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Temporally Discrete Three-species Lotka-Volterra Competitive Systems with Time Delays

Qiankun Bian, Weiguo Zhang* and Zhixian Yu

Abstract. In this paper, we investigate the existence of traveling wave solutions for three components temporally discrete reaction-diffusion systems with delays by using the cross iteration method and Schauder's fixed point theorem. The obtained results are well applied to a temporally discrete three-species Lotka-Volterra competitive systems with delays.

1. Introduction

It is well known that the Lotka-Volterra competitive system is one of the typical and important system in mathematical ecology. Particularly, when the objects investigated are two populations with spatial diffusion ability, the model is as follows:

$$\begin{cases} \frac{\partial}{\partial t}u_1(x,t) = d_1\frac{\partial^2}{\partial x^2}u_1(x,t) + r_1u_1(x,t)\left[1 - a_1u_1(x,t) - b_1u_2(x,t)\right], \\ \frac{\partial}{\partial t}u_2(x,t) = d_2\frac{\partial^2}{\partial x^2}u_2(x,t) + r_2u_2(x,t)\left[1 - b_2u_1(x,t) - a_2u_2(x,t)\right]. \end{cases} \tag{1.1}$$

Here $u_1(x,t)$ and $u_2(x,t)$ denote the density of the two competitors at time t and position x, respectively. d_1, d_2 are the diffusion rates for the two competitors and d_i, r_i, a_i, b_i, $(i = 1,2)$ are positive constants. In the study of various kinds of dynamic behaviors of (1.1), traveling wave solutions have been given rise to more and more researchers' attention in the mathematical theory and practical application, see [5–10, 13, 26, 27] and references cited therein.

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*Corresponding author.

Due to the influence of gestation, hatch and mature, the delay is always inevitable. Thus, Li et al. [14] investigated (1.1) with discrete delays, i.e.,

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + r_1 u_1(x,t) \left[1 - a_1 u_1(x, t-\tau_1) - b_1 u_2(x, t-\tau_2)\right], \\ \frac{\partial u_2}{\partial t} = d_2 \frac{\partial^2 u_2}{\partial x^2} + r_2 u_2(x,t) \left[1 - b_2 u_1(x, t-\tau_3) - a_2 u_2(x, t-\tau_4)\right] \end{cases} \tag{1.2}$$

and established the existence of traveling waves connecting a trivial (no species) equilibrium and a positive (two coexisting species) spatially homogeneous equilibrium by using a cross iteration scheme and Schauder's fixed point theorem, (see also [35]). Lv & Wang [19] further obtained the existence of traveling waves for (1.2) connecting two semi-trivial equilibria. Assume that each competitor's growth is governed by a Volterra integrodifferential equation with both instantaneous and delay self-regulatory terms (see Cushing [2, 4]). Then the delays in this type of model formulation are usually called spatiotemporal delays. The existence of travelling waves for two species Lotka-Volterra diffusion-competition systems with spatiotemporal delays is also widely investigated (see, [4, 15, 33, 39]). The problems on traveling waves for other types of evolution systems can be found in [15, 17, 20–22, 28, 30, 31, 34, 36–38].

Recently, Yu and Yuan [34] extended the results in [14] to n components reaction-diffusion systems with discrete delays and then applied to three-species competition models

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + r_1 u_1(x,t) \left[1 - a_1 u_1(x,t) - b_1 u_2(x, t-\tau_1) - c_1 u_3(x, t-\tau_2)\right], \\ \frac{\partial u_2}{\partial t} = d_2 \frac{\partial^2 u_2}{\partial x^2} + r_2 u_2(x,t) \left[1 - a_2 u_1(x, t-\tau_3) - b_2 u_2(x,t) - c_2 u_3(x, t-\tau_4)\right], \\ \frac{\partial u_3}{\partial t} = d_3 \frac{\partial^2 u_3}{\partial x^2} + r_3 u_3(x,t) \left[1 - a_3 u_1(x, t-\tau_5) - b_3 u_2(x, t-\tau_6) - c_3 u_3(x,t)\right], \end{cases} \tag{1.3}$$

and

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + r_1 u_1(x,t) \left[1 - a_1 u_1(x, t-\tau_1) - b_1 u_2(x, t-\tau_2) - c_1 u_3(x, t-\tau_3)\right], \\ \frac{\partial u_2}{\partial t} = d_2 \frac{\partial^2 u_2}{\partial x^2} + r_2 u_2(x,t) \left[1 - a_2 u_1(x, t-\tau_4) - b_2 u_2(x, t-\tau_5) - c_2 u_3(x, t-\tau_6)\right], \\ \frac{\partial u_3}{\partial t} = d_3 \frac{\partial^2 u_3}{\partial x^2} + r_3 u_3(x,t) \left[1 - a_3 u_1(x, t-\tau_7) - b_3 u_2(x, t-\tau_8) - c_3 u_3(x, t-\tau_9)\right], \end{cases} \tag{1.4}$$

where $r_1, r_2, c_1, c_2, a_1, a_2, b_1, b_2$ are positive constants and τ_i, $i = 1, 2, \ldots, 9$ are nonnegative constants. The existence of traveling waves for three-species Lotka-Volterra systems was also investigated in [3, 29].

However, when we described many phenomena in population dynamics [11, 12, 16], physical systems [25] and nervous systems [1,18,23], discrete-time models are more suitable than the continuous ones. By applying nonstandard finite difference schemes and Euler's method, Roger [24] obtained a discrete-time Lotka-Volterra competition model without the spatial diffusion

$$(1.5)\qquad \begin{cases} \frac{u(t+h)-u(t)}{h} = r_1u(t)\left[1-a_1u(t)-b_1v(t)\right], \\ \frac{v(t+h)-v(t)}{h} = r_2v(t)\left[1-a_2v(t)-b_2v(t)\right]. \end{cases}$$

Especially, let $t=n$ and $h=-1$, (1.5) can reduce to the following discrete-time model

$$(1.6)\qquad \begin{cases} u_n - u_{n-1} = r_1u_n(1-a_1u_n-b_1v_n), \\ v_n - v_{n-1} = r_2v_n(1-a_2u_n-b_2v_n). \end{cases}$$

Recently, Xia and Yu [32] applied nonstandard finite difference schemes and Euler's method to the model (1.2) and obtained the following discrete-time models with delays and spatial diffusion

$$(1.7)\qquad \begin{cases} u_n(x)-u_{n-1}(x) = d_1\triangle u_n(x) + r_1u_n(x)\left[1-a_1u_n(x)-b_1v_{n-\tau_1}(x)\right], \\ v_n(x)-v_{n-1}(x) = d_2\triangle v_n(x) + r_2v_n(x)\left[1-b_2u_{n-\tau_2}(x)-a_2v_n(x)\right] \end{cases}$$

and

$$(1.8)\qquad \begin{cases} u_n(x)-u_{n-1}(x) = d_1\triangle u_n(x) + r_1u_n(x)\left[1-a_1u_{n-\tau_1}(x)-b_1v_{n-\tau_2}(x)\right], \\ v_n(x)-v_{n-1}(x) = d_2\triangle v_n(x) + r_2v_n(x)\left[1-b_2u_{n-\tau_3}(x)-a_2v_{n-\tau_4}(x)\right]. \end{cases}$$

As far as the existence of traveling wave solution is concerned, Xia and Yu [32] obtained that the discrete time systems are dynamically consistent with the continuous-time systems.

Motivated by the above works, we apply nonstandard finite difference schemes and Euler's method to the models (1.3)-(1.4) and can obtain the discrete time models with the three species

(1.9)

$$\begin{cases} u_n(x)-u_{n-1}(x) = d_1\triangle u_n(x) + r_1u_n(x)\left[1-a_1u_n(x)-b_1v_{n-\tau_1}(x)-c_1w_{n-\tau_2}(x)\right], \\ v_n(x)-v_{n-1}(x) = d_2\triangle v_n(x) + r_2v_n(x)\left[1-a_2u_{n-\tau_3}(x)-b_2v_n(x)-c_2w_{n-\tau_4}(x)\right], \\ w_n(x)-w_{n-1}(x) = d_3\triangle w_n(x) + r_3w_n(x)\left[1-a_3u_{n-\tau_5}(x)-b_3v_{n-\tau_6}(x)-c_3w_n(x)\right], \end{cases}$$

and

(1.10)

$$\begin{cases} u_n(x)-u_{n-1}(x) = d_1\triangle u_n(x) + r_1u_n(x)\left[1-a_1u_{n-\tau_1}(x)-b_1v_{n-\tau_2}(x)-c_1w_{n-\tau_3}(x)\right], \\ v_n(x)-v_{n-1}(x) = d_2\triangle v_n(x) + r_2v_n(x)\left[1-a_2u_{n-\tau_4}(x)-b_2v_{n-\tau_5}(x)-c_2w_{n-\tau_6}(x)\right], \\ w_n(x)-w_{n-1}(x) = d_3\triangle w_n(x) + r_3w_n(x)\left[1-a_3u_{n-\tau_7}(x)-b_3v_{n-\tau_8}(x)-c_3w_{n-\tau_9}(x)\right], \end{cases}$$

where $u_n(x), v_n(x), w_n(x)$ are the densities of populations of three species at time n and location x. In this paper, our aim is to investigate the existence of traveling waves for (1.9) and (1.10). Thus, in terms of traveling waves, the discrete time systems (1.9) and (1.10) are dynamically consistent with the continuous-time systems (1.3) and (1.4), respectively.

In order to focus on the mathematical ideas and for the sake of simplicity, we consider more general equations with discrete delays, that is,

$$(1.11)\qquad \begin{cases} u_n(x) - u_{n-1}(x) = d_1 \triangle u_n(x) + f_1(u_{n-\tau_1}(x), v_{n-\tau_2}(x), w_{n-\tau_3}(x)), \\ v_n(x) - v_{n-1}(x) = d_2 \triangle v_n(x) + f_2(u_{n-\tau_4}(x), v_{n-\tau_5}(x), w_{n-\tau_6}(x)), \\ w_n(x) - w_{n-1}(x) = d_3 \triangle w_n(x) + f_3(u_{n-\tau_7}(x), v_{n-\tau_8}(x), w_{n-\tau_9}(x)). \end{cases}$$

Here $d_i > 0$, $\tau_i \geq 0$, $f_i \colon \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, $x \in (-\infty, +\infty)$. In addition, we make the following assumptions throughout this paper:

(A1) There exists $\boldsymbol{K} = (k_1, k_2, k_3)$ with $k_i > 0$ $(i = 1, 2, 3)$ such that

$$f_i(0, 0, 0) = f_i(k_1, k_2, k_3) = 0 \quad \text{for} \quad i = 1, 2, 3.$$

(A2) There exist positive constants L_i $(i = 1, 2, 3)$ such that

$$|f_i(\phi_1, \psi_1, \varphi_1) - f_i(\phi_2, \psi_2, \varphi_2)| \leq L_i \|\Phi - \Psi\|,$$

for $\Phi = (\phi_1, \psi_1, \varphi_1)$, $\Psi = (\phi_2, \psi_2, \varphi_2) \in C([-\tau, 0], \mathbb{R}^3)$ with $0 \leq \phi_i(s) \leq M_i$, $0 \leq \psi_i(s) \leq M_i$ and $0 \leq \varphi_i(s) \leq M_i$, $i = 1, 2, 3$, $s \in [-\tau, 0]$. Here $\tau = \max_{1 \leq i \leq 9} \{\tau_i\}$, $M_i \geq k_i$, $i = 1, 2, 3$, and $|\cdot|$ and $\|\cdot\|$ represent the Euclidean norm in $\mathbb{R}^3$ and the supremum norm in $C([-\tau, 0], \mathbb{R}^3)$, respectively.

The organization of this paper is as follows. In the next section, we introduce abstract results and obtain the existence of traveling wave solutions for system (1.11) under the condition of the weak quasi-monotonicity (WQM) and the exponential weak quasi-monotonicity (EWQM) reaction terms, respectively. Section 3 is invoked to derive the existence of travelling waves by constructing a pair of upper and lower solutions for temporally discrete diffusion-competition systems (1.9) and (1.10).

2. Abstract results

A traveling wave solution of (1.11) is a special solution of the form $u_n(x) = \phi(x + cn)$, $v_n(x) = \psi(x + cn)$, $w_n(x) = \varphi(x + cn)$ where $(\phi, \psi, \varphi) \in C^2(\mathbb{R}, \mathbb{R}^3)$ is the profile of the wave that propagates through the one-dimensional spatial domain at a constant speed c. Substituting $u_n(x) = \phi(x + cn)$, $v_n(x) = \psi(x + cn)$, $w_n(x) = \varphi(x + cn)$ into (1.11) and denoting $\phi_t(s) = \phi(t + s)$, $\psi_t(s) = \psi(t + s)$, $\varphi_t(s) = \varphi(t + s)$ and $x + cn$ by t, we obtain the following system:

$$(2.1) \qquad \begin{cases} d_1\phi''(t) - \phi(t) + \phi(t-c) + f_1(\phi(t-c\tau_1), \psi(t-c\tau_2), \varphi(t-c\tau_3)) = 0, \\ d_2\psi''(t) - \psi(t) + \psi(t-c) + f_2(\phi(t-c\tau_4), \psi(t-c\tau_5), \varphi(t-c\tau_6)) = 0, \\ d_3\varphi''(t) - \varphi(t) + \varphi(t-c) + f_3(\phi(t-c\tau_7), \psi(t-c\tau_8), \varphi(t-c\tau_9)) = 0. \end{cases}$$

Then (1.11) has a travelling waves if there exists a solution of the equation (2.1) satisfying asymptotic boundary conditions

$$(2.2) \qquad \lim_{t\to-\infty}(\phi(t), \psi(t), \varphi(t)) = (0,0,0) =: \mathbf{0}, \quad \lim_{t\to+\infty}(\phi(t), \psi(t), \varphi(t)) = (k_1, k_2, k_3).$$

Let

$$C_{[\mathbf{0},\boldsymbol{M}]}(\mathbb{R}, \mathbb{R}^3) = \left\{(\phi, \psi, \varphi) \in C(\mathbb{R}, \mathbb{R}^3), \mathbf{0} \le (\phi(t), \psi(t), \varphi(t)) \le \boldsymbol{M}, t \in \mathbb{R}\right\},$$

where $\boldsymbol{M} = (M_1, M_2, M_3)$ and $\boldsymbol{M} \ge \boldsymbol{K}$.

