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The Long Time Behavior of a Stochastic Logistic Model with Infinite Delay and Impulsive Perturbation

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Abstract. This paper considers a stochastic logistic model with infinite delay and impulsive perturbation. Firstly, with the space C_g as phase space, the definition of solution to a stochastic functional differential equation with infinite delay and impulsive perturbation is established. According to this definition, we show that our model has an unique global positive solution. Then we establish the sufficient and necessary conditions for extinction and stochastic permanence of the model. In addition, the effects of impulsive perturbation and delay on extinction and stochastic permanence are discussed, respectively.

1. Introduction

In recent years, stochastic delay population models have been investigated extensively by many authors. A famous logistic model with delays is expressed by

$$\begin{aligned} dx(t) = x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)\right] dt \\ + \sigma(t)x^{1+\nu}(t)\,dB(t), \end{aligned} \tag{1.1}$$

where $x(t)$ is the population size at time t, τ is a positive constant which represents the time delay, $\mu(\theta)$ is a measure on $(-\infty, 0]$, $dB(t)$ is white noise, $\sigma(t)$ denotes the intensity of the white noise and ν is a constant. There exist a multiple number of works concerned with the properties of model (1.1) or models similar to (1.1), and we cite [3, 4, 12, 14, 18, 19, 27] among many others here.

On the other hand, affected by a variety of factors both naturally and manly, such as earthquake, drought, flooding, fire, crop-dusting, planting, hunting and harvesting, the inner discipline of species or environment often suffers some dispersed changes over a relatively short time interval at the fixed times. In mathematics perspective, such sudden changes could be described by impulses(see e.g., [1, 5, 10, 11, 15–17, 20]). In particular, Liu

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and Wang incorporated the impulsive perturbation into stochastic population model at first time (see e.g., [15–17]), to the best of our knowledge. Motivated by these, we will study the following stochastic logistic system with infinite delay and impulsive perturbation

$$
(1.2)\qquad \begin{cases} dx(t) = x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)\,d\mu(\theta)\right]dt \\ \qquad\qquad + \sigma(t)x^{1+\nu}(t)\,dB(t), \quad t \neq t_k,\ k \in \mathbb{N} \\ x(t_k^+) - x(t_k) = I_k x(t_k), \quad k \in \mathbb{N} \end{cases}
$$

where $B(t)$ is Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, $\mathbb{N}$ denotes the set of positive integers, $0 < t_1 < t_2 < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$. Other parameters are defined and claimed as before.

For model (1.2), some important and significant topics appear spontaneously. Model (1.2) represents a population dynamics, then it is significant and critical to discuss the sufficient and necessary conditions for extinction and stochastic permanence of this model. In addition, what are the impacts of impulsive perturbation and delay on the extinction and stochastic permanence of model (1.2), respectively? The main aim of this paper is to deal with these problems.

For convenience and simplicity, we define the following notations:

$$
f^u = \sup_{t\in\overline{\mathbb{R}}_+} f(t), \qquad f^l = \inf_{t\in\overline{\mathbb{R}}_+} f(t), \qquad \langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)\,ds,
$$

$$
x_* = \liminf_{t\to+\infty} x(t), \qquad x^* = \limsup_{t\to+\infty} x(t), \qquad \mathbb{R}_+ = (0,+\infty).
$$

For model (1.2) we always assume:

A1. From biological meanings, we consider $1 + I_k > 0$, $k \in \mathbb{N}$. When $I_k > 0$, is satisfied, the perturbation turn to be the description process of planting of species and harvesting if not.

A2. $r(t)$, $a(t)$, $b(t)$, $c(t)$ and $\sigma(t)$ are continuous and bounded functions on $\overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}}_+ = [0,+\infty)$, $a^l > 0$ and $\sigma^l > 0$. Moreover, $\nu > 0.75$.

A3. $\boldsymbol{r}$ is a positive constant, μ satisfies that

$$
\mu_{\boldsymbol{r}} = \int_{-\infty}^{0} e^{-2\boldsymbol{r}\theta}\,d\mu(\theta) < +\infty.
$$

Here, $\boldsymbol{r}$ and $r(t)$ are different. Moreover, $\mu(\mathbb{R}^-) = 1$, where $\mathbb{R}^- = (-\infty, 0]$. The assumption **A3** above may be satisfied when $\mu(\theta) = e^{k\boldsymbol{r}\theta}$ $(k > 2)$ for $\theta \leq 0$, so there exist a large number of these probability measures. Throughout this paper, K stands for a generic positive constant whose values may be different at different places.

In sequel, we give the definitions about extinction and stochastic permanence.

Definition 1.1. The population $x(t)$ is said to be extinctive if $\lim_{t\to+\infty} x(t) = 0$. Population size $x(t)$ is said to be stochastic permanence [15] if for arbitrary $\varepsilon > 0$, there are constants $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\liminf_{t\to+\infty} \mathcal{P}\{x(t) \le \alpha_1\} \ge 1-\varepsilon \quad \text{and} \quad \liminf_{t\to+\infty} \mathcal{P}\{x(t) \ge \alpha_2\} \ge 1-\varepsilon.$$

2. Positive and global solutions

Now let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions. Let $W(t)$ denote a m-dimension standard Brownian motion defined on this probability space.

Definition 2.1. Considering the following impulsive stochastic functional differential equation with infinite delay:

$$\begin{cases} dX(t) = F_1(t, X(t-\tau), X_t)\,dt + F_2(t, X(t-\tau), X_t)\,dW(t), & t \ne t_k,\ k \in \mathbb{N} \\ X(t_k^+) - X(t_k) = I_k X(t_k), \quad k \in \mathbb{N} \end{cases} \tag{2.1}$$

Since phase space $BC((-\infty, 0]; \mathbb{R})$ may cause the usual well-posedness questions related to functional equations of unbounded delay (see e.g., [7, 9, 23]), we choose space C_g (see [2, 7, 8]) as phase place, which is defined by

$$C_g = \left\{ \varphi \in C((-\infty, 0]; \mathbb{R}) : \|\varphi\|_{c_g} = \sup_{-\infty<s\le 0} e^{\boldsymbol{r}s} |\varphi(s)| < +\infty \right\},$$

where we choose $g(s) = e^{-\boldsymbol{r}s}$, $\boldsymbol{r} > 0$. Furthermore, C_g is an admissible Banach space (see [8, 9]).

