Bifurcation and Stability for the Unstirred Chemostat Model with Beddington-DeAngelis Functional Response

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Abstract. In this paper, we consider a basic N-dimensional competition model in the unstirred chemostat with Beddington-DeAngelis functional response. The bifurcation solutions from a simple eigenvalue and a double eigenvalue are obtained respectively. In particular, for the double eigenvalue, we establish the existence and stability of coexistence solutions by the techniques of space decomposition and Lyapunov-Schmidt procedure. Moreover, we describe the global structure of these bifurcation solutions.

1. Introduction

The chemostat is a piece of laboratory apparatus, which captures the essentials of exploitative competition in an open system, such as a simple lake, a system of waste treatment and a commercial reactor of fermentation processes. Moreover, the chemostat is a standard kind of mathematical models that has been extensively applied in ecology to model biological behavior of a simple lake and in biotechnology to model bioreaction in commercial bioreactors. Generally, the chemostat consists of a nutrient input-with all nutrients needed for growth in abundance except one-pumped at a constant rate into a well-stirred culture vessel whose volume is kept constant by pumping the contents out at the same rate, and therefore its contents are spatially homogeneous. A detailed mathematical description of competition in the chemostat can be found in [19] and [22].

Current continuous culture theory is mainly based on the relationship between limiting substrate concentration and growth rate as originally introduced by Monod in [14]. A detailed theoretical derivation of the chemostat equations with Monod function can be found in [19, 22]. A lot of works for the chemostat models with Monod functions have been done in the past decades, one can refer to [6, 7, 9, 10, 15, 16, 23, 29, 30]. In particular, the basic unstirred chemostat model has received considerable attention. Hsu and Waltman [9] analyzed the asymptotic behavior of solutions by theory of uniform persistence in

infinite-dimensional dynamical system and the theory of strongly order-preserving semi-dynamical system. Dung and Smith [6] discussed the co-existence states of the general unstirred chemostat model with one nutrient by degree theory. In [23], the local coexistence of positive solutions was obtained by the standard bifurcation theorems in the one dimensional case. Later, Wu [29] obtained the corresponding results in the N-dimensional case. Moreover, the local stability for the local coexistence solutions and global structure of the coexistence solutions were established. Furthermore, Nie and Wu [16] studied the uniqueness and stability of coexistence solutions by Lyapunov-Schmidt procedure and perturbation technique. Recently, Guo et al. derived the bifurcation solution from a double eigenvalue in [7].

Unfortunately, as it has been pointed out in [2, 18, 28], the Monod relationship cannot be valid for substrates which are growth limiting at low concentrations but are inhibitory for the species at higher concentrations, and the functional response should depend not only on the prey density but also on the predator density. Beddington [1] and DeAngelis et al. [5] proposed a functional response that depends on the prey density and the predator density, i.e., Beddington-DeAngelis functional response. On the Beddington-DeAngelis functional response, a mathematical model of competition between two species in the unstirred chemostat was considered in [25–27]. However, they only focus on the reaction-diffusion system in one-dimensional domain case. There, the local and global coexistence solutions were derived and local stability for the coexistence solutions was established. Moreover, the effect of the parameter was considered in detailed in [27]. For more studies on the chemostat model with the Beddington-DeAngelis functional response, one can refer to [17, 20, 24, 33–35].

A common feature of all these studies on the chemostat model with the Beddington-DeAngelis functional response has been that the existence of the coexistence solutions is obtained by the simple eigenvalue bifurcation theorem. Naturally, we hope to know what happens to the existence of coexistence states when m_1, m_2 are some critical points where double eigenvalue appears. To our knowledge, there are few works about the bifurcation from a double eigenvalue in the unstirred chemostat with Beddington-DeAngelis functional response. Therefore, in this paper, we investigate the bifurcation from a double eigenvalue and establish the existence and stability of coexistence states on the unstirred chemostat model with Beddington-DeAngelis functional response, that is, we consider the basic N-dimensional competition model in the unstirred case:

$$(1.1) \quad \begin{cases} S_t = \Delta S - m_1 u f(S, u) - m_2 v g(S, v), & x \in \Omega, \ t > 0, \\ u_t = \Delta u + m_1 u f(S, u), & x \in \Omega, \ t > 0, \\ v_t = \Delta v + m_2 v g(S, v), & x \in \Omega, \ t > 0 \end{cases}$$

with boundary conditions

$$\begin{cases} \dfrac{\partial S}{\partial n} + d(x)S = h(x), & x \in \partial\Omega,\ t > 0, \\ \dfrac{\partial u}{\partial n} + d(x)u = 0, \quad \dfrac{\partial v}{\partial n} + d(x)v = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial\Omega$; $S(x,t)$ denotes the nutrient concentration at time t, and $u(x,t), v(x,t)$ denote the concentrations of the two organisms in the culture vessel, respectively; $f(S,u) = \frac{S}{1+k_1 S + \beta_1 u}$ and $g(S,v) = \frac{S}{1+k_2 S + \beta_2 v}$ are the Beddington-DeAngelis functions. Here, the parameters m_i, k_i, β_i ($i = 1,2$) are positive constants, where m_i ($i = 1,2$) are the maximal growth rates of the two competitors (without an inhibitor), respectively; k_i ($i = 1,2$) are the Michaelis-Menten constants; β_i ($i = 1,2$) model mutual interference between predators u, v respectively; $d(x)$, $h(x)$ are continuous and non-negative on $\partial\Omega$. Let $\Upsilon_1 = \{x \in \partial\Omega : d(x) = 0\}$. We assume that $\Upsilon_1 \neq \varnothing$, $\Upsilon_1 \neq \partial\Omega$ and $h(x)$ is positive on Υ_1.

In this paper, we are interested in the steady-state problem corresponding to the system (1.1):

$$(1.2) \quad \begin{cases} \Delta S - m_1 u f(S,u) - m_2 v g(S,v) = 0, & x \in \Omega, \\ \Delta u + m_1 u f(S,u) = 0, & x \in \Omega, \\ \Delta v + m_2 v g(S,v) = 0, & x \in \Omega, \\ \dfrac{\partial S}{\partial n} + d(x)S = h(x), & x \in \partial\Omega, \\ \dfrac{\partial u}{\partial n} + d(x)u = 0, \quad \dfrac{\partial v}{\partial n} + d(x)v = 0, & x \in \partial\Omega. \end{cases}$$

Let $z = S + u + v$. Then z satisfies

$$\Delta z = 0,\ x \in \Omega, \quad \frac{\partial z}{\partial n} + d(x)z = h(x),\ x \in \partial\Omega.$$

It follows from [29] that $z(x)$ exists uniquely and $z(x) > 0$ on $\overline{\Omega}$. Thus, any steady-state solutions of system (1.2) satisfies

$$S(x) + u(x) + v(x) = z(x), \quad x \in \overline{\Omega}.$$

We shall use this identity to eliminate S and concentrate on the following boundary value problem:

$$(1.3) \quad \begin{cases} \Delta u + m_1 u f(z - u - v, u) = 0, & x \in \Omega, \\ \Delta v + m_2 v g(z - u - v, v) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} + d(x)u = 0, \quad \dfrac{\partial v}{\partial n} + d(x)v = 0, & x \in \partial\Omega, \end{cases}$$

where

$$f(z-u-v,u) = \frac{z-u-v}{1+k_1(z-u-v)+\beta_1 u}, \quad g(z-u-v,v) = \frac{z-u-v}{1+k_2(z-u-v)+\beta_2 v}.$$

Since we only consider the case that S, u, v are nonnegative, we redefine the response functions as follows:

$$\widehat{f}(S,u) = \begin{cases} f(S,u), & S \geq 0,\ u \geq 0, \\ 0, & \text{otherwise,} \end{cases} \quad \widehat{g}(S,u) = \begin{cases} g(S,u), & S \geq 0,\ u \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For convenience, we still denote $\widehat{f}(S,u)$, $\widehat{g}(S,u)$ by $f(S,u)$, $g(S,u)$ respectively.