Define the operator $H = (H_1, H_2, H_3)\colon C_{[\mathbf{0},\boldsymbol{M}]}(\mathbb{R}, \mathbb{R}^3) \to C(\mathbb{R}, \mathbb{R}^3)$ by

$$\begin{cases} H_1(\phi, \psi, \varphi)(t) = (\beta_1 - 1)\phi(t) + \phi(t-c) + f_1(\phi(t-c\tau_1), \psi(t-c\tau_2), \varphi(t-c\tau_3)), \\ H_2(\phi, \psi, \varphi)(t) = (\beta_2 - 1)\psi(t) + \psi(t-c) + f_2(\phi(t-c\tau_4), \psi(t-c\tau_5), \varphi(t-c\tau_6)), \\ H_3(\phi, \psi, \varphi)(t) = (\beta_3 - 1)\varphi(t) + \varphi(t-c) + f_3(\phi(t-c\tau_7), \psi(t-c\tau_8), \varphi(t-c\tau_9)). \end{cases}$$

Then we can rewrite (2.1) by

$$(2.3) \qquad \begin{cases} d_1\phi''(t) - \beta_1\phi(t) + H_1(\phi, \psi, \varphi)(t) = 0, \\ d_2\psi''(t) - \beta_2\psi(t) + H_2(\phi, \psi, \varphi)(t) = 0, \\ d_3\varphi''(t) - \beta_3\varphi(t) + H_3(\phi, \psi, \varphi)(t) = 0. \end{cases}$$

Define the operator $F = (F_1, F_2, F_3)\colon C_{[\mathbf{0},\boldsymbol{M}]}(\mathbb{R}, \mathbb{R}^3) \to C(\mathbb{R}, \mathbb{R}^3)$ by

$$\begin{cases} F_1(\phi, \psi, \varphi)(t) = \frac{1}{d_1(\lambda_2-\lambda_1)}\left[\int_{-\infty}^t e^{\lambda_1(t-s)}H_1(\phi, \psi, \varphi)(s)\,ds + \int_t^{+\infty} e^{\lambda_2(t-s)}H_1(\phi, \psi, \varphi)(s)\,ds\right], \\ F_2(\phi, \psi, \varphi)(t) = \frac{1}{d_2(\lambda_4-\lambda_3)}\left[\int_{-\infty}^t e^{\lambda_3(t-s)}H_2(\phi, \psi, \varphi)(s)\,ds + \int_t^{+\infty} e^{\lambda_4(t-s)}H_2(\phi, \psi, \varphi)(s)\,ds\right], \\ F_3(\phi, \psi, \varphi)(t) = \frac{1}{d_3(\lambda_6-\lambda_5)}\left[\int_{-\infty}^t e^{\lambda_5(t-s)}H_3(\phi, \psi, \varphi)(s)\,ds + \int_t^{+\infty} e^{\lambda_6(t-s)}H_3(\phi, \psi, \varphi)(s)\,ds\right], \end{cases}$$

where

$$\lambda_1 = -\sqrt{\frac{\beta_1}{d_1}},\ \lambda_2 = \sqrt{\frac{\beta_1}{d_1}},\ \lambda_3 = -\sqrt{\frac{\beta_2}{d_2}},\ \lambda_4 = \sqrt{\frac{\beta_2}{d_2}},\ \lambda_5 = -\sqrt{\frac{\beta_3}{d_3}},\ \lambda_6 = \sqrt{\frac{\beta_3}{d_3}}.$$

We can easily check that the operator $F = (F_1, F_2, F_3)$ is well defined for $(\phi, \psi, \varphi) \in C_{[\mathbf{0},\boldsymbol{M}]}(\mathbb{R}, \mathbb{R}^3)$ and

$$\begin{cases} d_1F_1(\phi, \psi, \varphi)''(t) - \beta_1F_1(\phi, \psi, \varphi)(t) + H_1(\phi, \psi, \varphi)(t) = 0, \\ d_2F_2(\phi, \psi, \varphi)''(t) - \beta_2F_2(\phi, \psi, \varphi)(t) + H_2(\phi, \psi, \varphi)(t) = 0, \\ d_3F_3(\phi, \psi, \varphi)''(t) - \beta_3F_3(\phi, \psi, \varphi)(t) + H_3(\phi, \psi, \varphi)(t) = 0. \end{cases}$$

Therefore, a fixed point of F is a solution of (2.3). If this solution further satisfies the boundary condition (2.2), then it is a traveling wave solution of (1.11).

In order to obtain a fixed point of F, we propose two conditions on the reaction terms, which are to be called the weak quasimonotone condition (WQM) and the exponential weak quasimonotone condition (EWQM), respectively:

(WQM) There exist three positive constants β_1, β_2 and β_3 such that

$$\begin{aligned}
&f_1(\phi_1(-c\tau_1),\psi_1(-c\tau_2),\varphi_1(-c\tau_3))-f_1(\phi_2(-c\tau_1),\psi_1(-c\tau_2),\varphi_1(-c\tau_3)\\
&\quad+(\beta_1-1)(\phi_1(0)-\phi_2(0))\geq 0,\\
&f_1(\phi_1(-c\tau_1),\psi_1(-c\tau_2),\varphi_1(-c\tau_3))-f_1(\phi_1(-c\tau_1),\psi_2(-c\tau_2),\varphi_2(-c\tau_3))\leq 0,\\
&f_2(\phi_1(-c\tau_4),\psi_1(-c\tau_5),\varphi_1(-c\tau_6))-f_2(\phi_1(-c\tau_4),\psi_2(-c\tau_5),\varphi_1(-c\tau_6))\\
&\quad+(\beta_2-1)(\psi_1(0)-\psi_2(0))\geq 0,\\
&f_2(\phi_1(-c\tau_4),\psi_1(-c\tau_5),\varphi_1(-c\tau_6))-f_2(\phi_2(-c\tau_4),\psi_1(-c\tau_5),\varphi_2(-c\tau_6))\leq 0,\\
&f_3(\phi_1(-c\tau_7),\psi_1(-c\tau_8),\varphi_1(-c\tau_9))-f_3(\phi_1(-c\tau_7),\psi_1(-c\tau_8),\varphi_2(-c\tau_9))\\
&\quad+(\beta_3-1)(\varphi_1(0)-\varphi_2(0))\geq 0,\\
&f_3(\phi_1(-c\tau_7),\psi_1(-c\tau_8),\varphi_1(-c\tau_9))-f_3(\phi_2(-c\tau_7),\psi_2(-c\tau_8),\varphi_1(-c\tau_9))\leq 0
\end{aligned}$$

for any $\phi_1(s),\phi_2(s),\psi_1(s),\psi_2(s),\varphi_1(s),\varphi_2(s)\in C([-c\tau,0],\mathbb{R})$ with

$$0\leq\phi_2(s)\leq\phi_1(s)\leq M_1,\quad 0\leq\psi_2(s)\leq\psi_1(s)\leq M_2,\quad 0\leq\varphi_2(s)\leq\varphi_1(s)\leq M_3,$$

and $M_i>k_i$ $(i=1,2,3)$.

(EWQM) There exist three positive constants β_1, β_2 and β_3 such that

$$\begin{aligned}
&f_1(\phi_1(-c\tau_1),\psi_1(-c\tau_2),\varphi_1(-c\tau_3))-f_1(\phi_2(-c\tau_1),\psi_1(-c\tau_2),\varphi_1(-c\tau_3)\\
&\quad+(\beta_1-1)(\phi_1(0)-\phi_2(0))\geq 0,\\
&f_1(\phi_1(-c\tau_1),\psi_1(-c\tau_2),\varphi_1(-c\tau_3))-f_1(\phi_1(-c\tau_1),\psi_2(-c\tau_2),\varphi_2(-c\tau_3))\leq 0,\\
&f_2(\phi_1(-c\tau_4),\psi_1(-c\tau_5),\varphi_1(-c\tau_6))-f_2(\phi_1(-c\tau_4),\psi_2(-c\tau_5),\varphi_1(-c\tau_6))\\
&\quad+(\beta_2-1)(\psi_1(0)-\psi_2(0))\geq 0,\\
&f_2(\phi_1(-c\tau_4),\psi_1(-c\tau_5),\varphi_1(-c\tau_6))-f_2(\phi_2(-c\tau_4),\psi_1(-c\tau_5),\varphi_2(-c\tau_6))\leq 0,\\
&f_3(\phi_1(-c\tau_7),\psi_1(-c\tau_8),\varphi_1(-c\tau_9))-f_3(\phi_1(-c\tau_7),\psi_1(-c\tau_8),\varphi_2(-c\tau_9))\\
&\quad+(\beta_3-1)(\varphi_1(0)-\varphi_2(0))\geq 0,\\
&f_3(\phi_1(-c\tau_7),\psi_1(-c\tau_8),\varphi_1(-c\tau_9))-f_3(\phi_2(-c\tau_7),\psi_2(-c\tau_8),\varphi_1(-c\tau_9))\leq 0
\end{aligned}$$

for any $\phi_1(s),\phi_2(s),\psi_1(s),\psi_2(s),\varphi_1(s),\varphi_2(s)\in C([-c\tau,0],\mathbb{R})$ with

(i) $0\leq\phi_2(s)\leq\phi_1(s)\leq M_1$, $0\leq\psi_2(s)\leq\psi_1(s)\leq M_2$, $0\leq\varphi_2(s)\leq\varphi_1(s)\leq M_3$, and $M_i>k_i$ $(i=1,2,3)$.

(ii) $e^{\beta_1 s}[\phi_1(s)-\phi_2(s)]$, $e^{\beta_2 s}[\psi_1(s)-\psi_2(s)]$ and $e^{\beta_3 s}[\varphi_1(s)-\varphi_2(s)]$ are nondecreasing in $s\in[-c\tau,0]$.

In the following, we give the definition of upper and lower solutions of system (2.1) and the exponential decay norm.

Definition 2.1. A pair of continuous functions $\overline{\Phi}=(\overline{\phi},\overline{\psi},\overline{\varphi})$, $\underline{\Phi}=(\underline{\phi},\underline{\psi},\underline{\varphi})$ are called an upper solution and a lower solution of (2.1), respectively, if there exist constants T_i $(i=1,\ldots,m)$, where $T_i<T_j$, if $i<j$ such that $\overline{\Phi}$, $\underline{\Phi}$ are twice continuously differentiable in $T=\mathbb{R}\setminus\{T_i : i=1,\ldots,m\}$ and satisfy

$$\begin{cases} d_1\overline{\phi}''(t)-\overline{\phi}(t)+\overline{\phi}(t-c)+f_1(\overline{\phi}(t-c\tau_1),\underline{\psi}(t-c\tau_2),\underline{\varphi}(t-c\tau_3))\le 0, & t\in T,\\ d_2\overline{\psi}''(t)-\overline{\psi}(t)+\overline{\psi}(t-c)+f_2(\underline{\phi}(t-c\tau_4),\overline{\psi}(t-c\tau_5),\underline{\varphi}(t-c\tau_6))\le 0, & t\in T,\\ d_3\overline{\varphi}''(t)-\overline{\varphi}(t)+\overline{\varphi}(t-c)+f_3(\underline{\phi}(t-c\tau_7),\underline{\psi}(t-c\tau_8),\overline{\varphi}(t-c\tau_9))\le 0, & t\in T \end{cases}$$

and

$$\begin{cases} d_1\underline{\phi}''(t)-\underline{\phi}(t)+\underline{\phi}(t-c)+f_1(\underline{\phi}(t-c\tau_1),\overline{\psi}(t-c\tau_2),\overline{\varphi}(t-c\tau_3))\ge 0, & t\in T,\\ d_2\underline{\psi}''(t)-\underline{\psi}(t)+\underline{\psi}(t-c)+f_2(\overline{\phi}(t-c\tau_4),\underline{\psi}(t-c\tau_5),\overline{\varphi}(t-c\tau_6))\ge 0, & t\in T,\\ d_3\underline{\varphi}''(t)-\underline{\varphi}(t)+\underline{\varphi}(t-c)+f_3(\overline{\phi}(t-c\tau_7),\overline{\psi}(t-c\tau_8),\underline{\varphi}(t-c\tau_9))\ge 0, & t\in T. \end{cases}$$

For $\mu\in(0,\min\{\lambda_2,\lambda_4,\lambda_6\})$, define

$$B_\mu(\mathbb{R},\mathbb{R}^3)=\left\{\Phi\in C(\mathbb{R},\mathbb{R}^3),\sup_{t\in\mathbb{R}}|\Phi(t)|\,e^{-\mu|t|}<\infty\right\}.$$

Obviously, $B_\mu(\mathbb{R},\mathbb{R}^3)$ is a Banach space when it is equipped with the norm $|\cdot|_\mu$ defined by

$$|\Phi|_\mu=\sup_{t\in\mathbb{R}}|\Phi(t)|\,e^{-\mu|t|}\quad\text{for}\quad \Phi\in B_\mu(\mathbb{R},\mathbb{R}^3).$$

2.1. The Weak QM Case (WQM)

When the reaction terms f_1, f_2 and f_3 satisfy (WQM) condition, the existence of travelling wave solution of (2.1) will be obtained.