In (2.1), $X_t = \{X(t+\theta) : -\infty < \theta \le 0\}$ can be regarded as C_g-value stochastic process. The initial value $X(t)$, $t \le 0$ is nonrandom and positive, and belongs to the phase space C_g above. An $\mathbb{R}^d$-value stochastic process $X(t)$ defined on $\mathbb{R}$ is called a solution of the equation (2.1) with initial data above, if $X(t)$ has the following properties:

(i) $X(t)$ is $\mathcal{F}_t$-adapted and continuous on $(0, t_1)$ and (t_k, t_{k+1}), $k \in \mathbb{N}$; $t \to F_1(t, X_t) \in \mathcal{L}^1(\overline{\mathbb{R}}_+; \mathbb{R}^d)$ and $t \to F_2(t, X_t) \in \mathcal{L}^2(\overline{\mathbb{R}}_+; \mathbb{R}^{d\times m})$, where $\mathcal{L}^k(\overline{\mathbb{R}}_+, \mathbb{R}^d)$ is all $\mathbb{R}^d$ valued $\mathcal{F}_t$ adapted processes $f(t)$ such that $\int_0^T |f(t)|^k\,dt < \infty$ a.s. (almost surely) for all $T > 0$. $\mathcal{L}^k(\overline{\mathbb{R}}_+, \mathbb{R}^{d\times m})$ is defined similarly.

(ii) for each t_k, $k \in \mathbb{N}$, $X(t_k^+) = \lim_{t\to t_k^+} X(t)$ and $X(t_k^-) = \lim_{t\to t_k^-} X(t)$ exist and $X(t_k^-) = X(t_k)$ with probability one.

(iii) $X(t) = \xi(t)$ for $t \le 0$, for almost all $t \in [0, t_1]$, $X(t)$ obeys the integral equation

$$X(t) = \xi(0) + \int_0^t F_1(s, X(s-\tau), X_s)\, ds + \int_0^t F_2(s, X(s-\tau), X_s)\, dW(s). \tag{2.2}$$

And for almost all $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, $X(t)$ obeys the integral equation

$$X(t) = X(t_k^+) + \int_{t_k}^t F_1(s, X(s-\tau), X_s)\, ds + \int_{t_k}^t F_2(s, X(s-\tau), X_s)\, dW(s). \tag{2.3}$$

Moreover, $X(t)$ satisfies the impulsive conditions at each $t = t_k$, $k \in \mathbb{N}$ with probability one.

Remark 2.2. Now let us demonstrate the derivation procedure of Definition 2.1. First of all, noticing that the stochastic functional differential equation with infinite delay and impulsive perturbation (2.1) becomes the following stochastic functional differential equation with infinite delay:

$$dX(t) = F_1(t, X(t-\tau), X_t)\, dt + F_2(t, X(t-\tau), X_t)\, dW(t)$$

on $[0, t_1]$ and each interval $(t_k, t_{k+1}] \in \mathbb{R}_+$, $k \in \mathbb{N}$. From the definition of a solution of stochastic functional differential equations with infinite delay(see e.g., [24, 25, 28]), condition (i), (2.2) and (2.3) should be satisfied. Secondly, since there are impulsive perturbations in (2.1), then condition (ii) and impulsive conditions in (iii) should be satisfied. According to the two facts above, the Definition 2.1 is thus proposed.

For model (1.2), we let the initial value $x(t) = \xi(t)$, $t \le 0$ be nonrandom and positive, and belong to the phase space C_g above.

Now consider the following stochastic functional differential equation with infinite delay:

$$\begin{aligned} dy(t) = y(t) \Bigg[& r(t) - \prod_{0<t_k<t} (1+I_k) a(t) y(t) + \prod_{0<t_k<t-\tau} (1+I_k) b(t) y(t-\tau) \\ & + c(t) \int_{-\infty}^0 \prod_{0<t_k<t+\theta} (1+I_k) y(t+\theta)\, d\mu(\theta) \Bigg] dt \\ & + \sigma(t) \left(\prod_{0<t_k<t} (1+I_k) \right)^{\nu} y^{1+\nu}(t)\, dB(t) \end{aligned} \tag{2.4}$$

with the same initial condition as model (1.2). Obviously, $\forall\, t \le t_1$, $\prod_{0<t_k<t}(1+I_k) = 1$ holds.

Wei [24,25] and Xu [28] have proved that, in order for a stochastic functional differential equations with infinite delay to have a unique global solution for any given initial vale

$\xi \in C_g$, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition. The local Lipschitz condition guarantees that the unique solution exists in $(-\infty, \tau_e)$, where τ_e is the explosion time (see Mao [28]). Clearly, the coefficients of (2.4) satisfy the local Lipschitz condition, but do not satisfy the linear growth condition.

Lemma 2.3. *Let assumptions* **A1**–**A3** *hold. For model* (2.4)*, with any given initial value* $\xi \in C_g$*, there is a unique solution* $y(t)$ *on* $t \in R$ *and the solution will remain in* $\mathbb{R}_+$ *with probability* 1.

Proof. Since the coefficients of (2.4) are locally Lipschitz continuous, for any given initial condition $\xi \in C_g$, there is a unique local solution $y(t)$ on $t \in (-\infty, \tau_e)$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. For any given initial value $\xi \in C_g$, let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\infty<\theta\le 0} \xi(\theta) \le \max_{-\infty<\theta\le 0} \xi(\theta) < k_0.$$

For each integer $k \ge k_0$, we define a stopping time

$$\tau_k = \inf\left\{t \in (-\infty, \tau_e) : y(t) \le \frac{1}{k} \text{ or } y(t) \ge k\right\},$$

where we set $\inf \varnothing = +\infty$ (as usual $\varnothing$ denotes the empty set) throughout this paper. Clearly, τ_k is increasing as $k \to +\infty$. Set $\tau_{+\infty} = \lim_{k\to+\infty} \tau_k$, whence $\tau_{+\infty} \le \tau_e$ a.s. for all $t \ge 0$. If we can show that $\tau_{+\infty} = +\infty$ a.s., then $\tau_e = +\infty$ a.s. and $x(t) \in \mathbb{R}_+$ a.s. for all $t \ge 0$. In other words, to complete the proof all we need to show is that $\tau_{+\infty} = +\infty$ a.s. Now let us define a C^2-function $V\colon \mathbb{R}_+ \to \mathbb{R}_+$ by $V(y) = \sqrt{y} - 1 - 0.5 \ln y$. Let $k \ge k_0$ and $T > 0$ be arbitrary. For $0 \le t \le \tau_k \wedge T$, according to Itô's formula to $V(y(t))$, we have

$$\begin{aligned}
LV(y_t) &= 0.5\left(y^{-0.5}(t) - y^{-1}(t)\right) y(t) \\
&\quad \times \left(r(t) - \prod_{0<t_k<t} (1+I_k) a(t) y(t) + \prod_{0<t_k<t-\tau} (1+I_k) b(t) y(t-\tau) \right. \\
&\qquad \left. + c(t) \int_{-\infty}^{0} \prod_{0<t_k<t+\theta} (1+I_k) y(t+\theta)\, d\mu(\theta)\right) \\
&\quad + 0.5\left(-0.25 y^{-1.5}(t) + 0.5 y^{-2}(t)\right) \left(\prod_{0<t_k<t} (1+I_k)\right)^{2\nu} \sigma^2(t) y^{2+2\nu}(t) \\
&= 0.5 r(t)\left(y^{0.5}(t) - 1\right) - 0.5 \prod_{0<t_k<t} (1+I_k) a(t)\left(y^{0.5}(t) - 1\right) y(t) \\
&\quad + 0.5 \prod_{0<t_k<t-\tau} (1+I_k) b(t)\left(y^{0.5}(t) - 1\right) y(t-\tau)
\end{aligned}$$

