 1, 2, \ldots, k$ has been studied by Tarantello [14]. By using Ekeland's Variational Principle [5] and the Mountain Pass Lemma [1], she proved the existence of multiple solutions for $f \neq 0$ and satisfying a suitable assumption. They are nonnegative if f is also nonnegative.

For $k = 1$, Kang and Deng [11] proved the existence of at least two weak solutions in $H_0^1(\Omega)$ for the singular critical inhomogeneous problem:

$$-\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2_*(s)-2}\, u}{|x|^s} + \lambda u + f$$

under some sufficient assumptions on f, λ and μ, where $2_*(s) = \frac{2(N-s)}{N-2}$ is the critical Hardy-Sobolev exponent with $0 \leq s < 2$.

In [3], Chen studied the following problem:

$$\begin{cases} -\Delta u - \mu V(x) u = K(x) |u|^{2^*-2} u + \theta h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the linear weight V has m singular points, K is a positive bounded function defined on $\overline{\Omega}$ and $h \in H^{-1}$ (topological dual of $H_0^1(\Omega)$) is a positive function. Under some hypothesis on V, θ and μ, he obtained the existence of at least m positive solutions.

Chen and Rocha [4] proved the existence of at least four nontrivial solutions in $H_0^1(\Omega)$ for the following problem:

$$-\Delta u - \frac{\lambda}{|x|^2} u = |u|^{2_*-2} u + \frac{\mu}{|x|^{2-\alpha}} u + f$$

and showed that at least one of them is sign changing for $0 < \alpha < 2$.

The question is: can we have at least $2k$ solutions for the problem $(\mathcal{P})$? The answer is affirmative. More explicitly, important informations for the existence of multiple solutions of the considered problem are obtained. Our work generalizes the results obtained by Chen [3] and Chen and Rocha [4]. In our knowledge they are new and interesting.

In what follows, we state the main results for which we consider the following hypothesis

$$(\mathcal{F}) \qquad A_{\widetilde{\lambda},\widetilde{\mu}}(f) := \inf \left\{ C_N (T(u))^{(N+2)/4} - \int_\Omega f u \, dx : u \in H_0^1(\Omega), \int_\Omega |u|^{2^*} \, dx = 1 \right\} > 0$$

where $\widetilde{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\widetilde{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$, $C_N = \frac{4}{N-2} \left(\frac{N-2}{N+2} \right)^{(N+2)/4}$ and

$$T(u) = \int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2} u^2 - \sum_{i=1}^k \frac{\lambda_i}{|x-a_i|^{2-\alpha_i}} u^2 \right) dx.$$

Let $\overline{\mu} := \left(\frac{N-2}{2} \right)^2$ be the best Hardy constant. Now we are in measure to give our main results:

Theorem 1.1. *Assume that the parameters λ_i, μ_i are nonnegative numbers for $i = 1, \dots, k$ such that $\sum_{i=1}^k \lambda_i < \lambda^1$, $\sum_{i=1}^k \mu_i < \overline{\mu}$ and f is a bounded measurable function which is positive in each neighborhood of a_i and satisfies $(\mathcal{F})$. Then the problem $(\mathcal{P})$ has at least $2k$ solutions in $H_0^1(\Omega)$ if $0 < \alpha_i < \sqrt{\overline{\mu} - \mu_i}$.*

The positive constant λ^1 will be given later.

This paper is organized as follows: in the forthcoming section, we give some preliminaries used in our work. Section 3 is concerned by the proofs of our main results.

2. Preliminary results

We give some preliminaries which play important roles in sequel of this work.

2.1. Definitions and notations

In what follows we denote the norms of $L^s(\Omega)$, $(1 \le s < \infty)$ and H^{-1} by $|\cdot|_s$ and $\|\cdot\|_-$ respectively, $\int_\Omega u\,dx$ by $\int_\Omega u$, B_a^r is the ball in Ω with center a and radius r, $o_n(1)$ is any quantity which tends to zero as n goes to infinity and $\mathcal{O}(\varepsilon^s)$ verifies $|\mathcal{O}(\varepsilon^s)|/\varepsilon^s \le C$ for some a positive constant C.

Problem $(\mathcal{P})$ is related to the Hardy inequality [9]:

$$\int_{\mathbb{R}^N} \frac{u^2}{|x-a|^2} \le \frac{1}{\overline{\mu}} \int_{\mathbb{R}^N} |\nabla u|^2, \quad \text{for all } a \in \mathbb{R}^N,\ u \in C_0^\infty(\mathbb{R}^N),$$

where $\overline{\mu}$ is the best Hardy constant.

We define for $\mu \in (0, \overline{\mu})$ and $a \in \Omega$ the constant:

$$S_\mu(\Omega) := \inf_{u \in H \setminus \{0\}} \frac{|\nabla u|_2^2 - \mu \left| \frac{u}{|x-a|} \right|_2^2}{|u|_{2^*}^2}.$$

From [10], S_μ is independent of any $\Omega \subset \mathbb{R}^N$ in the sense that $S_\mu(\Omega) = S_\mu(\mathbb{R}^N) = S_\mu$. In addition, it is achieved by a family of functions

$$U_{\varepsilon,a}(x) := \frac{[4\varepsilon(\overline{\mu} - \mu)N/(N-2)]^{(N-2)/4}}{\left(\varepsilon\,|x-a|^{\gamma^-/\sqrt{\overline{\mu}}} + |x-a|^{\gamma^+/\sqrt{\overline{\mu}}}\right)^{(N-2)/2}}, \quad \varepsilon > 0,$$

where $\gamma^- = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$ and $\gamma^+ = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$.

Moreover the functions $U_{\varepsilon,a}$ satisfy

$$\begin{cases} -\Delta u - \mu \frac{u}{|x-a|^2} = |u|^{2^*-2} u & \text{in } \mathbb{R}^N \setminus \{a\} \\ u \to 0 & \text{as } |x| \to \infty \end{cases}$$

and

$$\int_{\mathbb{R}^N} |U_{\varepsilon,a}|^{2^*} = \int_{\mathbb{R}^N} \left(|\nabla U_{\varepsilon,a}|^2 - \mu \frac{U_{\varepsilon,a}^2}{|x-a|^2} \right) = S_\mu^{N/2}.$$

In the sequel of our work we consider $\lambda_i, \mu_i \geq 0$ for $i = 1, 2, \ldots, k$ such that $\sum_{i=1}^k \lambda_i < \lambda^1$ and $\sum_{i=1}^k \mu_i < \overline{\mu}$.