Assume that there exist a pair of the upper-lower solutions $\overline{\Phi}=(\overline{\phi},\overline{\psi},\overline{\varphi})$ and $\underline{\Phi}=(\underline{\phi},\underline{\psi},\underline{\varphi})$ of system (2.1) satisfying:

(P1) $\mathbf{0}\le(\underline{\phi}(t),\underline{\psi}(t),\underline{\varphi}(t))\le(\overline{\phi}(t),\overline{\psi}(t),\overline{\varphi}(t))\le\boldsymbol{M}$;

(P2) $\lim_{t\to-\infty}(\overline{\phi}(t),\overline{\psi}(t),\overline{\varphi}(t))=\mathbf{0}$, $\lim_{t\to+\infty}(\underline{\phi}(t),\underline{\psi}(t),\underline{\varphi}(t))=\lim_{t\to+\infty}(\overline{\phi}(t),\overline{\psi}(t),\overline{\varphi}(t))$ $=\boldsymbol{K}$.

Define the set

$$\Gamma = \left\{(\phi,\psi,\varphi) \in C_{[\mathbf{0},\boldsymbol{M}]}(\mathbb{R},\mathbb{R}^3) \mid (\underline{\phi},\underline{\psi},\underline{\varphi}) \le (\phi,\psi,\varphi) \le (\overline{\phi},\overline{\psi},\overline{\varphi})\right\}.$$

Obviously, Γ is a nonempty, closed and bounded convex set.

Lemma 2.2. *Suppose that* (A1), (A2) *and (WQM) hold. Then we have*

$$\begin{cases} H_1(\phi_2,\psi_1,\varphi_1)(t) \le H_1(\phi_1,\psi_2,\varphi_2)(t), \\ H_2(\phi_1,\psi_2,\varphi_1)(t) \le H_2(\phi_2,\psi_1,\varphi_2)(t), \\ H_3(\phi_1,\psi_1,\varphi_2)(t) \le H_3(\phi_2,\psi_2,\varphi_1)(t) \end{cases}$$

and

$$\begin{cases} F_1(\phi_2,\psi_1,\varphi_1)(t) \le F_1(\phi_1,\psi_2,\varphi_2)(t), \\ F_2(\phi_1,\psi_2,\varphi_1)(t) \le F_2(\phi_2,\psi_1,\varphi_2)(t), \\ F_3(\phi_1,\psi_1,\varphi_2)(t) \le F_3(\phi_2,\psi_2,\varphi_1)(t) \end{cases}$$

for any $\phi_1,\phi_2,\psi_1,\psi_2,\varphi_1,\varphi_2 \in C([-c\tau,0],\mathbb{R})$ *with*

$$0 \le \phi_2(s) \le \phi_1(s) \le M_1, \quad 0 \le \psi_2(s) \le \psi_1(s) \le M_2, \quad 0 \le \varphi_2(s) \le \varphi_1(s) \le M_3.$$

Lemma 2.3. *Suppose that* (A1), (A2) *and (WQM) hold. If* (2.1) *has a pair of upper and lower solutions* $\overline{\Phi} = (\overline{\phi},\overline{\psi},\overline{\varphi})$, $\underline{\Phi} = (\underline{\phi},\underline{\psi},\underline{\varphi})$, *respectively, satisfying* (P1), (P2) *and*

$$\begin{cases} \overline{\phi}'(t^+) \le \overline{\phi}'(t^-), \quad \overline{\psi}'(t^+) \le \overline{\psi}'(t^-), \quad \overline{\varphi}'(t^+) \le \overline{\varphi}'(t^-), \quad t \in \mathbb{R}, \\ \underline{\phi}'(t^+) \ge \underline{\phi}'(t^-), \quad \underline{\psi}'(t^+) \ge \underline{\psi}'(t^-), \quad \underline{\varphi}'(t^+) \ge \underline{\varphi}'(t^-), \quad t \in \mathbb{R}, \end{cases} \tag{2.4}$$

then

$$\underline{\Phi} \le (F_1(\underline{\phi},\overline{\psi},\overline{\varphi}), F_2(\overline{\phi},\underline{\psi},\overline{\varphi}), F_3(\overline{\phi},\overline{\psi},\underline{\varphi})) \le (F_1(\overline{\phi},\underline{\psi},\underline{\varphi}), F_2(\underline{\phi},\overline{\psi},\underline{\varphi}), F_3(\underline{\phi},\underline{\psi},\overline{\varphi})) \le \overline{\Phi}. \tag{2.5}$$

Moreover, $(F_1(\underline{\phi},\overline{\psi},\overline{\varphi}), F_2(\overline{\phi},\underline{\psi},\overline{\varphi}), F_3(\overline{\phi},\overline{\psi},\underline{\varphi}))$, $(F_1(\overline{\phi},\underline{\psi},\underline{\varphi}), F_2(\underline{\phi},\overline{\psi},\overline{\varphi}), F_3(\underline{\phi},\underline{\psi},\overline{\varphi}))$ *are a pair of lower and upper solutions of* (2.1).

Lemma 2.4. *Suppose that* (A1), (A2) *and (WQM) hold. If* (2.1) *has a pair of upper and lower solutions* $\overline{\Phi} = (\overline{\phi},\overline{\psi},\overline{\varphi})$, $\underline{\Phi} = (\underline{\phi},\underline{\psi},\underline{\varphi})$, *respectively, satisfying* (P1), (P2) *and* (2.4), *then* $F\colon \Gamma \to \Gamma$ *is completely continuous with respect to the decay norm* $|\cdot|_\mu$.

Since the proofs of Lemmas 2.2-2.4 are similar to Lemma 2.1 in [34], Lemma 3.9 in [14] and Lemma 3.4 in [3], respectively, we omit them here.

According to Lemmas 2.2-2.4, we easily obtain that $F(\Gamma) \subset \Gamma$ and F is completely continuous, which is similar to the proof of Lemma 2.2 in [34]. By Schauder's fixed point theorem, we can immediately obtain the following theorem.

Theorem 2.5. *Suppose that* (A1), (A2) *and (WQM) hold. If* (2.1) *has an upper solution* $(\overline{\phi},\overline{\psi},\overline{\varphi})$ *and a lower solution* $(\underline{\phi},\underline{\psi},\underline{\varphi})$ *satisfying* (P1), (P2) *and* (2.4), *then* (2.1) *has a solution satisfying* (2.2), *i.e.,* (1.11) *has a traveling wave solution connecting* $\mathbf{0}$ *and* $\boldsymbol{K}$.

2.2. The exponential WQM Case (EWQM)

We shall relax (WQM) to (EWQM) and employ the same idea to the exponential weak quasi-monotonicity. Suppose that there exist an upper solution $(\overline{\phi}(t),\overline{\psi}(t),\overline{\varphi}(t))$ and a lower solution $(\underline{\phi}(t),\underline{\psi}(t),\underline{\varphi}(t))$ satisfying both (P1), (P2), (2.4) and

(P3) $e^{\beta_1 t}\left[\overline{\phi}(t)-\underline{\phi}(t)\right]$, $e^{\beta_2 t}\left[\overline{\psi}(t)-\underline{\psi}(t)\right]$ and $e^{\beta_3 t}\left[\overline{\varphi}(t)-\underline{\varphi}(t)\right]$ are nondecreasing for $t\in\mathbb{R}$.

Define the set

$$\Gamma^*=\left\{(\phi,\psi,\varphi)\in C_{[0,M]}(\mathbb{R},\mathbb{R}^3):\begin{array}{l}\text{(i) } (\underline{\phi},\underline{\psi},\underline{\varphi})\le(\phi,\psi,\varphi)\le(\overline{\phi},\overline{\psi},\overline{\varphi})\\ \text{(ii) } e^{\beta_1 t}\left[\phi(t)-\underline{\phi}(t)\right],e^{\beta_1 t}\left[\overline{\phi}(t)-\phi(t)\right],\\ e^{\beta_2 t}\left[\psi(t)-\underline{\psi}(t)\right],e^{\beta_2 t}\left[\overline{\psi}(t)-\psi(t)\right],\\ e^{\beta_3 t}\left[\varphi(t)-\underline{\varphi}(t)\right],e^{\beta_3 t}\left[\overline{\varphi}(t)-\varphi(t)\right]\\ \text{are nondecreasing for every} t\in\mathbb{R}\end{array}\right\}.$$

By (P3) and the fact $e^{\beta_1 t}\left[\overline{\phi}(t)-\overline{\phi}(t)\right]=0$, $e^{\beta_2 t}\left[\overline{\psi}(t)-\overline{\psi}(t)\right]=0$, $e^{\beta_3 t}\left[\overline{\varphi}(t)-\overline{\varphi}(t)\right]=0$ we can see that $(\overline{\phi},\overline{\psi},\overline{\varphi})\in\Gamma^*$, namely, Γ^* is non-empty. Furthermore, we can easily learn that Γ^* is a closed, bounded, convex subset of $B_\mu(\mathbb{R},\mathbb{R}^3)$.

Theorem 2.6. *Suppose that* (A1), (A2) *and (EWQM) hold. If* (2.1) *has an upper solution* $(\overline{\phi},\overline{\psi},\overline{\varphi})$ *and a lower solution* $(\underline{\phi},\underline{\psi},\underline{\varphi})\in C_{[0,M]}(\mathbb{R},\mathbb{R}^3)$ *satisfying* (P1)-(P3) *and* (2.4). *Then for* $\min\{\beta_1 d_1,\beta_2 d_2,\beta_3 d_3\}>1$, (2.1) *has a traveling wave solution satisfying* (2.2).

Proof. Suppose that $(\phi,\psi,\varphi)\in\Gamma^*$, it is easy to see that

$$\begin{aligned}\underline{\phi}\le F_1(\underline{\phi},\overline{\psi},\overline{\varphi})\le F_1(\phi,\psi,\varphi)\le F_1(\overline{\phi},\underline{\psi},\underline{\varphi})\le\overline{\phi},\\ \underline{\psi}\le F_2(\overline{\phi},\underline{\psi},\overline{\varphi})\le F_2(\phi,\psi,\varphi)\le F_2(\underline{\phi},\overline{\psi},\underline{\varphi})\le\overline{\psi},\\ \underline{\varphi}\le F_3(\overline{\phi},\overline{\psi},\underline{\varphi})\le F_3(\phi,\psi,\varphi)\le F_3(\underline{\phi},\underline{\psi},\overline{\varphi})\le\overline{\varphi}.\end{aligned}$$

Hence, $(\underline{\phi},\underline{\psi},\underline{\varphi})\le(F_1(\underline{\phi},\overline{\psi},\overline{\varphi}),F_2(\underline{\phi},\overline{\psi},\overline{\varphi}),F_3(\underline{\phi},\overline{\psi},\overline{\varphi}))\le(\overline{\phi},\overline{\psi},\overline{\varphi})$.

Next, we will prove the second condition of Γ^*. Let $F_1(\phi,\psi,\varphi)(t)=\phi_1(t)$, then

$$\begin{aligned}&e^{\beta_1 t}\left[\overline{\phi}(t)-\phi_1(t)\right]\\ &=\frac{e^{\beta_1 t}}{d_1(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{t}e^{\lambda_1(t-s)}+\int_{t}^{+\infty}e^{\lambda_2(t-s)}\right]\\ &\quad\times\left\{\left[\beta_1\overline{\phi}(s)-d_1\overline{\phi}''(s)\right]-\left[\beta_1\phi_1(s)-d_1\phi_1''(s)\right]\right\}ds\\ &=\frac{e^{\beta_1 t}}{d_1(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{t}e^{\lambda_1(t-s)}+\int_{t}^{+\infty}e^{\lambda_2(t-s)}\right]\left\{\beta_1\overline{\phi}(s)-d_1\overline{\phi}''(s)-H_1(\phi,\psi,\varphi)(s)\right\}ds.\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\left\{e^{\beta_1 t}\left[ol\phi(t)-\phi_1(t)\right]\right\}\\
=&\frac{(\beta_1+\lambda_1)e^{\beta_1 t}}{d_1(\lambda_2-\lambda_1)}\int_{-\infty}^{t} e^{\lambda_1(t-s)}\left[\beta_1\overline{\phi}(s)-d_1\overline{\phi}''(s)-H_1(\phi,\psi,\varphi)(s)\right]ds\\
&+\frac{(\beta_1+\lambda_2)e^{\beta_1 t}}{d_1(\lambda_2-\lambda_1)}\int_{t}^{+\infty} e^{\lambda_2(t-s)}\left[\beta_1\overline{\phi}(s)-d_1\overline{\phi}''(s)-H_1(\phi,\psi,\varphi)(s)\right]ds\\
\geq&\frac{(\beta_1+\lambda_1)e^{\beta_1 t}}{d_1(\lambda_2-\lambda_1)}\int_{-\infty}^{t} e^{\lambda_1(t-s)}\left[\beta_1\overline{\phi}(s)-d_1\overline{\phi}''(s)-H_1(\overline{\phi},\underline{\psi},\underline{\varphi})(s)\right]ds\\
&+\frac{(\beta_1+\lambda_2)e^{\beta_1 t}}{d_1(\lambda_2-\lambda_1)}\int_{t}^{+\infty} e^{\lambda_2(t-s)}\left[\beta_1\overline{\phi}(s)-d_1\overline{\phi}''(s)-H_1(\overline{\phi},\underline{\psi},\underline{\varphi})(s)\right]ds.
\end{aligned}$$

Note that $\min\{\beta_1 d_1,\beta_2 d_2,\beta_3 d_3\}>1$ gives $\beta_1+\lambda_1>0$ and $\beta_1+\lambda_2>0$. Therefore, according to the definition of upper solution we obtain

$$\frac{d}{dt}\left\{e^{\beta_1 t}\left[\overline{\phi}(t)-\phi_1(t)\right]\right\}\geq 0.$$

Similarly, we can get that $e^{\beta_2 t}\left[\overline{\psi}(t)-F_2(\phi,\psi,\varphi)(t)\right]$, $e^{\beta_3 t}\left[\overline{\varphi}(t)-F_3(\phi,\psi,\varphi)(t)\right]$, $e^{\beta_1 t}\left[F_1(\phi,\psi,\varphi)(t)-\underline{\phi}(t)\right]$, $e^{\beta_2 t}\left[F_2(\phi,\psi,\varphi)(t)-\underline{\psi}(t)\right]$, $e^{\beta_3 t}\left[F_3(\phi,\psi,\varphi)(t)-\underline{\varphi}(t)\right]$ are non-decreasing in $t\in\mathbb{R}$. Hence, $F(\phi,\psi,\varphi)\in\Gamma^*$, and following the method of Lemmas 4.5 and 4.6 in [14], $F\Gamma^*\subset\Gamma^*$ and the map F is compact with respect to the norm $|\cdot|_\mu$. By Schauder's fixed point theorem and the assumption (P3), there exists $(\phi^*(t),\psi^*(t),\varphi^*(t))$ satisfying the asymptotic boundary condition (2.2). The proof is complete. □

Remark 2.7. The aim of this section is to discuss the existence of traveling wave solution of (1.11) under the condition (EWQM). We can choose $\beta_1>0$, $\beta_2>0$, $\beta_3>0$ large enough such that $\min\{\beta_1 d_1,\beta_2 d_2,\beta_3 d_3\}>1$. Therefore, Theorem 2.6 is now available and provide us a traveling wave solution.