$$
\begin{aligned}
&+ 0.5c(t)\left(y^{0.5}(t)-1\right)\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+I_k)y(t+\theta)\,d\mu(\theta)\\
&+ 0.5\left(-0.25y^{-1.5}(t)+0.5y^{-2}(t)\right)\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu}\sigma^2(t)y^{2+2\nu}(t)\\
\le\ & 0.5r(t)\left(y^{0.5}(t)-1\right)-0.5\prod_{0<t_k<t}(1+I_k)a(t)\left(y^{0.5}(t)-1\right)y(t)\\
&+ 0.0625\left(\prod_{0<t_k<t-\tau}(1+I_k)\right)^{2}b^2(t)\left(y^{0.5}(t)-1\right)^2+y^2(t-\tau)\\
&+ 0.0625c^2(t)\left(y^{0.5}(t)-1\right)^2+\left[\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+I_k)y(t+\theta)\,d\mu(\theta)\right]^2\\
&+ 0.5\left(-0.25y^{-1.5}(t)+0.5y^{-2}(t)\right)\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu}\sigma^2(t)y^{2+2\nu}(t)\\
\le\ & 0.5r(t)\left(y^{0.5}(t)-1\right)-0.5\prod_{0<t_k<t}(1+I_k)a(t)\left(y^{0.5}(t)-1\right)y(t)\\
&+ 0.0625\left(\prod_{0<t_k<t-\tau}(1+I_k)\right)^{2}b^2(t)\left(y^{0.5}(t)-1\right)^2+y^2(t-\tau)\\
&+ 0.0625c^2(t)\left(y^{0.5}(t)-1\right)^2+\int_{-\infty}^{0}\left(\prod_{0<t_k<t+\theta}(1+I_k)\right)^2 y^2(t+\theta)\,d\mu(\theta)\\
&+ 0.5\left(-0.25y^{-1.5}(t)+0.5y^{-2}(t)\right)\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu}\sigma^2(t)y^{2+2\nu}(t)\\
=\ & -0.125\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu}\sigma^2(t)y^{0.5+2\nu}(t)\\
&+ 0.25\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu}\sigma^2(t)y^{2\nu}(t)-0.5\prod_{0<t_k<t}(1+I_k)a(t)y^{1.5}(t)\\
&+ 0.5\prod_{0<t_k<t}(1+I_k)a(t)y(t)+0.0625\left(\prod_{0<t_k<t}(1+I_k)\right)^{2}b^2(t)y(t)\\
&+ 0.0625c^2(t)y(t)+0.5r(t)y^{0.5}(t)-0.125c^2(t)y^{0.5}(t)\\
&- 0.125\left(\prod_{0<t_k<t}(1+I_k)\right)^{2}b^2(t)y^{0.5}(t)+0.0625\left(\prod_{0<t_k<t-\tau}(1+I_k)\right)^{2}b^2(t)
\end{aligned}
$$

$$
\begin{aligned}
&- 0.5r(t) + 0.0625c^2(t) + \int_{-\infty}^{0} \left(\prod_{0<t_k<t+\theta} (1+I_k) \right)^2 y^2(t+\theta)\, d\mu(\theta) + y^2(t-\tau) \\
&= F(y(t)) + \int_{-\infty}^{0} \left(\prod_{0<t_k<t+\theta} (1+I_k) \right)^2 y^2(t+\theta)\, d\mu(\theta) \\
&- \left(\prod_{0<t_k<t} (1+I_k) \right)^2 y^2(t) + y^2(t-\tau) - y^2(t)
\end{aligned}
$$

where

$$
\begin{aligned}
F(y) = &-0.125 \left(\prod_{0<t_k<t} (1+I_k) \right)^{2\nu} \sigma^2(t) y^{0.5+2\nu} + 0.25 \left(\prod_{0<t_k<t} (1+I_k) \right)^{2\nu} \sigma^2(t) y^{2\nu} \\
&+ \left(\prod_{0<t_k<t} (1+I_k) \right)^2 y^2 + y^2 - 0.5 \prod_{0<t_k<t} (1+I_k) a(t) y^{1.5} \\
&+ 0.5 \prod_{0<t_k<t} (1+I_k) a(t) y + 0.0625 \left(\prod_{0<t_k<t} (1+I_k) \right)^2 b^2(t) y + 0.0625 c^2(t) y \\
&+ 0.5 r(t) y^{0.5} - 0.125 c^2(t) y^{0.5} - 0.125 \left(\prod_{0<t_k<t} (1+I_k) \right)^2 b^2(t) y^{0.5} \\
&+ 0.0625 \left(\prod_{0<t_k<t-\tau} (1+I_k) \right)^2 b^2(t) - 0.5 r(t) + 0.0625 c^2(t).
\end{aligned}
$$

Combining with assumption **A2**, it is easy to see that $F(y)$ is bounded above, say by K, in $\mathbb{R}_+$. We therefore obtain that

$$
\begin{aligned}
dV(y(t)) \le K\, dt &+ \int_{-\infty}^{0} \left(\prod_{0<t_k<t+\theta} (1+I_k) \right)^2 y^2(t+\theta)\, d\mu(\theta)\, dt \\
&- \left(\prod_{0<t_k<t} (1+I_k) \right)^2 y^2(t)\, dt + y^2(t-\tau)\, dt - y(t)\, dt \\
&+ 0.5 \left(y^{0.5+\nu}(t) - y^{\nu}(t) \right) \prod_{0<t_k<t} (1+I_k) \sigma(t)\, dB(t).
\end{aligned}
$$

Integrating both sides from 0 to t, and then taking expectations, we have

$$
\begin{aligned}
EV(y(t)) \le V(y(0)) + Kt &+ E \int_0^t \int_{-\infty}^{0} \left(\prod_{0<t_k<s+\theta} (1+I_k) \right)^2 y^2(s+\theta)\, d\mu(\theta)\, ds \\
&- E \int_0^t \left(\prod_{0<t_k<s} (1+I_k) \right)^2 y^2(s)\, ds + E \int_0^t y^2(s-\tau)\, ds \\
&- E \int_0^t y^2(s)\, ds.
\end{aligned} \tag{2.5}
$$