The contents of this paper is organized as follows. In Section 2, some preliminary results are given, which are used in the later sections. In Section 3, we investigate the stability of trivial and semitrivial solutions by spectral analysis. Moreover, some properties of the functions $\overline{m}_2(m_1)$ and $\overline{m}_1(m_2)$ are provided. Finally, in Section 4, we establish the existence and stability of coexistence solutions to (1.3) bifurcating from a simple eigenvalue and a double eigenvalue.

2. Preliminaries

In this section, we primarily introduce some basic notations and known results which will be used in this paper. For $p > N$, we denote $X_0 = \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial n} + d(x)u = 0 \text{ on } \partial\Omega \right\}$, $X = X_0 \times X_0$, $Y_0 = L^p(\Omega)$, $Y = Y_0 \times Y_0$. We denote the inner product by $(\cdot,\cdot)_2$ in $L^2(\Omega)$.

Consider the following eigenvalue problem

$$(2.1) \qquad \Delta\varphi + \lambda q(x)\varphi = 0,\ x \in \Omega, \quad \frac{\partial\varphi}{\partial n} + d(x)\varphi = 0,\ x \in \partial\Omega.$$

Then it is well known that

Lemma 2.1. [29,32] *Suppose that $q(x) \in C(\overline{\Omega})$ and $q(x) > 0$ on $\overline{\Omega}$. Then the eigenvalues of problem* (2.1) *can be listed in order*

$$0 < \lambda_1(q) < \lambda_2(q) \leq \cdots \to \infty$$

with the corresponding eigenfunctions $\varphi_1, \varphi_2, \ldots$, where the principal eigenvalue $\lambda_1(q)$ is given by

$$\lambda_1(q) = \inf_\varphi \frac{\int_\Omega |\nabla\varphi|^2\ dx + \int_{\partial\Omega} d(x)\varphi^2\ ds}{\int_\Omega q(x)\varphi^2\ dx}$$

and the principal eigenfunction φ_1 is positive on $\overline{\Omega}$. Here, ds represents the unit element on $\partial\Omega$. Moreover, the comparison principle holds: $\lambda_j(q_1) \leq \lambda_j(q_2)$ for $j \geq 1$ if $q_1(x) \geq q_2(x)$ on $\overline{\Omega}$ and the strict inequality holds if $q_1(x) \not\equiv q_2(x)$.

Consider the eigenvalue problem

(2.2) $$\Delta\varphi + q(x)\varphi = \mu\varphi, \ x \in \Omega, \quad \frac{\partial\varphi}{\partial n} + d(x)\varphi = 0, \ x \in \partial\Omega,$$

where $q(x) \in C(\overline{\Omega})$. Let $\mu_1(q)$ be the principal eigenvalue of problem (2.2). Then it is well known that

Lemma 2.2. [32] $\mu_1(q)$ *is strictly increasing with* $q(x)$ *in the sense that if* $q_1(x) \leq q_2(x)$ *and* $q_1(x) \not\equiv q_2(x)$, *then* $\mu_1(q_1) < \mu_1(q_2)$.

Let λ_1, σ_1 be, respectively, the principal eigenvalues of the problems (2.3) and (2.4):

(2.3) $$\Delta\phi + \lambda f(z,0)\phi = 0, \ x \in \Omega, \quad \frac{\partial\phi}{\partial n} + d(x)\phi = 0, \ x \in \partial\Omega,$$

(2.4) $$\Delta\psi + \sigma g(z,0)\psi = 0, \ x \in \Omega, \quad \frac{\partial\psi}{\partial n} + d(x)\phi = 0, \ x \in \partial\Omega$$

with the corresponding principal eigenfunctions $\phi_1, \psi_1 > 0$ on $\overline{\Omega}$, which are uniquely determined by the normalization $\|\phi_1\|_2^2 = 1$ and $\|\psi_1\|_2^2 = 1$.