3. Applications

In this section, we apply the results to temporally discrete for three-species competitive systems (1.9) and (1.10).

Suppose that

(3.1)

$$D=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}>0,\ D^{(1)}=\begin{vmatrix}1&b_1&c_1\\1&b_2&c_2\\1&b_3&c_3\end{vmatrix}>0,\ D^{(2)}=\begin{vmatrix}a_1&1&c_1\\a_2&1&c_2\\a_3&1&c_3\end{vmatrix}>0,\ D^{(3)}=\begin{vmatrix}a_1&b_1&1\\a_2&b_2&1\\a_3&b_3&1\end{vmatrix}>0.$$

We are interested in looking for travelling wave solutions of (1.9) and (1.10) connecting $(0,0,0)$ and a positive equilibrium $(k_1,k_2,k_3)>0$, where $k_i=\frac{D^{(i)}}{D}$, $i=1,2,3$ are the roots of the following equations:

$$\begin{cases} a_1k_1 + b_1k_2 + c_1k_3 = 1, \\ a_2k_1 + b_2k_2 + c_2k_3 = 1, \\ a_3k_1 + b_3k_2 + c_3k_3 = 1. \end{cases} \tag{3.2}$$

Example 3.1. Considering the following system with delays:

(3.3)

$$\begin{cases} u_n(x) - u_{n-1}(x) = d_1\triangle u_n(x) + r_1u_n(x)\left[1 - a_1u_n(x) - b_1v_{n-\tau_1}(x) - c_1w_{n-\tau_2}(x)\right], \\ v_n(x) - v_{n-1}(x) = d_2\triangle v_n(x) + r_2v_n(x)\left[1 - a_2u_{n-\tau_3}(x) - b_2v_n(x) - c_2w_{n-\tau_4}(x)\right], \\ w_n(x) - w_{n-1}(x) = d_3\triangle w_n(x) + r_3v_n(x)\left[1 - a_3u_{n-\tau_5}(x) - b_3v_{n-\tau_6}(x) - c_3w_n(x)\right]. \end{cases}$$

Substituting $u_n(x) = \phi(x+cn) = \phi(t)$, $v_n(x) = \psi(x+cn) = \psi(t)$, $w_n(x) = \varphi(x+cn) = \varphi(t)$ into (3.3), we can get the corresponding wave profile equations as follows:

(3.4)

$$\begin{cases} d_1\phi''(t) - \phi(t) + \phi(t-c) + r_1\phi(t)\left[1 - a_1\phi(t) - b_1\psi(t-c\tau_1) - c_1\varphi(t-c\tau_2)\right] = 0, \\ d_2\psi''(t) - \psi(t) + \psi(t-c) + r_2\psi(t)\left[1 - a_2\phi(t-c\tau_3) - b_2\psi(t) - c_2\varphi(t-c\tau_4)\right] = 0, \\ d_3\varphi''(t) - \varphi(t) + \varphi(t-c) + r_3\varphi(t)\left[1 - a_3\phi(t-c\tau_5) - b_3\psi(t-c\tau_6) - c_3\varphi(t)\right] = 0. \end{cases}$$

Define $f(\phi,\psi,\varphi) = (f_1(\phi,\psi,\varphi), f_2(\phi,\psi,\varphi), f_3(\phi,\psi,\varphi))$ by

$$\begin{aligned} f_1(\phi,\psi,\varphi) &= r_1\phi(0)\left[1 - a_1\phi(0) - b_1\psi(-c\tau_1) - c_1\varphi(-c\tau_2)\right], \\ f_2(\phi,\psi,\varphi) &= r_2\psi(0)\left[1 - a_2\phi(-c\tau_3) - b_2\psi(0) - c_2\varphi(-c\tau_4)\right], \\ f_3(\phi,\psi,\varphi) &= r_3\varphi(0)\left[1 - a_3\phi(-c\tau_5) - b_3\psi(-c\tau_6) - c_3\varphi(0)\right]. \end{aligned}$$

Obviously, $f(\phi,\psi,\varphi)$ satisfies (A1) and (A2). Now, let us prove that $f = (f_1, f_2, f_3)$ satisfies (WQM) by the following results.

Lemma 3.2. *For all delays τ_i, $i = 1, 2, \ldots, 6$, the function $f(\phi,\psi,\varphi)$ satisfies (WQM).*

Proof. For $\Phi = (\phi_1,\psi_1,\varphi_1)$, $\Psi = (\phi_2,\psi_2,\varphi_2) \in C([-c\tau,0],\mathbb{R}^3)$ with $0 \le \phi_2(s) \le \phi_1(s) \le M_1$, $0 \le \psi_2(s) \le \psi_1(s) \le M_2$, $0 \le \varphi_2(s) \le \varphi_1(s) \le M_3$, and $M_1 > k_1$, $M_2 > k_2$, $M_3 > k_3$, $\tau = \max\{\tau_i, i = 1, 2, \ldots, 6\}$, we can obtain

$$\begin{aligned} & f_1(\phi_1,\psi_1,\varphi_1) - f_1(\phi_2,\psi_1,\varphi_1) \\ &= r_1\phi_1(0)\left[1 - a_1\phi_1(0) - b_1\psi_1(-c\tau_1) - c_1\varphi_1(-c\tau_2)\right] \\ &\quad - r_1\phi_2(0)\left[1 - a_1\phi_2(0) - b_1\psi_1(-c\tau_1) - c_1\varphi_1(-c\tau_2)\right] \\ &= r_1\left[\phi_1(0) - \phi_2(0)\right] - r_1a_1\left[\phi_1^2(0) - \phi_2^2(0)\right] - r_1b_1\left[\phi_1(0)\psi_1(-c\tau_1) - \phi_2(0)\psi_1(-c\tau_1)\right] \\ &\quad - r_1c_1\left[\phi_1(0)\varphi_1(-c\tau_2) - \phi_2(0)\varphi_1(-c\tau_2)\right] \end{aligned}$$

$$
\begin{aligned}
&= r_1\left[\phi_1(0)-\phi_2(0)\right]\left\{1-a_1\left[\phi_1(0)+\phi_2(0)\right]-b_1\psi_1(-c\tau_1)-c_1\varphi_1(-c\tau_2)\right\}\\
&\geq -r_1(2a_1M_1+b_1M_2+c_1M_3-1)\left[\phi_1(0)-\phi_2(0)\right].
\end{aligned}
$$

Choosing $\beta_1 = r_1(2a_1M_1+b_1M_2+c_1M_3-1)+1 > r_1(2a_1k_1+b_1k_2+c_1k_3-1)+1>0$, we have $f_1(\phi_1,\psi_1,\varphi_1)-f_1(\phi_2,\psi_1,\varphi_1)+(\beta_1-1)(\phi_1(0)-\phi_2(0))\geq 0$. In addition,

$$
\begin{aligned}
&f_1(\phi_1,\psi_1,\varphi_1)-f_1(\phi_1,\psi_2,\varphi_2)\\
&= r_1\phi_1(0)\left[1-a_1\phi_1(0)-b_1\psi_1(-c\tau_1)-c_1\varphi_1(-c\tau_2)\right]\\
&\quad - r_1\phi_1(0)\left[1-a_1\phi_1(0)-b_1\psi_2(-c\tau_1)-c_1\varphi_2(-c\tau_2)\right]\\
&= r_1b_1\phi_1(0)\left[\psi_2(-c\tau_1)-\psi_1(-c\tau_1)\right]+r_1c_1\phi_1(0)\left[\varphi_2(-c\tau_2)-\varphi_1(-c\tau_2)\right]\\
&\leq 0.
\end{aligned}
$$

In a similar way, we can check that f_2, f_3 also satisfy (WQM). The proof is complete. □

In order to apply Theorem 2.5, we need to look for a pair of upper-lower solutions for (3.4). Define

$$
\Delta_{ic}(\lambda) = d_i\lambda^2 + e^{-\lambda c} + r_i - 1, \quad i = 1,2,3.
$$

By simple graphical arguments, we can easily obtain the following lemma.

Lemma 3.3. *Let $0 < r_i < 1$. Then there exists $c^* > 0$ such that for $c > c^*$, $\Delta_{ic}(\lambda)$, respectively, has two positive real roots λ_{2i-1}, λ_{2i} with $\lambda_{2i-1} < \lambda_{2i}$, $i = 1,2,3$. Moreover,*

$$
\Delta_{ic}(\lambda) = \begin{cases} >0 & \text{for } \lambda < \lambda_{2i-1};\\ <0 & \text{for } \lambda \in (\lambda_{2i-1},\lambda_{2i}); \quad i=1,2,3.\\ >0 & \text{for } \lambda > \lambda_{2i}.\end{cases}
$$

Suppose that $c > c^*$ with c^* given by Lemma 3.3. Let

$$
\eta \in \left(1, \min_{i=1,2,3}\left\{2, \frac{\lambda_{2i}}{\lambda_{2i-1}}, \frac{\lambda_1+\lambda_3}{\lambda_1}, \frac{\lambda_1+\lambda_5}{\lambda_1}, \frac{\lambda_3+\lambda_1}{\lambda_3}, \frac{\lambda_3+\lambda_5}{\lambda_3}, \frac{\lambda_5+\lambda_1}{\lambda_5}, \frac{\lambda_5+\lambda_3}{\lambda_5}\right\}\right)
$$

For a large constant $q > 0$, we define three functions $l_i(t) = e^{\lambda_{2i-1}t} - qe^{\eta\lambda_{2i-1}t}$, $i = 1,2,3$. Then, we can easily learn that $l_i(t)$, $i = 1,2,3$ have global maximum $m_i > 0$ respectively, and there exist $t_{2i-1} = \frac{1}{\lambda_{2i-1}(\eta-1)}\ln\frac{1}{q\eta} < 0$, such that $e^{\lambda_{2i-1}t_{2i-1}} - qe^{\eta\lambda_{2i-1}t_{2i-1}} = m_i$, $i = 1,2,3$, which means $l_i(t)$ are nondecreasing for $t \leq t_{2i-1}$, respectively. Therefore, for any given $\lambda > 0$, there exist $\varepsilon_{2i} > 0$ such that $k_i - \varepsilon_{2i}e^{-\lambda t_{2i-1}} = l_{2i-1}(t_{2i-1}) = m_i$, $i = 1,2,3$.