Moreover, we can derive that

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^0 \left(\prod_{0<t_k<s+\theta} (1+I_k) \right)^2 y^2(s+\theta)\, d\mu(\theta)\, ds \\
&= \int_0^t \Bigg[\int_{-\infty}^{-s} \left(\prod_{0<t_k<s+\theta} (1+I_k) \right)^2 y^2(s+\theta)\, d\mu(\theta) \\
&\qquad + \int_{-s}^{0} \left(\prod_{0<t_k<s+\theta} (1+I_k) \right)^2 y^2(s+\theta)\, d\mu(\theta) \Bigg] ds \\
&= \int_0^t ds \int_{-\infty}^{-s} e^{2\boldsymbol{r}(s+\theta)} y^2(s+\theta) e^{-2\boldsymbol{r}(s+\theta)}\, d\mu(\theta) \\
&\quad + \int_{-t}^{0} d\mu(\theta) \int_{-\theta}^{t} \left(\prod_{0<t_k<s+\theta} (1+I_k) \right)^2 y^2(s+\theta)\, ds \\
&\le \|\xi\|_{C_g}^2 \int_0^t e^{-2\boldsymbol{r}s}\, ds \int_{-\infty}^0 e^{-2\boldsymbol{r}\theta}\, d\mu(\theta) \\
&\quad + \int_{-\infty}^{0} d\mu(\theta) \int_0^t \left(\prod_{0<t_k<s} (1+I_k) \right)^2 y^2(s)\, ds \\
&\le \|\xi\|_{C_g}^2 \mu_{\boldsymbol{r}} t + \int_0^t \left(\prod_{0<t_k<s} (1+I_k) \right)^2 y^2(s)\, ds.
\end{aligned} \tag{2.6}
$$

On the other hand,

$$
\begin{aligned}
\int_0^t y^2(s-\tau)\, ds &= \int_{-\tau}^{t-\tau} y^2(s)\, ds = \int_{-\tau}^{0} \xi^2(s)\, ds + \int_0^{t-\tau} y^2(s)\, ds \\
&\le \int_{-\tau}^{0} \xi^2(s)\, ds + \int_0^t y^2(s)\, ds.
\end{aligned} \tag{2.7}
$$

Substituting (2.6) and (2.7) into (2.5) leads to

$$
EV(y(t)) \le V(y(0)) + Kt + \|\xi\|_{C_g}^2 \mu_{\boldsymbol{r}} t + \int_{-\tau}^{0} \xi^2(s)\, ds.
$$

Let $t = \tau_k \wedge T$, and we obtain that

$$
EV(y(\tau_k \wedge T)) \le V(y(0)) + KT + \|\xi\|_{C_g}^2 \mu_{\boldsymbol{r}} T + \int_{-\tau}^{0} \xi^2(s)\, ds. \tag{2.8}
$$

Note that for every $\omega \in \{\tau_k \le T\}$, $y(\tau_k, \omega)$ equals either k or $1/k$, and hence $V(y(\tau_k, \omega))$ is no less than either

$$\sqrt{k} - 1 - 0.5\log(k)$$

or

$$\sqrt{\frac{1}{k}} - 1 - 0.5\log\left(\frac{1}{k}\right) = \sqrt{\frac{1}{k}} - 1 + 0.5\log(k).$$

Thus,

$$V(y(\tau_k, \omega)) \geq \left[\sqrt{k} - 1 - 0.5\log(k)\right] \wedge \left[\sqrt{\frac{1}{k}} - 1 + 0.5\log(k)\right].$$

It then follows from (2.8) that

$$\begin{aligned} &V(y(0)) + KT + \|\xi\|_{C_g}^2 \mu_r T + \int_{-\tau}^{0} \xi^2(s)\,ds \\ &\geq E[1_{\{\tau_k \leq T\}}(\omega)V(y(\tau_k, \omega))] \\ &\geq \mathcal{P}\{\tau_k \leq T\}\left(\left[\sqrt{k} - 1 - 0.5\log(k)\right] \wedge \left[\sqrt{\frac{1}{k}} - 1 + 0.5\log(k)\right]\right), \end{aligned}$$

where $1_{\{\tau_k \leq T\}}$ is the indicator function of $\{\tau_k \leq T\}$. Letting $k \to +\infty$ gives

$$\lim_{k\to+\infty} \mathcal{P}\{\tau_k \leq T\} = 0$$

and hence

$$\mathcal{P}\{\tau_{+\infty} \leq T\} = 0.$$

Since $T > 0$ is arbitrary, we derive

$$\mathcal{P}\{\tau_{+\infty} < +\infty\} = 0.$$

Thus $\mathcal{P}\{\tau_{+\infty} = +\infty\} = 1$ as required. □

The following theorem is fundamental in this paper.

Theorem 2.4. *Let assumptions* **A1–A3** *hold. For model* (1.2)*, with any given initial condition* $\xi \in C_g$*, there is a unique solution* $x(t)$ *on* $t \in R$ *and the solution will remain in* $\mathbb{R}_+$ *with probability* 1*.*

Proof. Let

$$x(t) = \prod_{0<t_k<t} (1 + I_k)y(t),$$

where $y(t)$ is the solution of the system (2.4). We need only to clarify that $x(t)$ is the solution (1.2). As a matter of fact, $x(t)$ is continuous on $(t_k, t_{k+1}) \subset (0, +\infty)$, $k \in \mathbb{N}$ and for every $t \neq t_k$,

$$\begin{aligned} dx(t) &= d\left[\prod_{0<t_k<t}(1+I_k)y(t)\right] = \prod_{0<t_k<t}(1+I_k)dy(t) \\ &= \prod_{0<t_k<t}(1+I_k)y(t) \\ &\quad \times \left[r(t) - \prod_{0<t_k<t}(1+I_k)a(t)y(t) + \prod_{0<t_k<t-\tau}(1+I_k)b(t)y(t-\tau)\right. \\ &\qquad \left. + c(t)\int_{-\infty}^{0}\prod_{0<t_k<t+\theta}(1+I_k)y(t+\theta)\,d\mu(\theta)\right] dt \\ &\quad + \left(\prod_{0<t_k<t}(1+I_k)\right)^{1+\nu}\sigma(t)y^{1+\nu}(t)\,dB(t) \\ &= x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0}x(t+\theta)\,d\mu(\theta)\right] dt \\ &\quad + \sigma(t)x^{1+\nu}(t)\,dB(t). \end{aligned}$$

In addition, for every $k \in \mathbb{N}$ and $t_k \in [0,+\infty)$,

$$\begin{aligned} x(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+I_j)y(t) = \prod_{0<t_j\le t_k}(1+I_j)y(t_k^+) \\ &= (1+I_k)\prod_{0<t_j<t_k}(1+I_j)y(t_k) = (1+I_k)x(t_k). \end{aligned}$$

At the same time,

$$x(t_k^-) = \lim_{t\to t_k^-}\prod_{0<t_j<t}(1+I_j)y(t) = \prod_{0<t_j<t_k}(1+I_j)y(t_k^-) = \prod_{0<t_j<t_k}(1+I_j)y(t_k) = x(t_k).$$

Our next job is to prove the uniqueness of the solution. For $t \in [0,t_1]$, the model (1.2) becomes the following equation:

$$\begin{aligned} dx(t) &= x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0}x(t+\theta)\,d\mu(\theta)\right] dt \\ &\quad + \sigma(t)x^{1+\nu}(t)\,dB(t). \end{aligned} \tag{2.9}$$

Owing to the coefficients of (2.9) are locally Lipschitz continuous, by the theory of stochastic differential equation [24], the solution of (2.9) is unique. For $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, the model (1.2) becomes:

$$\begin{aligned} dx(t) &= x(t)\left[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0}x(t+\theta)\,d\mu(\theta)\right] dt \\ &\quad + \sigma(t)x^{1+\nu}(t)\,dB(t). \end{aligned} \tag{2.10}$$

Note that the coefficients of (2.10) are also locally Lipschitz continuous, then the solution of (2.10) is also unique. Consequently, the solution of the model (1.2) is unique. This completes the proof. □

3. Permanence and extinction

In this section, we shall study the extinction and stochastic permanence of model (1.2).