In (1.3), we let $v = 0$ and $u = 0$ respectively. Then we obtain two scaler equations:

(2.5) $$\Delta u + m_1 u f(z-u, u) = 0, \ x \in \Omega, \quad \frac{\partial u}{\partial n} + d(x)u = 0, \ x \in \partial\Omega,$$

(2.6) $$\Delta v + m_2 v g(z-v, v) = 0, \ x \in \Omega, \quad \frac{\partial v}{\partial n} + d(x)v = 0, \ x \in \partial\Omega.$$

For (2.5), the following results are proved in [27]:

Lemma 2.3. [27] *Suppose that* $m_1 \leq \lambda_1$. *Then* 0 *is the unique nonnegative solution of* (2.5). *Suppose that* $m_1 > \lambda_1$. *Then* (2.5) *has a unique positive solution, denoted by* θ, *satisfying the following properties:*

(1) $0 < \theta < z, \ x \in \overline{\Omega}$.

(2) θ *is continuously differentiable with respect to* $m_1 \in (\lambda_1, +\infty)$, *and is pointwisely increasing with respect to* m_1.

(3) $\lim_{m_1 \to \lambda_1} \theta(m_1) = 0$ *uniformly for* $x \in \overline{\Omega}$, *and* $\lim_{m_1 \to +\infty} \theta(m_1) = z$ *for almost every* $x \in \overline{\Omega}$.

(4) *Let* $L_1 = \Delta + m_1(f(z-\theta, \theta) - \theta f_1'(z-\theta, \theta) + \theta f_2'(z-\theta, \theta))$ *be the linearized operator of* (2.5) *at* θ. *Then all the eigenvalues of* L_1 *are strictly negative.*

Remark 2.4. The results in [27] are only proved in the one-dimensional case, but the results still hold in the N-dimensional case by the similar analysis.

Remark 2.5. For (2.6), we have the same conclusion as Lemma 2.3. If $m_2 \leq \sigma_1$, then 0 is the unique nonnegative solution of (2.6). If $m_2 > \sigma_1$, then (2.6) has a unique positive solution, denoted by Θ. Let $L_2 = \Delta + m_2(g(z-\Theta,\Theta) - \Theta g_1'(z-\Theta,\Theta) + \Theta g_2'(z-\Theta,\Theta))$ be the linearized operator of (2.6) at Θ. Then all the eigenvalues of L_2 are strictly negative.

3. Semitrivial solutions

In this section, we investigate the stability of trivial and semitrivial solutions by examining the spectrum of the corresponding linearized operator. By Theorems 5.1.1 and 5.1.3 in [8], one sees that any solution (u, v) of (1.3) is said to be asymptotically stable if the spectrum of the linearized operator of (1.3) at (u, v) lies in the left-hand side of the imaginary axis. If there are some points in the spectrum with positive real parts, we say that (u, v) is unstable.

The linearized operator of (1.3) at $(0, 0)$ is given by

$$T_0(m_1, m_2) = \begin{pmatrix} \Delta + m_1 f(z, 0) & 0 \\ 0 & \Delta + m_2 g(z, 0) \end{pmatrix}.$$

Thus, we easily obtain

Theorem 3.1. *The trivial solution* $(0, 0)$ *is asymptotically stable if* $m_1 < \lambda_1$ *and* $m_2 < \sigma_1$, *and unstable if* $m_1 > \lambda_1$ *or* $m_2 > \sigma_1$.