Now, we suppose that there exist $\varepsilon_0 > 0$, $\varepsilon_1 > 0$, $\varepsilon_3 > 0$ and $\varepsilon_5 > 0$ such that

$$\begin{cases} a_1\varepsilon_1 - b_1\varepsilon_4 - c_1\varepsilon_6 > \varepsilon_0, & a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5 > \varepsilon_0, \\ b_2\varepsilon_3 - a_2\varepsilon_2 - c_2\varepsilon_6 > \varepsilon_0, & b_2\varepsilon_4 - a_2\varepsilon_1 - c_2\varepsilon_5 > \varepsilon_0, \\ c_3\varepsilon_5 - a_3\varepsilon_2 - b_3\varepsilon_4 > \varepsilon_0, & c_3\varepsilon_6 - a_3\varepsilon_1 - b_3\varepsilon_3 > \varepsilon_0. \end{cases} \tag{3.5}$$

Letting $q > 0$ sufficiently large and $\lambda > 0$ sufficiently small be given, for the above constants and suitable constants t_2, t_4, t_6, we define the continuous functions as follows:

$$\overline{\phi}(t) = \begin{cases} e^{\lambda_1 t}, & t \le t_2, \\ k_1 + \varepsilon_1 e^{-\lambda t}, & t \ge t_2, \end{cases} \qquad \underline{\phi}(t) = \begin{cases} e^{\lambda_1 t} - q e^{\eta\lambda_1 t}, & t \le t_1, \\ k_1 - \varepsilon_2 e^{-\lambda t}, & t \ge t_1, \end{cases}$$

$$\overline{\psi}(t) = \begin{cases} e^{\lambda_3 t}, & t \le t_4, \\ k_2 + \varepsilon_3 e^{-\lambda t}, & t \ge t_4, \end{cases} \qquad \underline{\psi}(t) = \begin{cases} e^{\lambda_3 t} - q e^{\eta\lambda_3 t}, & t \le t_3, \\ k_2 - \varepsilon_4 e^{-\lambda t}, & t \ge t_3, \end{cases}$$

$$\overline{\varphi}(t) = \begin{cases} e^{\lambda_5 t}, & t \le t_6, \\ k_3 + \varepsilon_5 e^{-\lambda t}, & t \ge t_6, \end{cases} \qquad \underline{\varphi}(t) = \begin{cases} e^{\lambda_5 t} - q e^{\eta\lambda_5 t}, & t \le t_5, \\ k_3 - \varepsilon_6 e^{-\lambda t}, & t \ge t_5. \end{cases}$$

Obviously, $M_1 = \sup_{t\in\mathbb{R}} \overline{\phi}(t) > k_1$, $M_2 = \sup_{t\in\mathbb{R}} \overline{\psi}(t) > k_2$, $M_3 = \sup_{t\in\mathbb{R}} \overline{\varphi}(t) > k_3$, and $\overline{\phi}(t)$, $\overline{\psi}(t)$, $\overline{\varphi}(t)$, $\underline{\phi}(t)$, $\underline{\psi}(t)$, $\underline{\varphi}(t)$ satisfy (P1), (P2), (2.4) and

$$\min\{t_2, t_4, t_6\} - c\max\{1, \tau_i, i = 1, 2, \dots, 6\} \ge \max\{t_1, t_3, t_5\}.$$

In the following, we prove that $(\overline{\phi}(t), \overline{\psi}(t), \overline{\varphi}(t))$, $(\underline{\phi}(t), \underline{\psi}(t), \underline{\varphi}(t))$ are a pair of upper and lower solutions of (3.4), respectively.

Lemma 3.4. *Let $0 < r_i < 1$ $(i = 1, 2, 3)$, and assume that* (3.1) *and* (3.5) *hold. Then $(\overline{\phi}(t), \overline{\psi}(t), \overline{\varphi}(t))$ is an upper solution of* (3.4).

Proof. (i) For $t \le t_2$, $\overline{\phi}(t) = e^{\lambda_1 t}$, $\overline{\phi}(t-c) = e^{\lambda_1(t-c)}$, $\underline{\psi}(t - c\tau_1) \ge 0$ and $\underline{\varphi}(t - c\tau_2) \ge 0$. We have

$$\begin{aligned} & d_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) + r_1\overline{\phi}(t)\left[1 - a_1\overline{\phi}(t) - b_1\underline{\psi}(t - c\tau_1) - c_1\underline{\varphi}(t - c\tau_2)\right] \\ & \le d_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) + r_1\overline{\phi}(t) \\ & = e^{\lambda_1 t}\Delta_{1c}(\lambda_1) = 0. \end{aligned}$$

(ii) For $t_2 < t \le t_2 + c$, $\overline{\phi}(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\overline{\phi}(t-c) = e^{\lambda_1(t-c)}$, $\underline{\psi}(t - c\tau_1) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_1)}$, $\underline{\varphi}(t - c\tau_2) = k_3 - \varepsilon_6 e^{-\lambda(t-c\tau_2)}$. In view of $k_1 + \varepsilon_1 e^{-\lambda t_2} = e^{\lambda_1 t_2}$, we obtain

$$\begin{aligned} & d_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) + r_1\overline{\phi}(t)\left[1 - a_1\overline{\phi}(t) - b_1\underline{\psi}(t - c\tau_1) - c_1\underline{\varphi}(t - c\tau_2)\right] \\ & = d_1\varepsilon_1\lambda^2 e^{-\lambda t} - (k_1 + \varepsilon_1 e^{-\lambda t}) + e^{\lambda_1(t-c)} \end{aligned}$$

$$\begin{aligned}&+r_1(k_1+\varepsilon_1 e^{-\lambda t})\left[1-a_1(k_1+\varepsilon_1 e^{-\lambda t})-b_1(k_2-\varepsilon_4 e^{-\lambda(t-c\tau_1)})-c_1(k_3-\varepsilon_6 e^{-\lambda(t-c\tau_2)})\right]\\ \le\ & d_1\varepsilon_1\lambda^2 e^{-\lambda t}-(k_1+\varepsilon_1 e^{-\lambda t})+e^{\lambda_1 t_2}\\ &+r_1(k_1+\varepsilon_1 e^{-\lambda t})\left[-a_1\varepsilon_1 e^{-\lambda t}+b_1\varepsilon_4 e^{-\lambda(t-c\tau_1)}+c_1\varepsilon_6 e^{-\lambda(t-c\tau_2)}\right]\\ =\ & I_1(\lambda),\end{aligned}$$

where $I_1(\lambda)=d_1\varepsilon_1\lambda^2 e^{-\lambda t}-\varepsilon_1 e^{-\lambda t}+\varepsilon_1 e^{-\lambda t_2}+r_1(k_1+\varepsilon_1 e^{-\lambda t})[-a_1\varepsilon_1 e^{-\lambda t}+b_1\varepsilon_4 e^{-\lambda(t-c\tau_1)}+c_1\varepsilon_6 e^{-\lambda(t-c\tau_2)}]$. It follows $a_1\varepsilon_1-b_1\varepsilon_4-c_1\varepsilon_6>\varepsilon_0$ that $I_1(0)=r_1(k_1+\varepsilon_1)(-a_1\varepsilon_1+b_1\varepsilon_4+c_1\varepsilon_6)<0$. Thus, there exists a $\lambda_1^*>0$ such that $I_1(\lambda)<0$ for $\lambda\in(0,\lambda_1^*)$.

(iii) For $t>t_2+c$, $\overline{\phi}(t)=k_1+\varepsilon_1 e^{-\lambda t}$, $\overline{\phi}(t-c)=k_1+\varepsilon_1 e^{-\lambda(t-c)}$, $\underline{\psi}(t-c\tau_1)=k_2-\varepsilon_4 e^{-\lambda(t-c\tau_1)}$, $\underline{\varphi}(t-c\tau_2)=k_3-\varepsilon_6 e^{-\lambda(t-c\tau_2)}$, we get

$$\begin{aligned}&d_1\overline{\phi}''(t)-\overline{\phi}(t)+\overline{\phi}(t-c)+r_1\overline{\phi}(t)\left[1-a_1\overline{\phi}(t)-b_1\underline{\psi}(t-c\tau_1)-c_1\underline{\varphi}(t-c\tau_2)\right]\\ =\ & e^{-\lambda t}\left[d_1\varepsilon_1\lambda^2-\varepsilon_1+\varepsilon_1 e^{\lambda c}+r_1(k_1+\varepsilon_1 e^{-\lambda t})(b_1\varepsilon_4 e^{\lambda c\tau_1}+c_1\varepsilon_6 e^{\lambda c\tau_2}-a_1\varepsilon_1 e^{-\lambda t})\right]\\ :=\ & I_2(\lambda).\end{aligned}$$

Since $a_1\varepsilon_1-b_1\varepsilon_4-c_1\varepsilon_6>\varepsilon_0$, we can get $I_2(0)=r_1(k_1+\varepsilon_1)(b_1\varepsilon_4+c_1\varepsilon_6-a_1\varepsilon_1)<0$, which implies that there exists a $\lambda_2^*>0$ such that $I_2(\lambda)<0$ for $\lambda\in(0,\lambda_2^*)$.

Taking $\lambda^*=\min\{\lambda_1^*,\lambda_2^*\}$ and for $\lambda\in(0,\lambda^*)$, we get

$$d_1\overline{\phi}''(t)-\overline{\phi}(t)+\overline{\phi}(t-c)+r_1\overline{\phi}(t)\left[1-a_1\overline{\phi}(t)-b_1\underline{\psi}(t-c\tau_1)-c_1\underline{\varphi}(t-c\tau_2)\right]\le 0.$$

In a similar way, we can find a $\lambda^{**}>0$, such that for $\lambda\in(0,\lambda^{**})$,

$$d_2\overline{\psi}''(t)-\overline{\psi}(t)+\overline{\psi}(t-c)+r_2\overline{\psi}(t)\left[1-a_2\underline{\phi}(t-c\tau_3)-b_2\overline{\psi}(t)-c_2\underline{\varphi}(t-c\tau_4)\right]\le 0,$$

and a $\lambda^{***}>0$, such that for $\lambda\in(0,\lambda^{***})$,

$$d_3\overline{\varphi}''(t)-\overline{\varphi}(t)+\overline{\varphi}(t-c)+r_3\overline{\varphi}(t)\left[1-a_3\underline{\phi}(t-c\tau_5)-b_3\overline{\psi}(t-c\tau_6)-c_3\underline{\varphi}(t)\right]\le 0.$$

Therefore, $(\overline{\phi}(t),\overline{\psi}(t),\overline{\varphi}(t))$ is an upper solution of (3.4). □

Lemma 3.5. *Let* $0<r_i<1$ $(i=1,2,3)$*, and assume that* (3.1) *and* (3.5) *hold. Then* $(\underline{\phi}(t),\underline{\psi}(t),\underline{\varphi}(t))$ *is a lower solution of* (3.4).

Proof. (i) For $t\le t_1$, $\underline{\phi}(t)=e^{\lambda_1 t}-qe^{\eta\lambda_1 t}$, $\underline{\phi}(t-c)=e^{\lambda_1(t-c)}-qe^{\eta\lambda_1(t-c)}$, $\overline{\psi}(t-c\tau_1)=e^{\lambda_3(t-c\tau_1)}$, $\overline{\varphi}(t-c\tau_2)=e^{\lambda_5(t-c\tau_2)}$. Since $\Delta_{1c}(\lambda_1)=0$ and $\Delta_{1c}(\eta\lambda_1)<0$, we can choose an enough large number q such that

$$q>-\frac{r_1a_1+r_1b_1e^{-\lambda_3 c\tau_1}+r_1c_1e^{-\lambda_5 c\tau_2}}{\Delta_{1c}(\eta\lambda_1)}.$$

Thus, we obtain

$$\begin{aligned}
&d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t) - b_1\overline{\psi}(t-c\tau_1) - c_1\overline{\varphi}(t-c\tau_2)\right]\\
&= e^{\lambda_1 t}(d_1\lambda_1^2 + e^{-\lambda_1 c} + r_1 - 1) - qe^{\eta\lambda_1 t}\left[d_1(\eta\lambda_1)^2 + e^{-\eta\lambda_1 c} + r_1 - 1\right]\\
&\quad - r_1a_1(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})^2 - r_1b_1(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})e^{\lambda_3(t-c\tau_1)} - r_1c_1(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})e^{\lambda_5(t-c\tau_2)}\\
&= -qe^{\eta\lambda_1 t}\Delta_{1c}(\eta\lambda_1) - r_1a_1(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})^2 - r_1b_1(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})e^{\lambda_3(t-c\tau_1)}\\
&\quad - r_1c_1(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})e^{\lambda_5(t-c\tau_2)}\\
&\geq -qe^{\eta\lambda_1 t}\Delta_{1c}(\eta\lambda_1) - r_1a_1e^{2\lambda_1 t} - r_1b_1e^{\lambda_1 t}e^{\lambda_3(t-c\tau_1)} - r_1c_1e^{\lambda_1 t}e^{\lambda_5(t-c\tau_5)}\\
&= -qe^{\eta\lambda_1 t}\left[\Delta_{1c}(\eta\lambda_1) + \frac{r_1a_1}{q}e^{(2\lambda_1-\eta\lambda_1)t} + \frac{r_1b_1e^{-\lambda_3c\tau_1}}{q}e^{(\lambda_1+\lambda_3-\eta\lambda_1)t}\right.\\
&\qquad\qquad\left. + \frac{r_1c_1e^{-\lambda_5c\tau_2}}{q}e^{(\lambda_1+\lambda_5-\eta\lambda_1)t}\right]\\
&\geq -qe^{\eta\lambda_1 t}\left[\Delta_{1c}(\eta\lambda_1) + \frac{r_1a_1}{q} + \frac{r_1b_1e^{-\lambda_3c\tau_1}}{q} + \frac{r_1c_1e^{-\lambda_5c\tau_2}}{q}\right]\\
&> 0.
\end{aligned}$$

(ii) For $t_1 < t \leq t_1 + c$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t-c) = e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)}$, $\overline{\psi}(t-c\tau_1) = e^{\lambda_3(t-c\tau_1)}$, $\overline{\varphi}(t-c\tau_2) = e^{\lambda_5(t-c\tau_2)}$. Taking account of $e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1} = k_1 - \varepsilon_2 e^{-\lambda t_1}$, $e^{\lambda_3 t_4} = k_2 + \ varepsilon_3 e^{-\lambda t_4}$, $e^{\lambda_5 t_6} = k_3 + \varepsilon_5 e^{-\lambda t_6}$ and $t_1 + c \leq t_4$, $t_1 + c \leq t_6$, by calculation we have

$$\begin{aligned}
&d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t) - b_1\overline{\psi}(t-c\tau_1) - c_1\overline{\varphi}(t-c\tau_2)\right]\\
&= -d_1\varepsilon_2\lambda^2e^{-\lambda t} - (k_1 - \varepsilon_2e^{-\lambda t}) + e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)}\\
&\quad + r_1(k_1 - \varepsilon_2e^{-\lambda t})\left[1 - a_1(k_1 - \varepsilon_2e^{-\lambda t}) - b_1e^{\lambda_3(t-c\tau_1)} - c_1e^{\lambda_5(t-c\tau_2)}\right]\\
&\geq -d_1\varepsilon_2\lambda^2e^{-\lambda t} - \left[k_1 - \varepsilon_2e^{-\lambda(t_1+c)}\right] + \left[e^{\lambda_1(t_1-c)} - e^{\lambda_1 t_1}\right] + (e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1})\\
&\quad + r_1(k_1 - \varepsilon_2e^{-\lambda t})\left[1 - a_1(k_1 - \varepsilon_2e^{-\lambda t}) - b_1e^{\lambda_3 t_4} - c_1e^{\lambda_5 t_6}\right]\\
&= I_3(\lambda),
\end{aligned}$$

where

$$\begin{aligned}
I_3(\lambda) = {} & -d_1\varepsilon_2\lambda^2e^{-\lambda t} + \varepsilon_2e^{-\lambda(t_1+c)} + e^{\lambda_1 t_1}(e^{-\lambda_1 c} - 1) - \varepsilon_2e^{-\lambda t_1}\\
& + r_1(k_1 - \varepsilon_2e^{-\lambda t})(a_1\varepsilon_2e^{-\lambda t} - b_1\varepsilon_3e^{-\lambda t_4} - c_1\varepsilon_5e^{-\lambda t_6}).
\end{aligned}$$

Since $a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5 > \varepsilon_0$, we can easily check that $I_3(0) = e^{\lambda_1 t_1}(e^{-\lambda_1 c} - 1) + r_1(k_1 - \varepsilon_2)(a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5) > 0$ for sufficiently large q. Therefore, there exists a $\lambda_3^* > 0$ such that $I_3(\lambda) > 0$ for $\lambda \in (0, \lambda_3^*)$.