Denote H by the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2} u^2 \right) \right)^{1/2}.$$

Using Hardy's inequality, this norm is equivalent to the usual norm $\left(\int_\Omega |\nabla u|^2 \right)^{1/2}$.

For any $u \in H \setminus \{0\}$, define the positive value

$$t_{\max} = t_{\max}(u) = \left(\frac{T(u)}{(2^* - 1) |u|_{2^*}^{2^*}} \right)^{1/(2^*-2)}$$

and the functional $J\colon H \setminus \{0\} \to \mathbb{R}$ by

$$\begin{aligned} J(u) &= t_{\max} T(u) - t_{\max}^{2^*-1} |u|_{2^*}^{2^*} \\ &= C_N T(u)^{(N+2)/4} |u|_{2^*}^{-N/2}. \end{aligned}$$

The energy functional associated to $(\mathcal{P})$ is given by the following expression:

$$I(u) := \frac{1}{2} T(u) - \frac{1}{2^*} \int_\Omega |u|^{2^*} - \int_\Omega f u.$$

We see that I is well defined in H and belongs to $C^1(H, \mathbb{R})$.

A function $u \in H$ is said to be a weak solution to the problem $(\mathcal{P})$ if $\langle I'(u), \varphi \rangle = 0$, for all $\varphi \in H$, where

$$\begin{aligned} \langle I'(u), \varphi \rangle = \int_\Omega & \left(\nabla u \nabla \varphi - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2} u\varphi - \sum_{i=1}^k \frac{\lambda_i}{|x-a_i|^{2-\alpha_i}} u\varphi \right) \\ & + - \int_\Omega |u|^{2^*-2} u\varphi - \int_\Omega f\varphi. \end{aligned}$$

More standard elliptic regularity argument implies that a weak solution $u \in H$ is indeed in $C^2(\Omega \setminus \{a_1, a_2, \ldots, a_k\}) \cap C^0(\overline{\Omega} \setminus \{a_1, a_2, \ldots, a_k\})$ and we can say that u satisfies $(\mathcal{P})$ in the classical sense.

Definition 2.1. A functional $I \in C^1(H, \mathbb{R})$ satisfies the Palais–Smale condition at level c, ($(\mathrm{PS})_\mathrm{c}$ for short), if any sequence $(u_n) \subset H$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } H^{-1} \text{ (dual of } H\text{)},$$

contains a strongly convergent subsequence.

As I is not bounded from below on H, we consider it on the Nehari manifold:

$$\mathcal{N} = \left\{u \in H \setminus \{0\} : \left\langle I'(u), u \right\rangle = 0\right\}.$$

Thus $u \in \mathcal{N}$ if and only if:

$$T(u) - |u|_{2^*}^{2^*} - \int_\Omega fu = 0.$$

It is natural to split $\mathcal{N}$ into three subsets:

$$\mathcal{N}^+ = \left\{u \in \mathcal{N} : \left\langle I''(u), u \right\rangle > 0\right\}, \quad \mathcal{N}^- = \left\{u \in \mathcal{N} : \left\langle I''(u), u \right\rangle < 0\right\}$$

and

$$\mathcal{N}^0 = \left\{u \in \mathcal{N} : \left\langle I''(u), u \right\rangle = 0\right\},$$

with

$$\begin{aligned}\left\langle I''(u), u \right\rangle &= 2T(u) - 2^* \,|u|_{2^*}^{2^*} - \int_\Omega fu \\ &= T(u) - (2^* - 1)\,|u|_{2^*}^{2^*} \\ &= (2 - 2^*)T(u) + (2^* - 1)\int_\Omega fu.\end{aligned}$$

2.2. Eigenvalues problem

Due to the Hardy inequality, the operator

$$L_{\widetilde{\mu}} u = -\Delta u - \sum_{i=1}^{k} \frac{\mu_i}{|x - a_i|^2} u, \quad \text{with } \widetilde{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$$

is positive definite on H. Moreover the following eigenvalues problem with Hardy potentials and singular coefficient, for j fixed in $\{1, 2, \dots, k\}$,

$$(\mathcal{E}_j) \qquad \begin{cases} -\Delta u - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u = \lambda \frac{u}{|x-a_j|^{2-\alpha_j}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $0 < \alpha_j < 2$, $\lambda \in \mathbb{R}$, has a sequence of eigenvalues $\left\{\lambda_{\widetilde{\mu}}^k(|x - a_j|^{\alpha_j - 2})\right\}$ for $k \in \mathbb{N}^*$ such that

$$0 < \lambda_{\widetilde{\mu}}^1(|x - a_j|^{\alpha_j - 2}) < \lambda_{\widetilde{\mu}}^2(|x - a_j|^{\alpha_j - 2}) \leq \dots \leq \lambda_{\widetilde{\mu}}^k(|x - a_j|^{\alpha_j - 2}) \dots \to \infty \quad \text{as } k \to \infty.$$

The first eigenvalue of $(\mathcal{E}_j)$ is simple, positive and given by

$$\lambda_j^1 := \lambda_{\widetilde{\mu}}^1(|x - a_j|^{\alpha_j - 2}) = \inf_{u \in H \setminus \{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u^2\right)}{\int_\Omega \frac{u^2}{|x-a_j|^{2-\alpha_j}}}.$$

Let

$$\lambda^1 := \min_{i=1,2,\dots,k} \left\{\lambda_i^1\right\}. \tag{2.1}$$

2.3. Some lemmas

Lemma 2.2. *Define*

$$M := \inf \left\{ (T(u))^{1/2} : u \in H \text{ and } |u|_{2^*} = 1 \right\}$$

then M is positive.