Theorem 3.1. *Under the assumptions* **A1–A3***, if* $G^* < 0$*,* $b(t) \geq 0$*,* $c(t) \geq 0$ *and* $\inf_{t\in\overline{\mathbb{R}}_+} \{a(t) - b(t+\tau) - c^u\} \geq 0$*, where* $G^* = \limsup_{t\to+\infty} t^{-1} \left[\sum_{0<t_k<t} \ln(1+I_k) + \int_0^t r(s)\,ds\right]$*, then population* $x(t)$ *modeled by* (1.2) *will go to extinction a.s.*

Proof. For (2.4), by the generalized Itô's formula, we derive

$$
\begin{aligned}
d\ln y &= \frac{dy}{y} - \frac{(dy)^2}{2y^2} \\
&= \Bigg[r(t) - \prod_{0<t_k<t} (1+I_k) a(t) y + \prod_{0<t_k<t-\tau} (1+I_k) b(t) y(t-\tau) \\
&\quad + c(t) \int_{-\infty}^0 \prod_{0<t_k<t+\theta} (1+I_k) y(t+\theta)\, d\mu(\theta) - \frac{\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu} \sigma^2(t) y^{2\nu}(t)}{2} \Bigg] dt \\
&\quad + \sigma(t) \left(\prod_{0<t_k<t} (1+I_k) \right)^{\nu} y^{\nu}(t)\, dB(t).
\end{aligned}
$$

Integrating both sides from 0 to t, where $t \in [0, t_1]$ or $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$, we obtain

$$
\begin{aligned}
&\ln y(t) - \ln y(0) \\
&= \int_0^t \Bigg[r(s) - \prod_{0<t_k<s} (1+I_k) a(s) y(s) + \prod_{0<t_k<s-\tau} (1+I_k) b(s) y(s-\tau) \\
&\quad + c(s) \int_{-\infty}^0 \prod_{0<t_k<s+\theta} (1+I_k) y(s+\theta)\, d\mu(\theta) - \frac{\left(\prod_{0<t_k<s}(1+I_k)\right)^{2} \sigma^2(s) y^{2\nu}(s)}{2} \Bigg] ds \\
&\quad + \int_0^t \left(\prod_{0<t_k<s} (1+I_k) \right)^{\nu} \sigma(s) y^{\nu}(s)\, dB(s) \\
&= \int_0^t \left[r(s) - a(s)x(s) + b(s)x(s-\tau) + c(s) \int_{-\infty}^0 x(s+\theta)\, d\mu(\theta) - \frac{\sigma^2(s) x^{2\nu}(s)}{2} \right] ds \\
&\quad + \int_0^t \sigma(s) x^{\nu}(s)\, dB(s).
\end{aligned}
$$

On the other hand, from $b(t) \geq 0$, we obtain

$$\int_0^t b(s)x(s-\tau)\,ds = \int_{-\tau}^{t-\tau} b(s+\tau)x(s)\,ds = \int_{-\tau}^{0} b(s+\tau)x(s)\,ds + \int_0^{t-\tau} b(s+\tau)x(s)\,ds$$
$$\leq \int_{-\tau}^{0} b(s+\tau)x(s)\,ds + \int_0^{t} b(s+\tau)x(s)\,ds.$$

In other words, $\forall\, t \in \overline{\mathbb{R}}_+$, we have

$$\begin{aligned} &\ln y(t) - \ln y(0) \\ &\leq \int_0^t \left[r(s) - (a(s) - b(s+\tau))\,x(s) + c(s)\int_{-\infty}^0 x(s+\theta)\,d\mu(\theta) - \frac{\sigma^2(s)x^{2\nu}(s)}{2}\right] ds \\ &\quad + \int_{-\tau}^0 b(s+\tau)x(s)\,ds + M(t), \end{aligned} \tag{3.1}$$

where $M(t) = \int_0^t \sigma(s)x^\nu(s)\,dB(s)$. By assumption **A1** and $c(t) \geq 0$, we may compute that

$$\begin{aligned} &\int_0^t c(s)\int_{-\infty}^0 x(s+\theta)\,d\mu(\theta)\,ds \\ &= \int_0^t c(s)\left[\int_{-\infty}^{-s} x(s+\theta)\,d\mu(\theta)\,ds + \int_{-s}^0 x(s+\theta)\,d\mu(\theta)\right] ds \\ &= \int_0^t c(s)\,ds \int_{-\infty}^{-s} e^{\boldsymbol{r}(s+\theta)}x(s+\theta)e^{-\boldsymbol{r}(s+\theta)}\,d\mu(\theta) + \int_{-t}^0 d\mu(\theta)\int_{-\theta}^t c(s)x(s+\theta)\,ds \\ &\leq c^u\,\|\xi\|_{c_g}\int_0^t e^{-\boldsymbol{r}s}\,ds\int_{-\infty}^0 e^{-\boldsymbol{r}\theta}\,d\mu(\theta) + c^u\int_{-\infty}^0 d\mu(\theta)\int_0^t x(s)\,ds \\ &\leq c^u\,\|\xi\|_{c_g}\int_0^t e^{-\boldsymbol{r}s}\,ds\int_{-\infty}^0 e^{-2\boldsymbol{r}\theta}\,d\mu(\theta) + c^u\int_{-\infty}^0 d\mu(\theta)\int_0^t x(s)\,ds \\ &\leq \frac{1}{\boldsymbol{r}}c^u\,\|\xi\|_{c_g}\,\mu_{\boldsymbol{r}}(1-e^{-\boldsymbol{r}t}) + c^u\int_0^t x(s)\,ds. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_{t-\tau}^t b(s+\tau)x(s)\,ds - \int_{-\tau}^0 b(s+\tau)x(s)\,ds + \ln y(t) - \ln y(0) \\ &\leq \int_0^t \left[r(s) - (a(s) - b(s+\tau) - c^u)\,x(s) - \frac{\sigma^2(s)x^{2\nu}(s)}{2}\right] ds \\ &\quad + \frac{1}{\boldsymbol{r}}c^u\,\|\xi\|_{c_g}\,\mu_{\boldsymbol{r}}(1-e^{-\boldsymbol{r}t}) + M(t). \end{aligned}$$