Case 1. $t_6 + c\tau_2 \geq t_4 + c\tau_1$.

(iii) For $t_1+c<t\le t_4+c\tau_1$, $\underline{\phi}(t)=k_1-\varepsilon_2e^{-\lambda t}$, $\underline{\phi}(t-c)=k_1-\varepsilon_2e^{-\lambda(t-c)}$, $\overline{\psi}(t-c\tau_1)=e^{\lambda_3(t-c\tau_1)}$, $\overline{\varphi}(t-c\tau_2)=e^{\lambda_5(t-c\tau_2)}$. Taking $e^{\lambda_3t_4}=k_2+\varepsilon_3e^{-\lambda t_4}$, $e^{\lambda_5t_6}=k_3+\varepsilon_5e^{-\lambda t_6}$ into consideration, we can get

$$\begin{aligned}
&d_1\underline{\phi}''(t)-\underline{\phi}(t)+\underline{\phi}(t-c)+r_1\underline{\phi}(t)\left[1-a_1\underline{\phi}(t)-b_1\overline{\psi}(t-c\tau_1)-c_1\overline{\varphi}(t-c\tau_2)\right]\\
\ge&-d_1\varepsilon_2\lambda^2e^{-\lambda t}+\varepsilon_2e^{-\lambda t}-\varepsilon_2e^{-\lambda(t-c)}\\
&+r_1(k_1-\varepsilon_2e^{-\lambda t})\left[1-a_1(k_1-\varepsilon_2e^{-\lambda t})-b_1e^{\lambda_3t_4}-c_1e^{\lambda_5t_6}\right]\\
=&I_4(\lambda),
\end{aligned}$$

where

$$\begin{aligned}
I_4(\lambda)=&-d_1\varepsilon_2\lambda^2e^{-\lambda t}+\varepsilon_2e^{-\lambda t}-\varepsilon_2e^{-\lambda(t-c)}\\
&+r_1(k_1-\varepsilon_2e^{-\lambda t})(a_1\varepsilon_2e^{-\lambda t}-b_1\varepsilon_3e^{-\lambda t_4}-c_1\varepsilon_5e^{-\lambda t_6}).
\end{aligned}$$

Since $a_1\varepsilon_2-b_1\varepsilon_3-c_1\varepsilon_5>\varepsilon_0$, we learn that there exists a $\lambda_4^*>0$ such that $I_4(\lambda)>0$ for all $\lambda\in(0,\lambda_4^*)$.

(iv) For $t_4+c\tau_1<t\le t_6+c\tau_2$, $\underline{\phi}(t)=k_1-\varepsilon_2e^{-\lambda t}$, $\underline{\phi}(t-c)=k_1-\varepsilon_2e^{-\lambda(t-c)}$, $\overline{\psi}(t-c\tau_1)=k_2+\varepsilon_3e^{-\lambda(t-c\tau_1)}$, $\overline{\varphi}(t-c\tau_2)=e^{\lambda_5(t-c\tau_2)}$. Under the condition $e^{\lambda_5t_6}=k_3+\varepsilon_5e^{-\lambda t_6}$, by calculation we can derive that

$$\begin{aligned}
&d_1\underline{\phi}''(t)-\underline{\phi}(t)+\underline{\phi}(t-c)+r_1\underline{\phi}(t)\left[1-a_1\underline{\phi}(t)-b_1\overline{\psi}(t-c\tau_1)-c_1\overline{\varphi}(t-c\tau_2)\right]\\
\ge&-d_1\varepsilon_2\lambda^2e^{-\lambda t}+\varepsilon_2e^{-\lambda t}-\varepsilon_2e^{-\lambda(t-c)}\\
&+r_1(k_1-\varepsilon_2e^{-\lambda t})\left[1-a_1(k_1-\varepsilon_2e^{-\lambda t})-b_1(k_2+\varepsilon_3e^{-\lambda(t-c\tau_1)})-c_1e^{\lambda_5t_6}\right]\\
=&I_5(\lambda),
\end{aligned}$$

where

$$\begin{aligned}
I_5(\lambda)=&-d_1\varepsilon_2\lambda^2e^{-\lambda t}+\varepsilon_2e^{-\lambda t}-\varepsilon_2e^{-\lambda(t-c)}\\
&+r_1(k_1-\varepsilon_2e^{-\lambda t})(a_1\varepsilon_2e^{-\lambda t}-b_1\varepsilon_3e^{-\lambda(t-c\tau_1)}-c_1\varepsilon_5e^{-\lambda t_6}).
\end{aligned}$$

In view of $a_1\varepsilon_2-b_1\varepsilon_3-c_1\varepsilon_5>\varepsilon_0$, we obtain that there exists a $\lambda_5^*>0$ such that $I_5(\lambda)>0$ for all $\lambda\in(0,\lambda_5^*)$.

(v) For $t>t_6+c\tau_2$, $\underline{\phi}(t)=k_1-\varepsilon_2e^{-\lambda t}$, $\underline{\phi}(t-c)=k_1-\varepsilon_2e^{-\lambda(t-c)}$, $\overline{\psi}(t-c\tau_1)=k_2+\varepsilon_3e^{-\lambda(t-c\tau_1)}$, $\overline{\varphi}(t-c\tau_2)=k_3+\varepsilon_5e^{-\lambda(t-c\tau_2)}$.

$$\begin{aligned}
&d_1\underline{\phi}''(t)-\underline{\phi}(t)+\underline{\phi}(t-c)+r_1\underline{\phi}(t)\left[1-a_1\underline{\phi}(t)-b_1\overline{\psi}(t-c\tau_1)-c_1\overline{\varphi}(t-c\tau_2)\right]\\
=&-d_1\varepsilon_2\lambda^2e^{-\lambda t}+\varepsilon_2e^{-\lambda t}-\varepsilon_2e^{-\lambda(t-c)}\\
&+r_1(k_1-\varepsilon_2e^{-\lambda t})\left[a_1\varepsilon_2e^{-\lambda t}-b_1\varepsilon_3e^{-\lambda(t-c\tau_1)}-c_1\varepsilon_5e^{-\lambda(t-c\tau_2)}\right]\\
:=&I_6(\lambda).
\end{aligned}$$

As the result of $I_6(0) = r_1(k_1 - \varepsilon_2)(a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5)$ and $a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5 > \varepsilon_0$, we can get a $\lambda_6^* > 0$ such that $I_6(\lambda) > 0$ for all $\lambda \in (0, \lambda_6^*)$.

Case 2. $t_6 + c\tau_2 \le t_4 + c\tau_1$.

(iii) For $t_6 + c\tau_2 < t \le t_4 + c\tau_1$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t-c) = k_1 - \varepsilon_2 e^{-\lambda(t-c)}$, $\overline{\psi}(t - c\tau_1) = e^{\lambda_3(t-c\tau_1)}$, $\overline{\varphi}(t - c\tau_2) = k_3 - \varepsilon_5 e^{-\lambda(t-c\tau_2)}$. In view of $e^{\lambda_3 t_4} = k_2 + \varepsilon_3 e^{-\lambda t_4}$, we have

$$\begin{aligned} & d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t) - b_1\overline{\psi}(t-c\tau_1) - c_1\overline{\varphi}(t-c\tau_2)\right] \\ \ge & -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\ & + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1 e^{\lambda_3 t_4} - c_1(k_3 - \varepsilon_5 e^{-\lambda(t-c\tau_2)})\right] \\ = & I_7(\lambda), \end{aligned}$$

where

$$\begin{aligned} I_7(\lambda) = & -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\ & + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left(a_1\varepsilon_2 e^{-\lambda t} - b_1\varepsilon_3 e^{-\lambda t_4} - c_1\varepsilon_5 e^{-\lambda(t-c\tau_2)}\right). \end{aligned}$$

Under the condition $a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5 > \varepsilon_0$, we learn that there exists a $\lambda_7^* > 0$ such that $I_7(\lambda) > 0$ for all $\lambda \in (0, \lambda_7^*)$.

(iv) For the cases $t_1 + c < t \le t_6 + c\tau_2$ and $t > t_4 + c\tau_1$, their proofs are the same to (iii) and (v) of Case 1. We omit it here. Let

$$\lambda^* = \min\{\lambda_3^*, \lambda_4^*, \lambda_5^*, \lambda_6^*, \lambda_7^*\}.$$

We have

$$d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t) - b_1\overline{\psi}(t-c\tau_1)\right] \le 0.$$

Similarly, the rest inequalities are satisfied. The proof is complete. □

Therefore, by Theorem 2.5, we can get the following result.

Theorem 3.6. *Let* $0 < r_i < 1$, $i = 1,2,3$. *Suppose that* (3.1) *and* (3.5) *hold. If* $c > 2\max\left\{\sqrt{d_i r_i}, i = 1,2,3\right\}$, (3.3) *has a traveling wave solution* $(\phi(x+cn), \psi(x+cn), \varphi(x+cn))$ *connecting* $(0,0,0)$ *and* (k_1, k_2, k_3). *Furthermore,*

$$\lim_{\xi\to-\infty}\phi(\xi)e^{-\lambda_1\xi} = \lim_{\xi\to-\infty}\psi(\xi)e^{-\lambda_3\xi} = \lim_{\xi\to-\infty}\varphi(\xi)e^{-\lambda_5\xi} = 1.$$

Example 3.7. In the following, we discuss the following delayed diffusion system:

(3.6)
$$\begin{cases} u_n(x) - u_{n-1}(x) = d_1\triangle u_n(x) + r_1u_n(x)\left[1 - a_1u_{n-\tau_1}(x) - b_1v_{n-\tau_2}(x) - c_1w_{n-\tau_3}(x)\right], \\ v_n(x) - v_{n-1}(x) = d_2\triangle v_n(x) + r_2v_n(x)\left[1 - a_2u_{n-\tau_4}(x) - b_2v_{n-\tau_5}(x) - c_2w_{n-\tau_6}(x)\right], \\ w_n(x) - w_{n-1}(x) = d_3\triangle w_n(x) + r_3w_n(x)\left[1 - a_3u_{n-\tau_7}(x) - b_3v_{n-\tau_8}(x) - c_3w_{n-\tau_9}(x)\right]. \end{cases}$$

It is easy to obtain that the wave system corresponding to (3.6) is

(3.7)
$$\begin{cases} d_1\phi''(t) - \phi(t) + \phi(t-c) + r_1\phi(t)\left[1 - a_1\phi(t-c\tau_1) - b_1\psi(t-c\tau_2) - c_1\varphi(t-c\tau_3)\right] = 0, \\ d_2\psi''(t) - \psi(t) + \psi(t-c) + r_2\psi(t)\left[1 - a_2\phi(t-c\tau_4) - b_2\psi(t-c\tau_5) - c_2\varphi(t-c\tau_6)\right] = 0, \\ d_3\varphi''(t) - \varphi(t) + \varphi(t-c) + r_3\varphi(t)\left[1 - a_3\phi(t-c\tau_7) - b_3\psi(t-c\tau_8) - c_2\varphi(t-c\tau_9)\right] = 0. \end{cases}$$

Define $f(\phi,\psi,\varphi) = (f_1(\phi,\psi,\varphi), f_2(\phi,\psi,\varphi), f_3(\phi,\psi,\varphi))$ by

$$\begin{aligned} f_1(\phi,\psi,\varphi) &= r_1\phi(0)\left[1 - a_1\phi(-c\tau_1) - b_1\psi(-c\tau_2) - c_1\varphi(-c\tau_3)\right], \\ f_2(\phi,\psi,\varphi) &= r_2\psi(0)\left[1 - a_2\phi(-c\tau_4) - b_2\psi(-c\tau_5) - c_2\varphi(-c\tau_6)\right], \\ f_3(\phi,\psi,\varphi) &= r_3\psi(0)\left[1 - a_3\phi(-c\tau_7) - b_3\psi(-c\tau_8) - c_3\varphi(-c\tau_9)\right]. \end{aligned}$$

Obviously, (A1) and (A2) are satisfied.