Proof. We know that

$$\lambda_i^1 \int_\Omega \frac{u^2}{|x-a_i|^{2-\alpha_i}} \le \int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2} u^2 \right),$$

we deduce that

$$T(u) \ge \left(1 - \frac{1}{\lambda^1} \sum_{i=1}^k \lambda_i \right) \int_\Omega \left(|\nabla u|^2 - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2} u^2 \right) dx.$$

Thus by Hardy's inequality, we get

$$\int_\Omega |\nabla u|^2 \ge T(u) \ge K \int_\Omega |\nabla u|^2,$$

with

$$K := \left(1 - \frac{1}{\lambda^1} \sum_{i=1}^k \lambda_i \right) \left(1 - \frac{1}{\bar{\mu}} \sum_{i=1}^k \mu_i \right).$$

Then

$$T(u)^{1/2} \ge K^{1/2} S_0 > 0, \quad \text{for all } u \in H \text{ such that } |u|_{2^*} = 1.$$

Here S_0 is the best Sobolev constant for the embedding of H into $L^{2^*}(\Omega)$. □

Let $\delta > 0$ and φ_a be a smooth cut-off function centred at a such that

$$0 \le \varphi_a(x) \le 1, \quad \varphi_a(x) = \begin{cases} 0 & \text{if } |x-a| \ge 2\delta, \\ 1 & \text{if } |x-a| \le \delta, \end{cases} \quad \text{and} \quad |\nabla \varphi_a(x)| \le C.$$

Put $u_{\varepsilon,i} = \varphi_{a_i} U_{\varepsilon,a_i}$ for $i \in \{1, 2, \dots, k\}$.

Proposition 2.3. *Let $\omega \in H$ be a solution of the problem $(\mathcal{P})$. Then, for $\varepsilon > 0$ small enough, we have*

(i) $\displaystyle\int_\Omega \left(|\nabla u_{\varepsilon,i}|^2 - \frac{\mu_i}{|x-a_i|^2} u_{\varepsilon,i}^2 \right) = S_{\mu_i}^{N/2} + \mathcal{O}\left(\varepsilon^{(N-2)/2} \right),$

(ii) $\displaystyle\int_\Omega u_{\varepsilon,i}^{2^*} = S_{\mu_i}^{N/2} - \mathcal{O}\left(\varepsilon^{N/2} \right),$

(iii) $\displaystyle\int_\Omega \omega u_{\varepsilon,i}^{2^*-1} = \mathcal{O}\left(\varepsilon^{(N-2)/4} \right),$

(iv) $\displaystyle\int_\Omega |x-a_i|^{\alpha_i-2} u_{\varepsilon,i}^2 = \mathcal{O}\left(\varepsilon^{\alpha_i\sqrt{\overline{\mu}}/2\sqrt{\overline{\mu}-\mu_i}}\right)$ *when* $0<\alpha_i<2\sqrt{\overline{\mu}-\mu_i}$.

Proof. The proofs of (i), (ii) and (iii), (iv) are similar to the proofs of [8, Lemma 11.1] and [4, Proposition 2.4] respectively. □

Lemma 2.4. *Let $f \neq 0$ satisfying the condition $(\mathcal{F})$ then $\mathcal{N}^0 = \varnothing$.*

Proof. Suppose that $\mathcal{N}^0 \neq \varnothing$. Then for $u \in \mathcal{N}^0$ we have

$$T(u) = (2^*-1)\,|u|_{2^*}^{2^*},$$

thus

$$0 = T(u) - |u|_{2^*}^{2^*} - \int_\Omega fu = (2^*-2)\,|u|_{2^*}^{2^*} - \int_\Omega fu. \tag{2.2}$$

From $(\mathcal{F})$ and (2.2) we obtain

$$\begin{aligned} 0 &< C_N(T(u))^{(N+2)/4} - \int_\Omega fu \\ &= (2^*-2)\,|u|_{2^*}^{2N/(N-2)}\left[\left(\frac{T(u)}{(2^*-1)\,|u|_{2^*}^{2^*}}\right)^{(N+2)/4} - 1\right] = 0, \end{aligned}$$

which yields a contradiction. □

Lemma 2.5. *Suppose that $f \neq 0$ satisfies the condition $(\mathcal{F})$, then for each $u \in \mathcal{N}^+$ (or $u \in \mathcal{N}^-$), there exist $\varepsilon > 0$ and a differentiable function $t\colon B(0,\varepsilon) \subset H \to \mathbb{R}^+$ such that $t(0)=1$, $t(v)(u-v) \in \mathcal{N}^+$ for $\|v\| < \epsilon$ and*

$$\langle t'(0), v\rangle = \frac{\int_\Omega \left\{2\left(\nabla u\nabla v - \sum_{i=1}^k \left(\frac{\mu_i}{|x-a_i|^2} - \frac{\lambda_i}{|x-a_i|^{2-\alpha_i}}\right)uv\right) - 2^*\,|u|^{2^*-2}\,uv - fv\right\}}{T(u) - (2^*-1)\,|u|_{2^*}^{2^*}}.$$

Proof. Define the map $F\colon \mathbb{R}\times H \to \mathbb{R}$, by

$$F(s,v) = sT(u-v) - s^{2^*-1}\,|u-v|_{2^*}^{2^*} - \int_\Omega f(u-v).$$

Since $F(1,0)=0$, $\frac{\partial F}{\partial s}(1,0) = T(u) - (2^*-1)\,|u|_{2^*}^{2^*} \neq 0$ and applying the implicit function theorem at the point $(1,0)$, we get the desired result. □

Define, for $i \in \{1,2,\ldots,k\}$,

$$\beta_i(u) := \frac{\int_\Omega \psi_i(x)\,|\nabla u|^2}{|\nabla u|_2^2}$$

where $\psi_i(x) = \min\{\delta, |x-a_i|\}$ and $\delta > 0$. Take $r_0 = \frac{\delta}{3}$ with $\delta < \frac{1}{4}\min_{i\neq j}|a_i-a_j|$ and let

$$\mathcal{N}_i^+ = \left\{u \in \mathcal{N}^+ : \beta_i(u) \le r_0\right\} \quad\text{and}\quad \mathcal{N}_i^- = \left\{u \in \mathcal{N}^- : \beta_i(u) \le r_0\right\}.$$

Denote

$$m_i^+ := \inf_{u\in\mathcal{N}_i^+} I(u) \quad\text{and}\quad m_i^- := \inf_{u\in\mathcal{N}_i^-} I(u).$$

Lemma 2.6. [3] *Let $\delta > 0$ and r_0 defined as above. If $\beta_i(u) \le r_0$ then*

$$\int_{\Omega} |\nabla u|^2 \ge 3 \int_{\Omega \setminus B_i^{\delta}} |\nabla u|^2 .$$

Lemma 2.7. *Let f satisfy the condition $(\mathcal{F})$. Then for any $u \in H \setminus \{0\}$ there exists a unique $t^+ = t^+(u) > 0$ such that $t^+ u \in \mathcal{N}^-$,*

$$t^+ > \left(\frac{T(u)}{(2^* - 1) |u|_{2^*}^{2^*}} \right)^{(N-2)/4} := t_{\max}(u) = t_{\max}$$

and $I(t^+ u) = \max_{t \ge t_{\max}} I(tu)$.