Here, $M(t)$ is a continuous local martingale (in fact, it is a martingale as we can show that $E|x(t)|^{2\nu}$ is bounded in the same way as Lemma 4.1 in [21], Theorem 2.2 in [6], Theorem 4.2 in [26] and Theorem 5.1 in [4] did). The quadratic variation of $M(t)$ is

$\langle M(t), M(t)\rangle = \int_0^t \sigma^2(s)x^{2\nu}(s)\,ds$. By virtue of the exponential martingale inequality, for any positive constants T_0, α and β, we have

$$\mathcal{P}\left\{\sup_{0\le t\le T_0}\left[M(t) - \frac{\alpha}{2}\langle M(t), M(t)\rangle\right] > \beta\right\} \le e^{-\alpha\beta}.$$

Choose $T_0 = k$, $\alpha = 1$, $\beta = 2\ln n$. Then it follows that

$$\mathcal{P}\left\{\sup_{0\le t\le k}\left[M(t) - \frac{1}{2}\langle M(t), M(t)\rangle\right] > 2\ln n\right\} \le \frac{1}{n^2}.$$

On using the Borel-Cantelli lemma, for almost all $\omega \in \Omega$, there is a random integer $n_0 = n_0(\omega)$ such that for $n \ge n_0$,

$$\sup_{0\le t\le n}\left[M(t) - \frac{1}{2}\langle M(t), M(t)\rangle\right] \le 2\ln n.$$

This is to say

$$M(t) \le 2\ln n + \frac{1}{2}\langle M(t), M(t)\rangle = 2\ln n + \frac{1}{2}\int_0^t \sigma^2(s)x^{2\nu}(s)\,ds$$

for all $0 \le t \le n$, $n \ge n_0$ a.s. Substituting this inequality into (3.1), we can obtain that

$$\begin{aligned}
&\int_{t-\tau}^t b(s+\tau)x(s)\,ds + \ln y(t) - \ln y(0)\\
&\le \int_{-\tau}^0 b(s+\tau)x(s)\,ds + \int_0^t [r(s) - (a(s) - b(s+\tau) - c^u)\,x(s)]\,ds\\
&\quad + 2\ln n + \frac{1}{\boldsymbol{r}}c^u\,\|\xi\|_{c_g}\,\mu_{\boldsymbol{r}}(1 - e^{-\boldsymbol{r}t})
\end{aligned} \tag{3.2}$$

for all $0 \le t \le n$, $n \ge n_0$ a.s. On the other hand, it follows from (3.2) that

$$\begin{aligned}
&\sum_{0<t_k<t}\ln(1+I_k) + \ln y(t) - \ln y(0)\\
&\le \sum_{0<t_k<t}\ln(1+I_k) + \int_{-\tau}^0 b(s+\tau)x(s)\,ds\\
&\quad + \int_0^t [r(s) - (a(s) - b(s+\tau) - c^u)\,x(s)]\,ds + 2\ln n + \frac{1}{\boldsymbol{r}}c^u\,\|\xi\|_{c_g}\,\mu_{\boldsymbol{r}}(1 - e^{-\boldsymbol{r}t})
\end{aligned}$$

for all $0 \le t \le n$, $n \ge n_0$ a.s. In other words, we have shown that

$$\begin{aligned}
\ln x(t) - \ln x(0) \le &\sum_{0<t_k<t}\ln(1+I_k) + \int_{-\tau}^0 b(s+\tau)x(s)\,ds\\
&+ \int_0^t [r(s) - (a(s) - b(s+\tau) - c(s))\,x(s)]\,ds\\
&+ 2\ln n + \frac{1}{\boldsymbol{r}}c^u\,\|\xi\|_{c_g}\,\mu_{\boldsymbol{r}}(1 - e^{-\boldsymbol{r}t})
\end{aligned}$$

for all $n-1 \le t \le n$, $n \ge n_0$ a.s. Thus

$$\ln x(t) - \ln x(0) \le \sum_{0<t_k<t} \ln(1+I_k) + \int_{-\tau}^{0} b(s+\tau)x(s)\,ds + \int_0^t r(s)\,ds + 2\ln n + \frac{1}{\boldsymbol{r}} c^u \|\xi\|_{c_g} \mu_{\boldsymbol{r}} (1 - e^{-\boldsymbol{r}t})$$

for all $0 \le t \le n$, $n \ge n_0$ a.s. Then the required assertion by assumption **A2**. □

When it comes to the study of population system, the role of stochastic permanence indicating the eternal existence of the population,can never be ignorant with its theoretical and practical significance.And its importance has catched the eyes of scientists all over the world. So now let us show that $x(t)$ modeled by (1.2) is stochastic permanent in some cases.

A4. There are two positive constants m and M such that $m \le \prod_{0<t_k<t}(1+I_k) \le M$ for all $t > 0$.

Remark 3.2. Assumption **A4** is easy to be satisfied. For example, if $I_k = e^{(-1)^{k+1}/k} - 1$, then $e^{0.5} < \prod_{0<t_k<t}(1+I_k) < e$ for all $t > 0$. Thus $1 \le \prod_{0<t_k<t}(1+I_k) \le e$ for all $t > 0$.

Theorem 3.3. *Under the assumptions* **A1**–**A4***, if* $r_* > 0$*,* $\nu < 1$*,* $b(t) \ge 0$ *and* $c(t) \ge 0$*, then the population* $x(t)$ *represented by* (1.2) *will be stochastic permanence.*

Proof. First, we prove that for arbitrary $\varepsilon > 0$, there is a constant $\alpha_1 > 0$ such that $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \le \alpha_1\} \ge 1 - \varepsilon$. Let $0.5 < p = 2\nu - 1 < 1$ and choose $\varepsilon_1 \in (0, 2\boldsymbol{r})$, we computer