Lemma 3.8. *Suppose that τ_1, τ_5 and τ_9 are sufficiently small. Then the function $f(\phi,\psi,\varphi)$ satisfies (EWQM).*

Proof. For $\Phi = (\phi_1,\psi_1,\varphi_1)$, $\Psi = (\phi_2,\psi_2,\varphi_2) \in C([-c\tau,0],\mathbb{R}^3)$ with (i) $0 \le \phi_2(s) \le \phi_1(s) \le M_1$, $0 \le \psi_2(s) \le \psi_1(s) \le M_2$, $0 \le \varphi_2(s) \le \varphi_1(s) \le M_3$; (ii) $e^{\beta_1 s}\left[\phi_1(s) - \phi_2(s)\right]$, $e^{\beta_2 s}\left[\psi_1(s) - \psi_2(s)\right]$, $e^{\beta_3 s}\left[\varphi_1(s) - \varphi_2(s)\right]$ are nondecreasing for $s \in \mathbb{R}$.

$$\begin{aligned} & f_1(\phi_1,\psi_1,\varphi_1) - f_1(\phi_2,\psi_1,\varphi_1) \\ &= r_1\phi_1(0)\left[1 - a_1\phi_1(-c\tau_1) - b_1\psi_1(-c\tau_2) - c_1\varphi_1(-c\tau_3)\right] \\ &\quad - r_1\phi_2(0)\left[1 - a_1\phi_2(-c\tau_1) - b_1\psi_1(-c\tau_2) - c_1\varphi_1(-c\tau_3)\right] \\ &= r_1\left[\phi_1(0) - \phi_2(0)\right] - r_1a_1\left[\phi_1(0)\phi_1(-c\tau_1) - \phi_2(0)\phi_2(-c\tau_1)\right] \\ &\quad - r_1b_1\psi_1(-c\tau_2)\left[\phi_1(0) - \phi_2(0)\right] - r_1c_1\varphi_1(-c\tau_3)\left[\phi_1(0) - \phi_2(0)\right] \\ &\ge (r_1 - r_1b_1M_2 - r_1c_1M_3)\left[\phi_1(0) - \phi_2(0)\right] - r_1a_1\phi_1(0)\left[\phi_1(-c\tau_1) - \phi_2(-c\tau_1)\right] \\ &\quad - r_1a_1\phi_2(-c\tau_1)\left[\phi_1(0) - \phi_2(0)\right] \\ &\ge r_1(1 - b_1M_2 - c_1M_3 - a_1M_1)\left[\phi_1(0) - \phi_2(0)\right] \\ &\quad - r_1a_1\phi_1(0)e^{\beta_1c\tau_1}e^{-\beta_1c\tau_1}\left[\phi_1(-c\tau_1) - \phi_2(-c\tau_1)\right] \\ &\ge r_1(1 - b_1M_2 - c_1M_3 - a_1M_1)\left[\phi_1(0) - \phi_2(0)\right] - r_1a_1M_1e^{\beta_1c\tau_1}\left[\phi_1(0) - \phi_2(0)\right] \\ &= r_1(1 - b_1M_2 - c_1M_3 - a_1M_1 - a_1M_1e^{\beta_1c\tau_1})\left[\phi_1(0) - \phi_2(0)\right]. \end{aligned}$$

Choosing $\beta_1 > 0$ such that

(3.8)
$$\beta_1 - 1 > r_1(b_1M_2 + c_1M_3 + 2a_1M_1 - 1),$$

it follows from (3.8) that

$$\beta_1 - 1 \ge r_1(b_1M_2 + c_1M_3 + a_1M_1 - 1) + r_1a_1M_1e^{\beta_1c\tau_1}$$

for sufficiently small $\tau_1 > 0$. Therefore,

$$\begin{aligned}
&f_1(\phi_1,\psi_1,\varphi_1) - f_1(\phi_2,\psi_1,\varphi_1) + (\beta_1 - 1)\,[\phi_1(0) - \phi_2(0)] \\
&\geq \left[r_1(1 - b_1M_2 - c_1M_3 - a_1M_1 - a_1M_1e^{\beta_1 c\tau_1}) + (\beta_1 - 1)\right][\phi_1(0) - \phi_2(0)] \geq 0.
\end{aligned}$$

In addition,

$$\begin{aligned}
&f_1(\phi_1,\psi_1,\varphi_1) - f_1(\phi_1,\psi_2,\varphi_2) \\
&= r_1\phi_1(0)\,[1 - a_1\phi_1(-c\tau_1) - b_1\psi_1(-c\tau_2) - c_1\varphi_1(-c\tau_3)] \\
&\quad - r_1\phi_1(0)\,[1 - a_1\phi_1(-c\tau_1) - b_1\psi_2(-c\tau_2) - c_1\varphi_2(-c\tau_3)] \\
&= -r_1b_1\phi_1(0)\,[\psi_1(-c\tau_2) - \psi_2(-c\tau_2)] - r_1c_1\phi_1(0)\,[\varphi_1(-c\tau_3) - \varphi_2(-c\tau_3)] \leq 0.
\end{aligned}$$

Similarly, for sufficiently small $\tau_5 > 0$ and $\tau_9 > 0$, we can verify that $f_2(\phi,\psi,\varphi)$, $f_3(\phi,\psi,\varphi)$ satisfy (EWQM) by choosing appropriate $\beta_2 > 0$ and $\beta_3 > 0$. The proof is complete. □

Now, as in Example 3.1, we define $(\overline{\phi}(t),\overline{\psi}(t),\overline{\varphi}(t))$ and $(\underline{\phi}(t),\underline{\psi}(t),\underline{\varphi}(t))$, respectively.

Lemma 3.9. *Let* $0 < r_i < 1$, $i = 1,2,3$ *and suppose that* (3.1) *and* (3.5) *hold. If* $\tau_1,\tau_5,\tau_9 > 0$ *are small enough, then* $(\overline{\phi}(t),\overline{\psi}(t),\overline{\varphi}(t))$ *is an upper solution of* (3.7).

Proof. For $\overline{\phi}(t)$, we need to verify that

$$d_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) + r_1\overline{\phi}(t)\left[1 - a_1\overline{\phi}(t-c\tau_1) - b_1\underline{\psi}(t-c\tau_2) - c_1\underline{\psi}(t-c\tau_3)\right] \leq 0. \tag{3.9}$$

Suppose that τ_1 is small enough such that $0 < \tau_1 \ll 1$. We divide the proof into the following steps.

(i) For $t \leq t_2$ and $t > t_2 + c$, the proof of (3.9) is similar to that of Lemma 3.4. We omit it here.

(ii) For $t_2 < t \leq t_2 + c\tau_1$, $\overline{\phi}(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\overline{\phi}(t - c\tau_1) = e^{\lambda_1(t-c\tau_1)}$, $\overline{\phi}(t-c) = e^{\lambda_1(t-c)}$ $\underline{\psi}(t-c\tau_2) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)}$, $\underline{\varphi}(t-c\tau_3) = k_3 - \varepsilon_6 e^{-\lambda(t-c\tau_3)}$. In view of $k_1 + \varepsilon_1 e^{-\lambda t_2} = e^{\lambda_1 t_2}$, therefore,

$$\begin{aligned}
&d_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) + r_1\overline{\phi}(t)\left[1 - a_1\overline{\phi}(t-c\tau_1) - b_1\underline{\psi}(t-c\tau_2) - c_1\underline{\varphi}(t-c\tau_3)\right] \\
&\leq d_1\varepsilon_1\lambda^2e^{-\lambda t} - \left[k_1 + \varepsilon_1 e^{-\lambda(t_2+c\tau_1)}\right] + e^{\lambda_1 t_2} \\
&\quad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})\left\{1 - a_1e^{\lambda_1(t-c\tau_1)} - b_1\left[k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)}\right] - c_1\left[k_3 - \varepsilon_6 e^{-\lambda(t-c\tau_3)}\right]\right\} \\
&= d_1\varepsilon_1\lambda^2e^{-\lambda t} - \varepsilon_1 e^{-\lambda(t_2+c\tau_1)} + \varepsilon_1 e^{-\lambda t_2} \\
&\quad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})\left\{1 - a_1e^{\lambda_1(t-c\tau_1)} - b_1\left[k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)}\right] - c_1\left[k_3 - \varepsilon_6 e^{-\lambda(t-c\tau_3)}\right]\right\} \\
&\to \quad d_1\varepsilon_1\lambda^2e^{-\lambda t} - \varepsilon_1 e^{-\lambda t_2} + \varepsilon_1 e^{-\lambda t_2}
\end{aligned}$$

$$
\begin{aligned}
&\quad + r_1(k_1+\varepsilon_1 e^{-\lambda t_2})\left\{1 - a_1 e^{\lambda_1 t_2} - b_1\left[k_2 - \varepsilon_4 e^{-\lambda(t_2 - c\tau_2)}\right] - c_1\left[k_3 - \varepsilon_6 e^{-\lambda(t-c\tau_3)}\right]\right\}\\
&= d_1\varepsilon_1\lambda^2 e^{-\lambda t} + r_1(k_1+\varepsilon_1 e^{-\lambda t_2})\left[-a_1\varepsilon_1 e^{-\lambda t_2} + b_1\varepsilon_4 e^{-\lambda(t_2-c\tau_2)} + c_1\varepsilon_6 e^{-\lambda(t-c\tau_3)}\right]\\
&:= I_1(\lambda),
\end{aligned}
$$

which implies $I_1(0) = r_1(k_1+\varepsilon_1)(b_1\varepsilon_4 + c_1\varepsilon_6 - a_1\varepsilon_1) < 0$. Hence, for sufficiently small τ_1, there exists a $\lambda_1^* > 0$, such that $I_1(\lambda) < 0$ for $\lambda \in (0, \lambda_1^*)$.

(iii) For $t_2 + c\tau_1 < t \le t_2 + c$, $\overline{\phi}(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\overline{\phi}(t - c\tau_1) = k_1 + \varepsilon_1 e^{-\lambda(t-c\tau_1)}$, $\overline{\phi}(t-c) = e^{\lambda_1(t-c)}$, $\underline{\psi}(t-c\tau_2) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)}$, $\underline{\varphi}(t-c\tau_3) = k_3 - \varepsilon_6 e^{-\lambda(t-c\tau_3)}$. Since $k_1 + \varepsilon_1 e^{-\lambda t_2} = e^{\lambda_1 t_2}$, we have

$$
\begin{aligned}
&\quad d_1\overline{\phi}''(t) - \overline{\phi}(t) + \overline{\phi}(t-c) + r_1\overline{\phi}(t)\left[1 - a_1\overline{\phi}(t-c\tau_1) - b_1\underline{\psi}(t-c\tau_2) - c_1\underline{\varphi}(t-c\tau_3)\right]\\
&= d_1\varepsilon_1\lambda^2 e^{-\lambda t} - (k_1 + \varepsilon_1 e^{-\lambda t}) + e^{\lambda_1(t-c)}\\
&\quad + r_1(k_1+\varepsilon_1 e^{-\lambda t})\\
&\qquad \times\left\{1 - a_1\left[k_1 + \varepsilon_1 e^{-\lambda(t-c\tau_1)}\right] - b_1\left[k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)}\right] - c_1\left[k_3 - \varepsilon_6 e^{-\lambda(t-c\tau_3)}\right]\right\}\\
&\le d_1\varepsilon_1\lambda^2 e^{-\lambda t} - \left[k_1 + \varepsilon_1 e^{-\lambda(t_2+c)}\right] + e^{\lambda_1 t_2}\\
&\quad + r_1(k_1+\varepsilon_1 e^{-\lambda t})\left[-a_1\varepsilon_1 e^{-\lambda(t-c\tau_1)} + b_1\varepsilon_4 e^{-\lambda(t-c\tau_2)} + c_1\varepsilon_6 e^{-\lambda(t-c\tau_3)}\right]\\
&= d_1\varepsilon_1\lambda^2 e^{-\lambda t} - \varepsilon_1 e^{-\lambda(t_2+c)} + \varepsilon_1 e^{-\lambda t_2}\\
&\quad + r_1(k_1+\varepsilon_1 e^{-\lambda t})\left[-a_1\varepsilon_1 e^{-\lambda(t-c\tau_1)} + b_1\varepsilon_4 e^{-\lambda(t-c\tau_2)} + c_1\varepsilon_6 e^{-\lambda(t-c\tau_3)}\right]\\
&:= I_2(\lambda).
\end{aligned}
$$

Just like the case (ii), we can find a $\lambda_2^* > 0$ such that $I_2(\lambda) < 0$ for $\lambda \in (0, \lambda_2^*)$. Thus, (3.9) holds.

Similarly, we obtain

$$
\begin{aligned}
d_2\overline{\psi}''(t) - \overline{\psi}(t) + \overline{\psi}(t-c) + r_2\overline{\psi}(t)\left[1 - a_2\underline{\phi}(t-c\tau_4) - b_2\overline{\psi}(t-c\tau_5) - c_2\overline{\varphi}(t-c\tau_6)\right] \le 0,\\
d_3\overline{\varphi}''(t) - \overline{\varphi}(t) + \overline{\varphi}(t-c) + r_3\overline{\varphi}(t)\left[1 - a_3\underline{\phi}(t-c\tau_7) - b_3\overline{\psi}(t-c\tau_8) - c_3\overline{\varphi}(t-c\tau_9)\right] \le 0.
\end{aligned}
$$

The proof is complete. □

Lemma 3.10. *Let $0 < r_i < 1$, $i = 1, 2, 3$. Suppose that (3.1) and (3.5) hold. If $\tau_1, \tau_5, \tau_9 > 0$ are small enough, then $(\underline{\phi}(t), \underline{\psi}(t), \underline{\varphi}(t))$ is a lower solution of (3.7).*

Proof. For $\underline{\phi}(t)$, we need to prove that

$$
d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t-c\tau_1) - b_1\overline{\psi}(t-c\tau_2)\right] \ge 0. \tag{3.10}
$$

(i) For $t \le t_1$, $\underline{\phi}(t-c\tau_1) \le \underline{\phi}(t)$, the proof is similar to the cases (iv) in Lemma 3.4.