Moreover, if $\int_{\Omega} fu\, dx > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^- u \in \mathcal{N}^+$, $t^- < t_{\max}$ and $I(t^- u) = \min_{0 \le t \le t_{\max}} I(tu)$.

Proof. The lemma is proved in the same way as in [14]. □

3. Proof of Theorem 1.1

From now we consider j fixed in $\{1, 2, \ldots, k\}$.

3.1. Existence of solutions in $\mathcal{N}^+$

Using Ekeland's Variational Principle we will prove the existence of k solutions in $\mathcal{N}^+$.

Proposition 3.1. *Let f be a bounded measurable function, locally positive in each neighborhood of a_i and satisfying $(\mathcal{F})$. Then $m_i^+ = \inf_{v \in \mathcal{N}_i^+} I(v)$ is achieved at a point $u_i \in \mathcal{N}_i^+$ which is a critical point and even a local minimum for I.*

Proof. We start by showing that I is bounded from below in $\mathcal{N}$. Indeed, using Holder's inequality and the fact that $u \in \mathcal{N}$ we get

$$\begin{aligned} I(u) &= \frac{1}{2} T(u) - \frac{1}{2^*} |u|_{2^*}^{2^*} - \int_{\Omega} fu \\ &\ge \frac{-1}{16NK} \left[(N+2) \|f\|_- \right]^2 \end{aligned}$$

in particular

$$m_j^+ \ge m_0 \ge \frac{-1}{16NK} \left[(N+2) \|f\|_- \right]^2 ,$$

where $m_0 = \inf_{u \in \mathcal{N}} I(u)$.

We claim that $m_j^+ < 0$. In fact, we have for some $0 < \varepsilon < \varepsilon_1$, $\int_{B_j^{\varepsilon}} f u_{\varepsilon,j} > 0$.

Let $0 < t_{\varepsilon,j}^- < t_{\varepsilon,j,\max}$ defined in Lemma 2.7 such that $t_{\varepsilon,j}^- u_{\varepsilon,j} \in \mathcal{N}^+$. Since $\beta_j(t_{\varepsilon,j}^- u_{\varepsilon,j})$ tends to 0 as ε goes to 0, we get for $r_0 > 0$ the existence of ε_2 such that $\beta_j(t_{\varepsilon,j}^- u_{\varepsilon,j}) \le r_0$

for $0 < \varepsilon < \varepsilon_2 < \varepsilon_1$. Then $t^-_{\varepsilon,j} u_{\varepsilon,j} \in \mathcal{N}^+_j$ whence

$$\begin{aligned} I(t^-_{\varepsilon,j} u_{\varepsilon,j}) &= \frac{(t^-_{\varepsilon,j})^2}{2} T(u_{\varepsilon,j}) - \frac{(t^-_{\varepsilon,j})^{2^*}}{2^*} |u_{\varepsilon,j}|^{2^*}_{2^*} - t^-_{\varepsilon,j} \int_\Omega f u_{\varepsilon,j} \\ &= -\frac{(t^-_{\varepsilon,j})^2}{2} T(u_{\varepsilon,j}) + \frac{N+2}{2N} (t^-_{\varepsilon,j})^{2^*} |u_{\varepsilon,j}|^{2^*}_{2^*} \\ &< -\frac{(t^-_{\varepsilon,j})^2}{N} T(u_{\varepsilon,j}) < 0, \end{aligned}$$

this leads to $-\infty < m_0 \le m^+_j < 0$.

Claim 3.2. *$\mathcal{N}^+_j$ is a closed set in $H^1_0(\Omega)$, $j \in \{1, 2, \ldots, k\}$.*

Proof. By Lemma 2.4, we deduce that $\mathcal{N} = \mathcal{N}^- \cup \mathcal{N}^+$ ($\mathcal{N}^\pm$ are closed subsets in $H^1_0(\Omega) \setminus \{0\}$).

We have $\mathcal{N}^+_j = \mathcal{N}^+ \cap \beta_j^{-1}([0, r_0])$. For this, it suffices to prove that β_j is a continuous function on $\mathcal{N}^+$.

Let $(u_n) \subset \mathcal{N}^+$ such that $u_n \to u$ in $H^1_0(\Omega)$ i.e., $\forall \varepsilon > 0$, $\exists N_0(\varepsilon) > 0$, $\forall n \ge N_0$, $|\nabla(u_n - u)|_2 < \varepsilon$,

$$\begin{aligned} |\beta_j(u_n) - \beta_j(u)| \le \frac{1}{|\nabla u_n|_2^2} \int_\Omega \psi_j(x) \left| |\nabla u_n(x)|^2 - |\nabla u(x)|^2 \right| dx \\ + \int_\Omega \psi_j(x) |\nabla u(x)|^2 \left| \frac{1}{|\nabla u_n|_2^2} - \frac{1}{|\nabla u|_2^2} \right| dx. \end{aligned}$$

Using the Hölder inequality, we obtain

$$|\beta_j(u_n) - \beta_j(u)| \le 4 \frac{\delta \varepsilon}{|\nabla u|_2}. \qquad \square$$

Ekeland's Variational Principle gives us a minimizing sequence $(u_{j,n})_n \subset \mathcal{N}^+_j$ with the following properties:

(i) $I(u_{j,n}) < m^+_j + \dfrac{1}{n}$

(ii) $I(w) \ge I(u_{j,n}) - \dfrac{1}{n} |\nabla(w - u_{j,n})|_2$, for all $w \in \mathcal{N}^+_j$.

By taking n large, we have for some $\varepsilon \in (0, \varepsilon_2)$

$$I(u_{j,n}) = \frac{1}{N} T(u_{j,n}) - \frac{N+2}{2N} \int_\Omega f u_{j,n} < m^+_j + \frac{1}{n} \le -\frac{(t^-_{\varepsilon,j})^2}{N} T(u_{\varepsilon,j}).$$

This implies that

$$\int_\Omega f u_{j,n} \ge \frac{2}{N+2} (t^-_{\varepsilon,j})^2 T(u_{\varepsilon,j}) > 0.$$

Consequently, $u_{j,n} \neq 0$ and we get

$$\frac{2}{N+2}\frac{(t^-_{\varepsilon,j})^2}{\|f\|_-}T(u_{\varepsilon,j}) \leq \|u_{j,n}\| \leq \frac{N+2}{2K}\|f\|_- .$$

Thus there exists a subsequence labeled $(u_{j,n})_n$ such that $u_{j,n} \rightharpoonup u_j$ weakly in H.