The linearized operator of (1.3) at $(\theta, 0)$ is given by

$$T_1(m_1, m_2) = \begin{pmatrix} \Delta + m_1(f(z-\theta,\theta) - \theta f_1'(z-\theta,\theta) + \theta f_2'(z-\theta,\theta)) & -m_1\theta f_1'(z-\theta,\theta) \\ 0 & \Delta + m_2 g(z-\theta,0) \end{pmatrix}$$

$$= \begin{pmatrix} L_1 & -m_1\theta f_1'(z-\theta,\theta) \\ 0 & \Delta + m_2 g(z-\theta,0) \end{pmatrix}.$$

It follows from the Riesz-Schauder theory that the spectrum $\sigma(T_1(m_1, m_2))$ of $T_1(m_1, m_2)$ consists of real eigenvalues. Moreover, we see that $\sigma(T_1(m_1, m_2)) = \sigma(L_1) \cup \sigma(\Delta + m_2 g(z - \theta, 0))$ by Lemma 3.5 in [11]. By Lemma 2.3, one finds that $\sigma(L_1)$ lies on the negative real axis. On the other hand, by Lemma 2.2, $\sigma(\Delta + m_2 g(z - \theta, 0))$ lies on the negative real axis when $m_2 < \overline{m}_2(m_1)$ (note that: $\mu_1(\overline{m}_2(m_1)g(z - \theta, 0)) = 0$), where $\overline{m}_2(m_1)$ is the principal eigenvalue of the following problem

$$(3.1) \qquad \Delta\psi + m_2 g(z-\theta,0)\psi = 0, \ x \in \Omega, \quad \frac{\partial\psi}{\partial n} + d(x)\psi = 0, \ x \in \partial\Omega.$$

Therefore, we have

Theorem 3.2. *Suppose that $m_1 > \lambda_1$. Then the semitrivial solution $(\theta, 0)$ is asymptotically stable if $m_2 < \overline{m}_2(m_1)$ and unstable if $m_2 > \overline{m}_2(m_1)$.*

Next, we follow the ideas used in [11, 31] to analyze some properties of $\overline{m}_2(m_1)$. It follows from Lemma 2.1 that

$$(3.2) \qquad \overline{m}_2(m_1) = \inf_{\psi} \frac{\int_\Omega |\nabla \psi|^2 \, dx + \int_{\partial\Omega} d(x)\psi^2 \, ds}{\int_\Omega g(z - \theta, 0)\psi^2 \, dx}.$$

Lemma 3.3. *The function $\overline{m}_2(m_1)$ defined by (3.2) satisfies*

(1) $\overline{m}_2(m_1) \in C([\lambda_1, +\infty))$ *and* $\overline{m}_2(\lambda_1) = \sigma_1$;

(2) $\overline{m}_2(m_1) \in C^1((\lambda_1, +\infty))$ *and* $\overline{m}_2'(m_1) > 0$;

(3) $\lim_{m_1 \to \lambda_1} \overline{m}_2'(m_1) = \frac{\sigma_1}{\lambda_1} \frac{\int_\Omega f(z,0)\phi_1^2 \, dx}{\int_\Omega (f_1'(z,0) - f_2'(z,0))\phi_1^3 \, dx} \frac{\int_\Omega g_1'(z,0)\phi_1\psi_1^2 \, dx}{\int_\Omega g(z,0)\psi_1^2 \, dx}$.

Proof. Since $\overline{m}_2(m_1)$ is the principal eigenvalue of problem (3.1), we may take the corresponding principal eigenfunction by $\overline{\psi}(m_1)$ such that $\|\overline{\psi}(m_1)\|_2^2 = 1$ and $\overline{\psi}(m_1) > 0$ on $\overline{\Omega}$. Then, the infimum in (3.2) is attained by $\overline{\psi}(m_1)$, and so

$$\overline{m}_2(m_1) = \frac{\int_\Omega |\nabla \overline{\psi}(m_1)|^2 \, dx + \int_{\partial\Omega} d(x)\overline{\psi}^2(m_1) \, ds}{\int_\Omega g(z - \theta_{m_1}, 0)\overline{\psi}^2(m_1) \, dx}$$