(ii) For $t_1 < t \le t_1 + c\tau_1$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t-c) = e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)}$, $\underline{\phi}(t - c\tau_1) = e^{\lambda_1(t-c\tau_1)} - qe^{\eta\lambda_1(t-c\tau_1)}$, $\overline{\psi}(t-c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, $\overline{\varphi}(t-c\tau_3) = e^{\lambda_5(t-c\tau_3)}$. Since $e^{\lambda_1(t-c\tau_1)} - qe^{\eta\lambda_1(t-c\tau_1)} \le k_1 - \varepsilon_2 e^{-\lambda t}$ and $t_1 + c\tau_1 \le t_4$, $t_1 + c\tau_1 \le t_6$, hence,

$$\begin{aligned}
&d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t-c\tau_1) - b_1\overline{\psi}(t-c\tau_2) - c_1\overline{\varphi}(t-c\tau_3)\right] \\
&\ge -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1 e^{\lambda_3 t_4} - c_1 e^{\lambda_5 t_6}\right].
\end{aligned}$$

Therefore, the proof returns to the case (ii) in Lemma 3.5 and is omitted.

(iii) For $t_1 + c\tau_1 < t \le t_1 + c$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t - c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}$, $\underline{\phi}(t-c) = e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)}$, $\overline{\psi}(t-c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, $\overline{\varphi}(t-c\tau_5) = e^{\lambda_5(t-c\tau_3)}$. Since $e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1} = k_1 - \varepsilon_2 e^{-\lambda t_1}$, $e^{\lambda_3 t_4} = k_2 + \varepsilon_3 e^{-\lambda t_4}$, $e^{\lambda_5 t_6} = k_3 + \varepsilon_5 e^{-\lambda t_6}$, we have

$$\begin{aligned}
&d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t-c\tau_1) - b_1\overline{\psi}(t-c\tau_2) - c_1\overline{\varphi}(t-c\tau_3)\right] \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + e^{\lambda_1(t-c)} - qe^{\eta\lambda_1(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}\right] - b_1 e^{\lambda_3(t-c\tau_2)} - c_1 e^{\lambda_5(t-c\tau_3)}\right\} \\
&\ge -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - k_1 + \varepsilon_2 e^{-\lambda(t_1+c)} + e^{\lambda_1(t_1+c\tau_1-c)} - e^{\lambda_1 t_1} + e^{\lambda_1 t_1} - qe^{\eta\lambda_1 t_1} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t_1+c-c\tau_1)}\right] - b_1 e^{\lambda_3 t_4} - c_1 e^{\lambda_5 t_6}\right\} \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda(t_1+c)} + e^{\lambda_1 t_1}\left[e^{\lambda_1(c\tau_1-c)} - 1\right] - \varepsilon_2 e^{-\lambda t_1} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left[a_1\varepsilon_2 e^{-\lambda(t_1+c-c\tau_1)} - b_1\varepsilon_3 e^{-\lambda t_4} - c_1\varepsilon_5 e^{-\lambda t_6}\right] \\
&:= I_3(\lambda).
\end{aligned}$$

Therefore, $I_3(0) = e^{\lambda_1 t_1}\left[e^{\lambda_1(c\tau_1-c)} - 1\right] + r_1(k_1 - \varepsilon_2)(a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5)$. When $q > 0$ is large enough, $-t_1$ is sufficiently large. Since $a_1\varepsilon_1 - b_1\varepsilon_3 - c_1\varepsilon_5 > \varepsilon_0$, there exists a $\lambda_3^* > 0$ such that $I_3(\lambda) > 0$ for all $\lambda \in (0, \lambda_3^*)$.

Case 1. $t_4 + c\tau_2 < t_6 + c\tau_3$.

(iv) For $t_1 + c < t < t_4 + c\tau_2$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t - c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}$, $\underline{\phi}(t-c) = k_1 - \varepsilon_2 e^{-\lambda(t-c)}$, $\overline{\psi}(t-c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, $\overline{\varphi}(t-c\tau_3) = e^{\lambda_5(t-c\tau_3)}$. According to $e^{\lambda_3 t_4} = k_2 + \varepsilon_3 e^{-\lambda t_4}$, $e^{\lambda_5 t_6} = k_3 + \varepsilon_5 e^{-\lambda t_6}$, we obtain

$$\begin{aligned}
&d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t-c\tau_1) - b_1\overline{\psi}(t-c\tau_2) - c_1\overline{\varphi}(t-c\tau_3)\right] \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + k_1 - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}\right] - b_1 e^{\lambda_3(t-c\tau_2)} - c_1 e^{\lambda_5(t-c\tau_3)}\right\} \\
&\ge -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}\right] - b_1 e^{\lambda_3 t_4} - c_1 e^{\lambda_5 t_6}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left[a_1\varepsilon_2 e^{-\lambda(t-c\tau_1)} - b_1\varepsilon_3 e^{-\lambda t_4} - c_1\varepsilon_5 e^{-\lambda t_6}\right] \\
&:= I_4(\lambda).
\end{aligned}
$$

Hence, $I_4(0) = r_1(k_1 - \varepsilon_2)(a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5)$. Note that $a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5 > \varepsilon_0$, we can get a $\lambda_4^* > 0$ such that $I_4(\lambda) > 0$ for all $\lambda \in (0, \lambda_4^*)$.

(v) For $t_4 + c\tau_2 < t < t_6 + c\tau_3$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t - c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}$, $\underline{\phi}(t - c) = k_1 - \varepsilon_2 e^{-\lambda(t-c)}$, $\overline{\psi}(t - c\tau_2) = k_2 + \varepsilon_3 e^{-\lambda(t-c\tau_2)}$, $\overline{\varphi}(t - c\tau_3) = e^{\lambda_5(t-c\tau_3)}$. Since $e^{\lambda_5 t_6} = k_3 + \varepsilon_5 e^{-\lambda t_6}$, we have

$$
\begin{aligned}
&d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t-c\tau_1) - b_1\overline{\psi}(t-c\tau_2) - c_1\overline{\varphi}(t-c\tau_3)\right] \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + k_1 - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)} - b_1(k_2 + \varepsilon_3 e^{-\lambda(t-c\tau_2)}) - c_1 e^{\lambda_5(t-c\tau_3)}\right]\right\} \\
&\geq -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}\right] - b_1(k_2 + \varepsilon_3 e^{-\lambda(t-c\tau_2)}) - c_1 e^{\lambda_5 t_6}\right\} \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left[a_1\varepsilon_2 e^{-\lambda(t-c\tau_1)} - b_1\varepsilon_3 e^{-\lambda(t-c\tau_2)} - c_1\varepsilon_5 e^{-\lambda t_6}\right] \\
&:= I_5(\lambda).
\end{aligned}
$$

Therefore, $I_5(0) = r_1(k_1-\varepsilon_2)(a_1\varepsilon_2-b_1\varepsilon_3-c_1\varepsilon_5)$. Taking account of $a_1\varepsilon_2-b_1\varepsilon_3-c_1\varepsilon_5 > \varepsilon_0$, we can obtain a $\lambda_5^* > 0$ such that $I_5(\lambda) > 0$ for all $\lambda \in (0, \lambda_5^*)$.

(vi) For $t > t_6 + c\tau_3$, in view of $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t - c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}$, $\underline{\phi}(t - c) = k_1 - \varepsilon_2 e^{-\lambda(t-c)}$, $\overline{\psi}(t - c\tau_2) = k_2 + \varepsilon_3 e^{-\lambda(t-c\tau_2)}$, $\overline{\varphi}(t - c\tau_3) = k_3 + \varepsilon_5 e^{-\lambda(t-c\tau_3)}$, we obtain

$$
\begin{aligned}
&d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t-c\tau_1) - b_1\overline{\psi}(t-c\tau_2) - c_1\overline{\varphi}(t-c\tau_3)\right] \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + k_1 - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t}) \\
&\qquad \times \left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}\right] - b_1(k_2 + \varepsilon_3 e^{-\lambda(t-c\tau_2)}) - c_1(k_3 + \varepsilon_5 e^{-\lambda(t-c\tau_3)})\right\} \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left[a_1\varepsilon_2 e^{-\lambda(t-c\tau_1)} - b_1\varepsilon_3 e^{-\lambda(t-c\tau_2)} - c_1\varepsilon_5 e^{-\lambda(t-c\tau_3)}\right] \\
&:= I_6(\lambda),
\end{aligned}
$$

which implies that $I_6(0) = r_1(k_1-\varepsilon_2)(a_1\varepsilon_2-b_1\varepsilon_3-c_1\varepsilon_5)$. Noting that $a_1\varepsilon_2-b_1\varepsilon_3-c_1\varepsilon_5 > \varepsilon_0$, we can get a $\lambda_6^* > 0$ such that $I_6(\lambda) > 0$ for $\lambda \in (0, \lambda_6^*)$.

Case 2. $t_6 + c\tau_3 < t_4 + c\tau_2$.

(iv) For $t_1 + c < t < t_6 + c\tau_3$, and $t > t_4 + c\tau_2$, the proof is similar to (iv) and (vi) of Case 1.

(v) For $t_6 + c\tau_3 < t < t_4 + c\tau_2$, $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t - c\tau_1) = k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}$, $\underline{\phi}(t - c) = k_1 - \varepsilon_2 e^{-\lambda(t-c)}$, $\overline{\psi}(t - c\tau_2) = e^{\lambda_3(t-c\tau_2)}$, $\overline{\varphi}(t - c\tau_3) = k_3 + \varepsilon_5 e^{-\lambda(t-c\tau_3)}$. Since $e^{\lambda_3 t_4} = k_2 + \varepsilon_3 e^{-\lambda t_4}$, then

$$
\begin{aligned}
& d_1\underline{\phi}''(t) - \underline{\phi}(t) + \underline{\phi}(t-c) + r_1\underline{\phi}(t)\left[1 - a_1\underline{\phi}(t-c\tau_1) - b_1\overline{\psi}(t-c\tau_2) - c_1\overline{\varphi}(t-c\tau_3)\right] \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - (k_1 - \varepsilon_2 e^{-\lambda t}) + k_1 - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}\right] - b_1 e^{\lambda_3(t-c\tau_2)} - c_1(k_3 + \varepsilon_5 e^{-\lambda(t-c\tau_3)})\right\} \\
&\geq -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t}) \\
&\qquad \times \left\{1 - a_1\left[k_1 - \varepsilon_2 e^{-\lambda(t-c\tau_1)}\right] - b_1(k_2 + \varepsilon_3 e^{-\lambda t_4}) - c_1(k_3 + \varepsilon_5 e^{-\lambda(t-c\tau_3)})\right\} \\
&= -d_1\varepsilon_2\lambda^2 e^{-\lambda t} + \varepsilon_2 e^{-\lambda t} - \varepsilon_2 e^{-\lambda(t-c)} \\
&\quad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left[a_1\varepsilon_2 e^{-\lambda(t-c\tau_1)} - b_1\varepsilon_3 e^{-\lambda t_4} - c_1\varepsilon_5 e^{-\lambda(t-c\tau_3)}\right] \\
&:= I_7(\lambda).
\end{aligned}
$$

Thus, $I_7(0) = r_1(k_1 - \varepsilon_2)(a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5)$. In view of $a_1\varepsilon_2 - b_1\varepsilon_3 - c_1\varepsilon_5 > \varepsilon_0$, there exists a $\lambda_7^* > 0$ such that $I_7(\lambda) > 0$ for $\lambda \in (0, \lambda_7^*)$.

According to the above argument, we can prove that (3.10) is satisfied. Similarly, we can obtain

$$
\begin{aligned}
d_2\underline{\psi}''(t) - \underline{\psi}(t) + \underline{\psi}(t-c) + r_1\underline{\psi}(t)\left[1 - a_2\overline{\phi}(t-c\tau_4) - b_2\underline{\psi}(t-c\tau_5) - c_2\underline{\varphi}(t-c\tau_6)\right] \geq 0, \\
d_3\underline{\varphi}''(t) - \underline{\varphi}(t) + \underline{\varphi}(t-c) + r_1\underline{\varphi}(t)\left[1 - a_3\overline{\phi}(t-c\tau_7) - b_3\underline{\psi}(t-c\tau_8) - c_3\underline{\varphi}(t-c\tau_9)\right] \geq 0.
\end{aligned}
$$

The proof is complete. □

Theorem 3.11. *Let* $0 < r_i < 1$, $i = 1, 2, 3$. *Suppose that* (3.1) *and* (3.5) *hold and* $\tau_1, \tau_5, \tau_9 > 0$ *are small enough. Then for every* $c > 2\max\left\{\sqrt{d_i r_i}, i = 1, 2, 3\right\}$, (3.6) *has a travelling wave solution* $(\phi(x + cn), \psi(x + cn), \varphi(x + cn))$ *with the wave speed* c, *which connects* $(0, 0, 0)$ *and* (k_1, k_2, k_3). *Furthermore,*

$$
\lim_{\xi\to-\infty} \phi(\xi)e^{-\lambda_1\xi} = \lim_{\xi\to-\infty} \psi(\xi)e^{-\lambda_3\xi} = \lim_{\xi\to-\infty} \varphi(\xi)e^{-\lambda_5\xi} = 1.
$$

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Qiankun Bian, Weiguo Zhang and Zhixian Yu
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, P. R. China
E-mail address: qkbian@163.com, zwgzwm@126.com, zxyu0902@163.com