Lemma 3.3. *Let f satisfy the condition $(\mathcal{F})$, then $\|I'(u_{j,n})\|$ tends to 0 as n goes to $+\infty$.*

Proof. Assume that $\|I'(u_{j,n})\| > 0$ for n large. By applying Lemma 2.5 with $u = u_{j,n}$ and $w = \delta\frac{I'(u_{j,n})}{\|I'(u_{j,n})\|}$, $\delta > 0$ small, we find $t_n(\delta) := t\left[\delta\frac{I'(u_{j,n})}{\|I'(u_{j,n})\|}\right]$ such that

$$w_\delta = t_n(\delta)\left[u_{j,n} - \delta\frac{I'(u_{j,n})}{\|I'(u_{j,n})\|}\right] \in \mathcal{N}^+.$$

Thus there exists δ_0 such that $w_\delta \in \mathcal{N}^+_j$ for any $0 < \delta < \delta_0$.

From (ii), we have

$$\begin{aligned}\frac{1}{n}\|w_\delta - u_{j,n}\| &\geq I(u_{j,n}) - I(w_\delta)\\ &= (1 - t_{j,n}(\delta))\left\langle I'(w_\delta), u_{j,n}\right\rangle + \delta t_{j,n}(\delta)\left\langle I'(w_\delta), \frac{I'(u_{j,n})}{\|I'(u_{j,n})\|}\right\rangle + o_n(\delta).\end{aligned}$$

Dividing by δ and passing to the limit as δ goes to zero we derive that

$$\frac{1}{n}(1 + \left|t'_{j,n}(0)\right|\|u_{j,n}\|) \geq -t'_{j,n}(0)\left\langle I'(u_{j,n}), u_{j,n}\right\rangle + \left\|I'(u_{j,n})\right\| = \left\|I'(u_{j,n})\right\|,$$

where $t'_{j,n}(0) = \left\langle t'(0), \frac{I'(u_{j,n})}{\|I'(u_{j,n})\|}\right\rangle$. As $(u_{j,n})$ is a bounded sequence, we conclude that

$$\left\|I'(u_{j,n})\right\| \leq \frac{C}{n}\left(1 + \left|t'_{j,n}(0)\right|\right).$$

Claim 3.4. *The sequence $\left(\left|t'_{j,n}(0)\right|\right)_n$ is bounded uniformly on n.*

Proof of Claim 3.2. Indeed, $(u_{j,n})$ is a bounded sequence and $w \in B_\delta$, we have

$$\left|t'_{j,n}(0)\right| \leq \frac{C}{\left|T(u_{j,n}) - (2^*-1)\left|u_{j,n}\right|^{2^*}_{2^*}\right|}.$$

Hence we must prove that $\left|T(u_{j,n}) - (2^*-1)\left|u_{j,n}\right|^{2^*}_{2^*}\right|$ is bounded away from zero. Arguing by contradiction, assume that for a subsequence still called $(u_{j,n})$, we have

$$T(u_{j,n}) - (2^*-1)\left|u_{j,n}\right|^{2^*}_{2^*} = o_n(1). \tag{3.1}$$

From (3.1) we derive that $|u_{j,n}|_{2^*} \geq \gamma$, for a suitable constant γ and the fact that $u_{j,n} \in \mathcal{N}$ also gives

$$\int_\Omega f u_{j,n} = (2^*-2)\left|u_{j,n}\right|^{2^*}_{2^*} + o_n(1).$$

□

This together with (3.1) imply that

$$0<\gamma A_{\widetilde{\lambda},\widetilde{\mu}}(f)\le C_N(T(u_{j,n}))^{(N+2)/4}\,|u_{j,n}|_{2^*}^{-N/2}-\int_\Omega fu_{j,n}=o_n(1),$$

which is absurd. Thus $\|I'(u_{j,n})\|$ tends to 0 as n goes to ∞. □

From the previous lemma we deduce that

$$\big\langle I'(u_j),w\big\rangle=0,\quad \text{for all } w\in H \tag{3.2}$$

i.e., u_j is a weak solution of $(\mathcal{P})$.

In particular $u_j\in\mathcal{N}$, and we have

$$\int_\Omega fu_j=\lim_{n\to+\infty}\int_\Omega fu_{j,n}\ge\frac{2}{N+2}(t^-_{\varepsilon,j})^2T(u_{\varepsilon,j})>0.$$

Thus $u_j\ne 0$. Also, from Lemma 2.6 and (3.2) it follows that necessarily $u_j\in\mathcal{N}^+$.

By the fact that $\beta_j(u_j)=\lim_{n\to\infty}\beta_j(u_{j,n})\le r_0$, then $u_j\in\mathcal{N}_j^+$. Hence

$$m_j^+\le I(u_j)=\frac{1}{N}T(u_j)-\frac{N+2}{2N}\int_\Omega fu_j\le\lim_{n\to\infty}\inf I(u_{j,n})=m_j^+.$$

Then $u_{j,n}\to u_j$ strongly in H and $I(u_j)=m_j^+$. By Lemma 2.6, we deduce the existence of k solutions to the problem $(\mathcal{P})$. □

3.2. Existence of solutions in $\mathcal{N}^-$

In this subsection, we shall find the range of c where I verifies the $(\mathrm{PS})_\mathrm{c}$ condition.