$$\begin{aligned}
& dy^p(t) \\
&= py^{p-1}(t)\,dy(t) + \frac{1}{2}p(p-1)y^{p-2}(t)(dy(t))^2 \\
&= py^{p-1}(t)\Bigg[\Bigg(y(t)\Bigg(r(t) - \prod_{0<t_k<t}(1+I_k)a(t)y(t) + \prod_{0<t_k<t-\tau}(1+I_k)b(t)y(t-\tau) \\
&\qquad + c(t)\int_{-\infty}^{0} \prod_{0<t_k<t+\theta}(1+I_k)y(t+\theta)\,d\mu(\theta)\Bigg)\Bigg)dt \\
&\qquad + \Bigg(\prod_{0<t_k<t}(1+I_k)\Bigg)^{\nu} \sigma(t)y^{1+\nu}(t)\,dB(t)\Bigg] \\
&\quad + \frac{1}{2}p(p-1)\Bigg(\prod_{0<t_k<t}(1+I_k)\Bigg)^{2\nu} \sigma^2(t)y^{p+2\nu}(t)\,dt \\
&\le py^{p-1}(t)\Bigg[\Bigg(y(t)\Bigg(r(t) - ma(t)y(t) + Mb(t)y(t-\tau) + Mc(t)\int_{-\infty}^{0} y(t+\theta)\,d\mu(\theta)\Bigg)\Bigg)dt \\
&\qquad + \Bigg(\prod_{0<t_k<t}(1+I_k)\Bigg)^{\nu} \sigma(t)y^{1+\nu}(t)\,dB(t)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}p(p-1)\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu}\sigma^2(t)y^{p+2\nu}(t)\,dt \\
&\le \left[r(t)py^p(t) + \frac{p^2M^2b^2(t)y^{2p}(t)}{4} + y^2(t-\tau)\right. \\
&\quad \left. + \frac{p^2M^2c^2(t)y^{2p}(t)}{4} + \int_{-\infty}^{0} y^2(t+\theta)\,d\mu(\theta)\right] dt \\
&\quad + \left(\prod_{0<t_k<t}(1+I_k)\right)^{\nu} p\sigma(t)y^{p+\nu}(t)\,dB(t) - \frac{1}{2}p(1-p)m^{2\nu}\sigma^2(t)y^{p+2\nu}(t)\,dt \\
&= F(y(t))\,dt - \left[\varepsilon_1 y^p(t) + e^{\varepsilon_1\tau}y^2(t) - y^2(t-\tau) - \int_{-\infty}^{0} y^2(t+\theta)\,d\mu(\theta) + \mu_{\boldsymbol{r}} y^2(t)\right] dt \\
&\quad + \left(\prod_{0<t_k<t}(1+I_k)\right)^{\nu} p\sigma(t)y^{p+\nu}(t)\,dB(t),
\end{aligned}
$$

where

$$
\begin{aligned}
F(y) = {} & e^{\varepsilon_1\tau}y^2 + \mu_{\boldsymbol{r}} y^2 + (\varepsilon_1 + r(t)p)\,y^p + p^2b^2(t)y^{2p} \\
& + p^2c^2(t)y^{2p} - \frac{1}{2}p(1-p)m^{2\nu}\sigma^2(t)y^{p+2\nu}.
\end{aligned}
$$

From $p > 0$ and assumption **A2**, we have $F(y)$ is bounded above in $\mathbb{R}_+$, namely

$$
K = \sup_{y\in\mathbb{R}_+} F(y) < +\infty.
$$

We therefore obtain

$$
\begin{aligned}
dy^p(t) = {} & \left[K - \varepsilon_1 y^p(t) - e^{\varepsilon_1\tau}y^2(t) + y^2(t-\tau)\right] dt \\
& + \int_{-\infty}^{0} y^2(t+\theta)\,d\mu(\theta)\,dt - \mu_{\boldsymbol{r}} y^2(t)\,dt \\
& + \left(\prod_{0<t_k<t}(1+I_k)\right)^{\nu} p\sigma(t)y^{p+\nu}(t)\,dB(t).
\end{aligned}
$$

Once more by the Itô's formula we have

$$
\begin{aligned}
d\left[e^{\varepsilon_1 t}y^p(t)\right] &= e^{\varepsilon_1 t}\left[\varepsilon_1 y^p(t)\,dt + dy^p(t)\right] \\
&\le e^{\varepsilon_1 t}\left[K - e^{\varepsilon_1\tau}y^2(t) + y^2(t-\tau) + \int_{-\infty}^{0} y^2(t+\theta)\,d\mu(\theta) - \mu_{\boldsymbol{r}} y^2(t)\right] dt \\
&\quad + e^{\varepsilon_1 t}\left(\prod_{0<t_k<t}(1+I_k)\right)^{\nu} p\sigma(t)y^{p+\nu}(t)\,dB(t).
\end{aligned}
$$

Hence we derive that

$$\begin{aligned}
e^{\varepsilon_1 t}E[y^p(t)] &\le \xi^p(0) + \frac{e^{\varepsilon_1 t}K}{\varepsilon_1} - \frac{K}{\varepsilon_1} - E\int_0^t e^{\varepsilon_1 s+\varepsilon_1\tau} y^2(s)\,ds + E\int_0^t e^{\varepsilon_1 s} y^2(s-\tau)\,ds \\
&\quad + E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 y^2(s+\theta)\,d\mu(\theta)\,ds - E\int_0^t \mu_{\boldsymbol{r}} e^{\varepsilon_1 s} y^2(s)\,ds \\
&= \xi^p(0) + \frac{e^{\varepsilon_1 t}K}{\varepsilon_1} - \frac{K}{\varepsilon_1} - E\int_0^t e^{\varepsilon_1 s+\varepsilon_1\tau} y^2(s)\,ds + E\int_{-\tau}^{t-\tau} e^{\varepsilon_1 s+\varepsilon_1\tau} y^2(s)\,ds \\
&\quad + E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 y^2(s+\theta)\,d\mu(\theta)\,ds - E\int_0^t \mu_{\boldsymbol{r}} e^{\varepsilon_1 s} y^2(s)\,ds \\
&\le \xi^p(0) + \frac{e^{\varepsilon_1 t}K}{\varepsilon_1} - \frac{K}{\varepsilon_1} + \int_{-\tau}^0 e^{\varepsilon_1 s+\varepsilon_1\tau} y^2(s)\,ds \\
&\quad + E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 y^2(s+\theta)\,d\mu(\theta)\,ds - E\mu_{\boldsymbol{r}}\int_0^t e^{\varepsilon_1 s} y^2(s)\,ds.
\end{aligned}$$

From hypothesis **A1**, we have

$$\begin{aligned}
&\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 y^2(s+\theta)\,d\mu(\theta)\,ds \\
&= \int_0^t e^{\varepsilon_1 s}\left[\int_{-\infty}^{-s} y^2(s+\theta)\,d\mu(\theta) + \int_{-s}^0 y^2(s+\theta)\,d\mu(\theta)\right] ds \\
&= \int_0^t e^{\varepsilon_1 s}\,ds\int_{-\infty}^{-s} e^{2\boldsymbol{r}(s+\theta)} y^2(s+\theta) e^{-2\boldsymbol{r}(s+\theta)}\,d\mu(\theta) + \int_{-t}^0 d\mu(\theta)\int_{-\theta}^t e^{\varepsilon_1(s)} y^2(s+\theta)\,ds \\
&= \int_0^t e^{\varepsilon_1 s}\,ds\int_{-\infty}^{-s} e^{2\boldsymbol{r}(s+\theta)} y^2(s+\theta) e^{-2\boldsymbol{r}(s+\theta)}\,d\mu(\theta) + \int_{-t}^0 d\mu(\theta)\int_0^{t+\theta} e^{\varepsilon_1(s-\theta)} y^2(s)\,ds \\
&\le \|\xi\|_{c_g}^2\int_0^t e^{(\varepsilon_1-2\boldsymbol{r})s}\,ds\int_{-\infty}^0 e^{-2\boldsymbol{r}\theta}\,d\mu(\theta) + \int_{-\infty}^0 e^{-\varepsilon_1\theta}\,d\mu(\theta)\int_0^t e^{\varepsilon_1 s} y^2(s)\,ds \\
&\le \|\xi\|_{c_g}^2\,\mu_{\boldsymbol{r}} t + \mu_{\boldsymbol{r}}\int_0^t e^{\varepsilon_1 s} y^2(s)\,ds.
\end{aligned}$$