$$\leq \frac{\int_\Omega |\nabla \overline{\psi}(m_1 + h)|^2 \, dx + \int_{\partial\Omega} d(x)\overline{\psi}^2(m_1 + h) \, ds}{\int_\Omega g(z - \theta_{m_1}, 0)\overline{\psi}^2(m_1 + h) \, dx}$$

$$= \frac{\overline{m}_2(m_1 + h) \int_\Omega g(z - \theta_{m_1+h}, 0)\overline{\psi}^2(m_1 + h) \, dx}{\int_\Omega g(z - \theta_{m_1}, 0)\overline{\psi}^2(m_1 + h) \, dx},$$

where $\theta_{m_1} = \theta$. This implies

$$\frac{\overline{m}_2(m_1) \int_\Omega g(z - \theta_{m_1}, 0)\overline{\psi}^2(m_1 + h) \, dx}{\int_\Omega g(z - \theta_{m_1+h}, 0)\overline{\psi}^2(m_1 + h) \, dx} \leq \overline{m}_2(m_1 + h).$$

Similarly, if m_1 and $m_1 + h$ are exchanged, then we get

$$\overline{m}_2(m_1 + h) = \frac{\int_\Omega |\nabla \overline{\psi}(m_1 + h)|^2 \, dx + \int_{\partial\Omega} d(x)\overline{\psi}^2(m_1 + h) \, ds}{\int_\Omega g(z - \theta_{m_1+h}, 0)\overline{\psi}^2(m_1 + h) \, dx}$$

$$\leq \frac{\int_\Omega |\nabla \overline{\psi}(m_1)|^2 \, dx + \int_{\partial\Omega} d(x)\overline{\psi}^2(m_1) \, ds}{\int_\Omega g(z - \theta_{m_1+h}, 0)\overline{\psi}^2(m_1) \, dx}$$

$$= \frac{\overline{m}_2(m_1) \int_\Omega g(z - \theta_{m_1}, 0)\overline{\psi}^2(m_1) \, dx}{\int_\Omega g(z - \theta_{m_1+h}, 0)\overline{\psi}^2(m_1) \, dx}.$$

Let $h \to 0$. Then we obtain $\overline{m}_2(m_1) \in C([\lambda_1, +\infty))$. Furthermore, by Lemma 2.3(3), we easily check that $\overline{m}_2(m_1) \to \sigma_1$ as $m_1 \to \lambda_1$ since

$$\overline{m}_2(m_1) = \frac{\int_\Omega \left|\nabla\overline{\psi}(m_1)\right|^2 dx + \int_{\partial\Omega} d(x)\overline{\psi}^2(m_1)\, ds}{\int_\Omega g(z - \theta_{m_1}, 0)\overline{\psi}^2(m_1)\, dx}.$$

Therefore, we complete the proof of (1).

Continuing the calculation above, we find

$$\frac{\overline{m}_2(m_1 + h)\int_\Omega (g(z - \theta_{m_1}, 0) - g(z - \theta_{m_1+h}, 0))\overline{\psi}^2(m_1 + h)\, dx}{\int_\Omega g(z - \theta_{m_1}, 0)\overline{\psi}^2(m_1 + h)\, dx}$$

(3.3) $\qquad \leq \overline{m}_2(m_1 + h) - \overline{m}_2(m_1)$

$$\leq \frac{\overline{m}_2(m_1)\int_\Omega (g(z - \theta_{m_1}, 0) - g(z - \theta_{m_1+h}, 0))\overline{\psi}^2(m_1)\, dx}{\int_\Omega g(z - \theta_{m_1+h}, 0)\overline{\psi}^2(m_1)\, dx}.$$

Dividing both two sides of the inequality (3.3) by h and letting $h \to 0$, we derive