Lemma 3.5. *The functional I satisfies the condition $(\mathrm{PS})_\mathrm{c}$ for all $c<\frac{1}{N}S_{\mu_l}^{N/2}$, where $S_{\mu_l}^{N/2}=\min\Big(S_{\mu_1}^{N/2},S_{\mu_2}^{N/2},\dots,S_{\mu_k}^{N/2}\Big)$.*

Proof. Let (u_n) be a $(\mathrm{PS})_\mathrm{c}$ sequence for I with $c<\frac{1}{N}S_{\mu_l}^{N/2}$. We know that (u_n) is bounded in H, and there exist a subsequence of (u_n) (still denoted by (u_n)) and $u_0\in H$ such that

$$\begin{aligned}
&u_n\rightharpoonup u_0 && \text{weakly in } H,\\
&u_n\rightharpoonup u_0 && \text{weakly in } L^2(\Omega,|x-a_i|^{-2}) \text{ for } 1\le i\le k \text{ and in } L^{2^*}(\Omega),\\
&u_n\to u_0 && \text{strongly in } L^2(\Omega,|x-a_i|^{\alpha_i-2}) \text{ for } 1\le i\le k,\\
&u_n\to u_0 && \text{strongly in } L^s(\Omega) \text{ for all } s,\ 1\le s<2^*.
\end{aligned}$$

By a standard argument, we deduce that u_0 is a solution of problem $(\mathcal{P})$. Thus

$$I'(u_0)=0\quad\text{and}\quad \int_\Omega fu_n=\int_\Omega fu_0+o_n(1).$$

Next we verify that $u_0 \neq 0$. Arguing by contradiction, we assume that $u_0 \equiv 0$. By the Concentration-Compactness Principle [12, 13], there exist a subsequence, still denoted by (u_n), an at most countable set $\Im$ of different $(x_j)_{j\in\Im} \subset \Omega \setminus \bigcup_{j\in\Im\setminus\{1,2,\dots,k\}} \{a_j\}$ and sets of nonnegative numbers μ_{x_j}, ν_{x_j} for $j \in \Im$; $\mu_{a_i}, \gamma_{a_i}, \nu_{a_i}$ for $1 \leq i \leq k$ such that:

$$|\nabla u_n|^2 \rightharpoonup d\mu \geq \sum_{j\in\Im} \mu_{x_j}\delta_{x_j} + \sum_{i=1}^{k} \mu_{a_i}\delta_{a_i}, \quad \frac{|u_n|^2}{|x-a_i|^2} \rightharpoonup d\gamma = \gamma_{a_i}\delta_{a_i}$$

and

$$|u_n|^{2^*} \rightharpoonup d\nu = \sum_{j\in\Im} \nu_{x_j}\delta_{x_j} + \sum_{i=1}^{k} \nu_{a_i}\delta_{a_i}$$

where δ_x is the Dirac mass at x.

By the Sobolev-Hardy inequalities, we get

$$\mu_{a_i} - \mu_i\gamma_{a_i} \geq S_{\mu_i}\nu_{a_i}^{2/2^*}, \quad 1 \leq i \leq k. \tag{3.3}$$

Claim 3.6. *The set $\Im$ is finite and either $\nu_{x_j} = 0$ or $\nu_{x_j} \geq S_0^{N/2}$ for any $j \in \Im$.*

Proof of Claim 3.4. In fact, let $\varepsilon > 0$ be small enough such that $a_i \notin B^{\varepsilon}_{x_j}$ for all $1 \leq j \leq k$ and $B^{\varepsilon}_{x_i} \cap B^{\varepsilon}_{x_j} = \varnothing$ for $i \neq j$, and $i, j \in \Im$.

Let ϕ^j_ε be a smooth cut-off function centred at x_j such that

$$0 \leq \phi^j_\varepsilon \leq 1, \quad \phi^j_\varepsilon = \begin{cases} 1 & \text{if } |x-x_j| < \frac{\varepsilon}{2}, \\ 0 & \text{if } |x-x_j| > \varepsilon, \end{cases} \quad \text{and} \quad \left|\nabla\phi^j_\varepsilon\right| \leq \frac{4}{\varepsilon},$$

then

$$\begin{aligned} \lim_{\varepsilon\to0}\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 \phi^j_\varepsilon &= \lim_{\varepsilon\to0} \int_\Omega \phi^j_\varepsilon \, d\mu \geq \mu_{x_j}, \\ \lim_{\varepsilon\to0}\lim_{n\to\infty} \int_\Omega \frac{|u_n|^2}{|x-a_i|^2}\phi^j_\varepsilon &= \lim_{\varepsilon\to0} \int_\Omega \phi^j_\varepsilon \, d\gamma = 0, \\ \lim_{\varepsilon\to0}\lim_{n\to\infty} \int_\Omega |u_n|^{2^*} \phi^j_\varepsilon &= \lim_{\varepsilon\to0} \int_\Omega \phi^j_\varepsilon \, d\nu = \nu_{x_j}, \\ \lim_{\varepsilon\to0}\lim_{n\to\infty} \int_\Omega u_n\nabla u_n\nabla\phi^j_\varepsilon &= 0, \end{aligned}$$

thus we have

$$0 = \lim_{\varepsilon\to0}\lim_{n\to\infty} \left\langle I'(u_n), u_n\phi^j_\varepsilon \right\rangle \geq \mu_{x_j} - \nu_{x_j}.$$

By the Sobolev inequality, we get

$$S_0\nu_{x_i}^{2/2^*} \leq \mu_{x_j},$$

hence we deduce that

$$\nu_{x_j} = 0 \quad \text{or} \quad \nu_{x_j} \geq S_0^{N/2},$$

which implies that $\Im$ is finite.

Consider the possibility of concentration at points a_i, with $1 \leq i \leq k$. For $\varepsilon > 0$ be small enough such that $x_j \notin B_{a_j}^{\varepsilon}$ for all $j \in \Im$ and $B_{a_i}^{\varepsilon} \cap B_{a_j}^{\varepsilon} = \varnothing$ for $i \neq j$ and $1 \leq i, j \leq k$.

Let ψ_ε^i be a smooth cut-off function centred at x_i such that

$$0 \leq \psi_\varepsilon^i \leq 1, \quad \psi_\varepsilon^i = \begin{cases} 1 & \text{if } |x - x_i| < \frac{\varepsilon}{2}, \\ 0 & \text{if } |x - x_i| > \varepsilon, \end{cases} \quad \text{and} \quad \left|\nabla \psi_\varepsilon^i\right| \leq \frac{4}{\varepsilon},$$

then

$$\begin{aligned}
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_\Omega |\nabla u_n|^2 \psi_\varepsilon^i &= \lim_{\varepsilon \to 0} \int_\Omega \psi_\varepsilon^i \, d\mu \geq \mu_{a_i}, \\
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_\Omega |u_n|^{2^*} \psi_\varepsilon^i &= \lim_{\varepsilon \to 0} \int_\Omega \psi_\varepsilon^i \, d\nu = \nu_{a_i}, \\
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_\Omega \frac{|u_n|^2}{|x - a_i|^2} \psi_\varepsilon^i &= \lim_{\varepsilon \to 0} \int_\Omega \psi_\varepsilon^i \, d\gamma = \gamma_{a_i}, \\
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_\Omega \frac{|u_n|^2}{|x - a_j|^2} \psi_\varepsilon^i &= 0 \quad \text{for } j \neq i, \\
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_\Omega u_n \nabla u_n \nabla \psi_\varepsilon^i &= 0,
\end{aligned}$$

thus we have

$$0 = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left\langle I'(u_n), u_n \psi_\varepsilon^i \right\rangle \geq \mu_{a_i} - \mu_i \gamma_{a_i} - \nu_{a_i}. \tag{3.4}$$