This immediately implies that

$$\limsup_{t\to+\infty} E[y^p(t)] \le \frac{K}{\varepsilon_1} = H. \tag{3.3}$$

Consequently,

$$\limsup_{t\to+\infty} E(x^p(t)) = \limsup_{t\to+\infty}\left[\prod_{0<t_k<t}(1+I_k)\right]^p E(x^p(t)) \le \left[M^p\frac{K}{\varepsilon_1}\right] = M_1. \tag{3.4}$$

Thus for any given $\varepsilon > 0$, let $\alpha_1 = M_1^{1/p}/\varepsilon^{1/p}$, by virtue of Chebyshev's inequality, we can derive that

$$\mathcal{P}\left\{x(t) > \alpha_1\right\} = \mathcal{P}\left\{x^p(t) > \alpha_1^p\right\} \le \frac{E[x^p(t)]}{\alpha_1^p}.$$

That is to say $\limsup_{t\to+\infty} \mathcal{P}\{x(t) > \alpha_1\} \le \varepsilon$. Consequently, $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \le \alpha_1\} \ge 1-\varepsilon$.

Next, we claim that for arbitrary $\varepsilon > 0$, there is a constant $\alpha_2 > 0$ such that $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \ge \alpha_2\} \ge 1-\varepsilon$. One can show that

$$\begin{aligned} d\left(\frac{1}{y(t)}\right) = \Bigg[& -\frac{r(t)}{y(t)} - \frac{b(t)y(t-\tau)}{y(t)} - \frac{c(t)\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+I_k)y(t+\theta)\,d\mu(\theta)}{y(t)} \\ & + a(t) + \sigma^2(t)\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu} y^{2\nu-1}(t)\Bigg]\,dt \\ & - \sigma(t)\left(\prod_{0<t_k<s}(1+I_k)\right)^{\nu} y^{\nu-1}(t)\,dB(t). \end{aligned} \tag{3.5}$$

Integrating from 0 to t and taking expectations on the both sides of (3.5), we get

$$\begin{aligned} E\left[\frac{1}{y(t)}\right] = E\left[\frac{1}{y(0)}\right] & \\ + \int_0^t \Bigg(& E\left[\frac{-r(s)}{y(s)}\right] + a(s) + \sigma^2(s)\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu} E[y^{2\nu-1}(s)] \\ & - b(s)E\left[\frac{y(s-\tau)}{y(s)}\right] - c(s)E\left[\frac{\int_{-\infty}^0 \prod_{0<t_k<t+\theta}(1+I_k)y(s+\theta)\,d\mu(\theta)}{y(s)}\right]\Bigg)\,ds. \end{aligned}$$

Hence

$$\begin{aligned} \frac{dE[1/y(t)]}{dt} &= -r(t)E[1/y(t)] + a(t) + \sigma^2(t)\left(\prod_{0<t_k<t}(1+I_k)\right)^{2\nu} E[y^{2\nu-1}(t)] \\ &\quad - b(t)E\left[\frac{y(t-\tau(t))}{y(t)}\right] - c(t)E\left[\frac{\int_{-\infty}^0 \prod_{0<t_k<t+\theta} y(t+\theta)\,d\mu(\theta)}{y(t)}\right]. \end{aligned} \tag{3.6}$$

Considering the following equation:

$$\frac{dE[1/y_1(t)]}{dt} = -r(t)E[1/y_1(t)] + a(t) + \sigma^2(t)M^{2\nu}H. \tag{3.7}$$

For any $\varepsilon > 0$, there is $T_1 > 0$, such that $r(t) > r_* - \varepsilon$ for all $t > T_1$. Using the same method as Theorem 4.5 in [13], we derive

$$\lim_{t\to+\infty} E\left[\frac{1}{y_1(t)}\right] \le d,$$

where

$$d = \frac{2(\sigma^2)^u M^{2\nu} H}{r_*} + \frac{\exp\left\{-\int_0^T r(s)\,ds\right\}}{y(0)}.$$

In view of (3.3), there exists a $T > T_1 > 0$ such that $E[y^{2\nu-1}(t)] \le H$ for all $t > T$. Thus from (3.6) and (3.7), using the comparison theorem for ODEs yields

$$E\left[\frac{1}{y(t)}\right] \le E\left[\frac{1}{y_1(t)}\right]$$

for all $t > T$, which implies that

$$\limsup_{t\to+\infty} E\left[\frac{1}{y(t)}\right] \le \limsup_{t\to+\infty} E\left[\frac{1}{y_1(t)}\right] = \lim_{t\to+\infty} E\left[\frac{1}{y_1(t)}\right] \le d.$$

Thus

$$\limsup_{t\to+\infty} E\left[\frac{1}{x(t)}\right] = \lim_{t\to+\infty} E\left[\frac{1}{\prod_{0<t_k<t}(1+I_k)y(t)}\right] = m^{-1}d.$$

So for any $\varepsilon > 0$, letting $\alpha_2 = m\varepsilon/d$, then the desired assertion follows from the Chebyshev inequality. □

Remark 3.4. In view of $G^* = \limsup_{t\to+\infty} t^{-1}\left[\sum_{0<t_k<t}\ln(1+I_k) + \int_0^t r(s)\,ds\right]$ in Theorem 3.1 and Theorem 3.3, we can find that the impulse does not affect extinction and stochastic permanence if the impulsive perturbations are bounded and some changes significantly if not.

Remark 3.5. Obviously, if assumptions **A1**–**A4** hold, $b(t) \ge 0$, $c(t) \ge 0$, $\nu \in [0.75, 1]$, $\inf_{t\in\overline{\mathbb{R}}_+}\{a(t) - b(t+\tau) - c^u\} \ge 0$ and $r(t) \equiv r \ne 0$, where r is a constant, then Theorem 3.1 and Theorem 3.3 establish the sufficient and necessary conditions for stochastic permanence and extinction of model (1.2).

Remark 3.6. From Theorem 3.1 and Theorem 3.3, we found that the delay has no effect on the extinction and stochastic permanence of the stochastic model in autonomous case.

4. Conclusions and remarks

With the space C_g as phase space, we investigate the permanence and extinction of a stochastic logistic model with infinite delay and impulsive perturbation. Sufficient and necessary conditions for extinction and stochastic permanence are given.

Several interesting topics deserve our further engagement. One may introduces the colored noise [22] into the model, which is more feasible and more accordance with the actual. Another significant and interesting problem is devoted to multidimensional impulsive stochastic model with infinite delay, and such investigations are in advance.

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