From (3.3) and (3.4) we deduce that

$$S_{\mu_i} \nu_{x_i}^{2/2^*} \leq \nu_{a_i}$$

and then either $\nu_{a_i} = 0$ or $\nu_{a_i} \geq S_{\mu_i}^{N/2}$ for all $1 \leq i \leq k$. □

Consequently, from the above argument and (3.1), we conclude that

$$c = \lim_{n \to \infty} \left(I(u_n) - \frac{1}{2} \left\langle I'(u_n), u_n \right\rangle \right) = \frac{1}{N} \lim_{n \to \infty} \int_\Omega |u_n|^{2^*} = \frac{1}{N} \left(\sum_{j \in \Im} \nu_{x_j} + \sum_{i=1}^{k} \nu_{a_i} \right).$$

If $\nu_{a_i} = \nu_{x_j} = 0$ for all $i \in \{1, 2, \ldots, k\}$, $j \in \Im$, then $c = 0$ which contradicts the assumption that $c > 0$. On the other hand, if there exists an $i \in \{1, 2, \ldots, k\}$ such that $\nu_{a_i} \neq 0$ or there exists an $j \in \Im$ with $\nu_{x_j} \neq 0$ then we infer that

$$c \geq \frac{1}{N} S_{\mu_l}^{N/2} = c^*.$$

Therefore u_0 is a nonzero solution of the problem $(\mathcal{P})$. □

Lemma 3.7. *Under the hypotheses of Lemma 3.3 then for $0 < \alpha_l < \sqrt{\overline{\mu} - \mu_l}$ and all $s > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ we have*

$$\sup_{s>0} I(u_j + su_{\varepsilon,l}) < m_j^+ + \frac{1}{N} S_{\mu_l}^{N/2}.$$

Proof. Since u_j is a solution of $(\mathcal{P})$, then we obtain

$$\begin{aligned}&\int_\Omega \left(\nabla u_j \nabla u_{\varepsilon,l} - \sum_{i=1}^k \frac{\mu_i}{|x-a_i|^2} u_j u_{\varepsilon,l} - \sum_{i=1}^k \frac{\lambda_i}{|x-a_i|^{2-\alpha_i}} u_j u_{\varepsilon,l} \right) \\ &= \int_\Omega \left(|u_j|^{2^*-2} u_j u_{\varepsilon,l} + f u_{\varepsilon,l} \right).\end{aligned}$$

From the estimates given in [2], we have

$$\begin{aligned}|u_j + su_{\varepsilon,l}|_{2^*}^{2^*} &= |u_j|_{2^*}^{2^*} + |su_{\varepsilon,l}|_{2^*}^{2^*} + 2^* s \int_\Omega |u_j|^{2^*-2} u_j u_{\varepsilon,l} \\ &\quad + 2^* s^{2^*-1} \int_\Omega u_{\varepsilon,l}^{2^*-1} u_j + o\left(\varepsilon^{(N-2)/2}\right).\end{aligned}$$

Then we get

$$I(u_j + su_{\varepsilon,l}) = I(u_j) + J(su_{\varepsilon,l}) - s^{2^*-1} \int_\Omega u_{\varepsilon,l}^{2^*-1} u_j \, dx + o(\varepsilon^{(N-2)/2})$$

where

$$J(su_{\varepsilon,l}) = I(su_{\varepsilon,l}) + s \int_\Omega f u_{\varepsilon,l} \, dx.$$

Note that

$$\begin{aligned}\sup_{s>0} J(su_{\varepsilon,l}) &= \sup_{s>0} \left(\frac{s^2}{2} T(u_{\varepsilon,l}) - \frac{s^{2^*}}{2^*} \int_\Omega |u_{\varepsilon,l}|^{2^*} \right) \\ &= \left(\frac{1}{2} - \frac{1}{2^*} \right) (T(u_{\varepsilon,l}))^{2^*/(2^*-2)} \left(\int_\Omega |u_{\varepsilon,l}|^{2^*} \right)^{2/(2^*-2)} \\ &= \frac{1}{N} \left(S_{\mu_l}^{N/2} + \mathcal{O}\left(\varepsilon^{(N-2)/4}\right) - \mathcal{O}\left(\varepsilon^{\alpha_l \sqrt{\overline{\mu}}/2\sqrt{\overline{\mu}-\mu_l}}\right) \right)^{N/2} \\ &\quad \times \left(S_{\mu_l}^{N/2} - \mathcal{O}\left(\varepsilon^{N/2}\right)^{1-(N/2)} \right) \\ &= \frac{1}{N} S_{\mu_l}^{N/2} - \mathcal{O}\left(\varepsilon^{\alpha_l \sqrt{\overline{\mu}}/2\sqrt{\overline{\mu}-\mu_l}}\right).\end{aligned}$$

From Proposition 2.3, we obtain

$$\begin{aligned}\sup_{s>0} I(u_j + su_{\varepsilon,j}) &\le I(u_j) + \sup_{s_\varepsilon>0} J(s_\varepsilon u_{\varepsilon,l}) - s_\varepsilon^{2^*-1} \int_\Omega u_{\varepsilon,l}^{2^*-1} u_j \\ &\le m_j^+ + \mathcal{O}\left(\varepsilon^{(N-2)/4}\right) + \frac{1}{N} S_{\mu_l}^{N/2} \\ &\quad - \mathcal{O}\left(\varepsilon^{\alpha_l \sqrt{\overline{\mu}}/2\sqrt{\overline{\mu}-\mu_l}}\right) - \mathcal{O}\left(\varepsilon^{(N-2)/4}\right) \\ &= m_j^+ + \frac{1}{N} S_{\mu_l}^{N/2} - \mathcal{O}\left(\varepsilon^{\alpha_l \sqrt{\overline{\mu}}/2\sqrt{\overline{\mu}-\mu_l}}\right) \\ &< m_j^+ + \frac{1}{N} S_{\mu_l}^{N/2} \quad \text{since } \alpha_l < \sqrt{\overline{\mu} - \mu_l}.\end{aligned}$$

□

We know by Lemma 2.6 that for $u \in H$ such that $\|u\| = 1$ there exists a unique $t^+(u) > 0$ such that $t^+(u)u \in \mathcal{N}^-$ and $I(t^+(u)u) = \max_{t \geq t_{\max}} I(tu)$. The uniqueness of $t^+(u)$ and its extremal property give that $t^+(u)$ is a continuous function of u. Let

$$U_1 = \{0\} \cup \left\{ \frac{v}{\|v\|} < t^+\left(\frac{v}{\|v\|}\right) \right\} \quad \text{and} \quad U_2 = \left\{ \frac{v}{\|v\|} > t^+\left(\frac{v}{\|v\|}\right) \right\}.$$

We remark that $H \setminus \mathcal{N}^- = U_1 \cup U_2$ and $\mathcal{N}^+ \subset U_1$. In particular $u_j \in U_1$ for all $j \in \{1, 2, \ldots, k\}$. As in [14], for s_l carefully chosen and for any $0 < \varepsilon < \varepsilon_0$, we have $\widehat{u_j} = u_j + s_l u_{\varepsilon,l} \in U_2$. Set

$$\pounds_j = \{h \colon [0,1] \to H \text{ continuous with } h(0) = u_j, h(1) = \widehat{u_j}\}.$$

Let $h \in \pounds_j$ defined by $h(t) = u_j + ts_l u_{\varepsilon,l}$ for $t \in [0,1]$.

We get the following lemma.

Lemma 3.8. *For a suitable choice of $s_l > 0$ and $0 < \varepsilon < \varepsilon_0$ the value*

$$c_j^* = \inf_{h \in \pounds_j} \max_{t \in [0,1]} I(h(t))$$

defines a critical value for I and $c_j^* \geq m_j^-$.

Proof. We have

$$I(h(t)) < m_j^+ + \frac{1}{N} S_{\mu_l}^{N/2}, \quad \text{for } h \in \pounds_j$$

and hence

$$c_j^* < m_j^+ + \frac{1}{N} S_{\mu_l}^{N/2}$$

Also, since the range of any $h \in \pounds_j$ intersects $\mathcal{N}^-$ we obtain:

$$c_j^* \geq m_j^-.$$

Lemma 3.8 results by applying the Mountain Pass Lemma. □

Proposition 3.9. *Suppose that f verifies the condition $(\mathcal{F})$ and $0 < \alpha_l < \sqrt{\overline{\mu} - \mu_l}$ then I has a minimizer $v_j \in \mathcal{N}_j^-$ such that $m_j^- = I(v_j)$. Moreover, v_j is a solution of the problem $(\mathcal{P})$.*

Proof. There exists a minimizing sequence $(v_{j,n}) \subset \mathcal{N}_j^-$ such that $I(v_{j,n}) \to m_j^-$ and $I'(v_{j,n}) \to 0$ in H.

By Lemma 3.8, we have $m_j^- < m_j^+ + \frac{1}{N} S_{\mu_l}^{N/2}$. Using Lemma 3.7, we deduce that $v_{j,n}$ converges strongly to v_j in H. Thus $v_j \in \mathcal{N}_j^-$ ($v_j \in \mathcal{N}^-$, $\mathcal{N}^-$ is closed and $\beta_j(v_j) = \lim_{n\to\infty} \beta_j(v_{j,n}) \leq r_0$) and $m_j^- = I(v_j)$.

Then $I'(v_j) = 0$ and so v_j is a solution of the problem $(\mathcal{P})$ thus we conclude that $(\mathcal{P})$ admits also k solutions in $\mathcal{N}^-$. □

Proof of Theorem 1.1. By Propositions 3.1 and 3.9, we deduce that the problem $(\mathcal{P})$ admits at least $2k$ distinct solutions in H. □

References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Anallysis **14** (1973), 349–381. http://dx.doi.org/10.1016/0022-1236(73)90051-7

[2] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477. http://dx.doi.org/10.1002/cpa.3160360405

[3] J. Chen, *Multiple positive solutions for a semilinear equation with prescribed singularity*, J. Math. Anal. Appl. **305** (2005), no. 1, 140–157. http://dx.doi.org/10.1016/j.jmaa.2004.10.057

[4] J. Chen and E. M. Rocha, *Four solutions of an inhomogeneous elliptic equation with critical exponent and singular term*, Nonlinear Anal. **71** (2009), no. 10, 4739–4750. http://dx.doi.org/10.1016/j.na.2009.03.048

[5] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **17** (1974), 324–353.

[6] V. Felli and S. Terracini, *Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity*, Comm. Partial Differential Equations **31** (2006), no. 1-3, 469–495. http://dx.doi.org/10.1080/03605300500394439

[7] W. M. Frank, D. J. Land and R. M. Spector, *Singular potentials*, Rev. Modern Phys. **43** (1971), no. 1, 36–98. http://dx.doi.org/10.1103/revmodphys.43.36

[8] N. Ghoussoub and C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5703–5743.

[9] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Reprint of the 1952 edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.

[10] E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equations **156** (1999), no. 2, 407–426. http://dx.doi.org/10.1006/jdeq.1998.3589

[11] D. Kang and Y. Deng, *Multiple solutions for inhomogeneous elliptic problems involving critical Sobolev–Hardy exponents*, Nonlinear Anal. **60** (2005), no. 4, 729–753. http://dx.doi.org/10.1016/j.na.2004.09.048

[12] P.-L. Lions, *The concentration-compactness principle in the calculus of variations*, The limit case I, Rev. Mat. Iberoamericana **1** (1985), no. 1, 145–201. http://dx.doi.org/10.4171/rmi/6

[13] ______, *The concentration-compactness principle in the calculus of variations*, The limit case II, Rev. Mat. Iberoamericana, **1** (1985), no. 2, 45–121. http://dx.doi.org/10.4171/rmi/12

[14] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), no. 3, 281–309.

Mohammed Bouchekif and Sofiane Messirdi
Laboratoire Systèmes Dynamiques et Applications, Faculté des Sciences, Université de Tlemcen BP 119 Tlemcen 13000, Algérie
E-mail address: m_bouchekif@yahoo.fr, messirdi.sofiane@hotmail.fr