(3.4) $\qquad \overline{m}_2'(m_1) = \overline{m}_2(m_1)\dfrac{\int_\Omega g_1'(z - \theta_{m_1}, 0)\theta_{m_1}'\overline{\psi}^2(m_1)\, dx}{\int_\Omega g(z - \theta_{m_1}, 0)\overline{\psi}^2(m_1)\, dx}$

for almost every $m_1 \in (\lambda_1, \infty)$, where θ_{m_1}' is the differential of θ_{m_1} with respect to m_1. Since $m_1 \to \theta_{m_1}$ is continuously differentiable in $C(\overline{\Omega})$ by Lemma 2.3(2) and $m_1 \to \overline{\psi}(m_1)$ is continuous in $L^2(\Omega)$ by the perturbation result of Kato [12] (see Chap. 4, §5), the right hand of (3.4) is continuous with respect to m_1. So (3.4) holds for all $m_1 \in (\lambda_1, \infty)$. By Lemma 2.3(2) and the expression of $g(z - \theta_{m_1}, 0)$, the positivity of $\overline{m}_2'(m_1)$ is easily derived from (3.4). So the assertion (2) holds.

Finally, by using the identity of (3.4), we shall prove (3). It follows from Lemma 2.3(3) that

(3.5) $\qquad \lim\limits_{m_1 \to \lambda_1} g_1'(z - \theta_{m_1}, 0) = g_1'(z, 0) \quad \text{in } C^1(\overline{\Omega}).$

Moreover, it follows from the results of Kato [12] (see Chap. 4, §5) that

(3.6) $\qquad \lim\limits_{m_1 \to \lambda_1} \overline{\psi}(m_1) = \psi_1 \quad \text{in } L^2(\Omega).$

It remains to show the dependence of θ_{m_1}' on m_1. By Lemma 2.3, we see that θ_{m_1} is the unique nontrivial solution of (2.5) for $m_1 > \lambda_1$. We can apply the Crandall-Rabinowitz bifurcation theory [3] to derive the expression of θ_{m_1} near $m_1 = \lambda_1$. For this, we define the operator $F \colon \mathbb{R} \times X_0 \to Y_0$ by

$$F(m_1; u) = \Delta u + m_1 u f(z - u, u).$$

It is clear that $F(\lambda_1; 0) = 0$, $F_u(\lambda_1; 0) = \Delta + \lambda_1 f(z, 0)$ and $N(F_u(\lambda_1; 0)) = \text{span}\{\phi_1\}$. Since $F_u^*(\lambda_1; 0) = F_u(\lambda_1; 0)$, where $F_u^*(\lambda_1; 0)$ is the adjoint operator of $F_u(\lambda_1; 0)$, we obtain $\dim N(F_u(\lambda_1; 0)) = \text{codim}\, R(F_u(\lambda_1; 0)) = 1$. Moreover, $F_{m_1 u}\phi_1 = f(z, 0)\phi_1 \notin R(F_u(\lambda_1; 0))$. Therefore, by the Crandall-Rabinowitz bifurcation theory [3], there exists a function $(w(s), m_1(s)) \in C^1([-s_0, s_0]; Y_0 \times \mathbb{R})$ for sufficiently small $s_0 > 0$ satisfying $w(0) = 0$, $m_1(0) = \lambda_1$ and $F(m_1(s); u(s)) = 0$, where $u(s) = s(\phi_1 + w(s))$ and $w(s) \in R(F_u(\lambda_1; 0)) \cap X_0$. Due to the uniqueness of nontrivial solution of (2.5) near $m_1 = \lambda_1$, we derive

$$\theta_{m_1}(s) = s(\phi_1 + w(s)).$$

Substituting $(m_1(s); \theta_{m_1}(s))$ into (2.5), dividing by s, differentiating with respect to s and setting $s = 0$, we have

$$\Delta w'(0) + m_1'(0)\phi_1 f(z, 0) + \lambda_1 w'(0)f(z, 0) - \lambda_1 \phi_1^2 (f_1'(z, 0) - f_2'(z, 0)) = 0.$$

Multiplying by ϕ_1 and integrating over Ω by parts, we get

$$(3.7) \qquad m_1'(0) \int_\Omega f(z, 0)\phi_1^2 \, dx = \lambda_1 \int_\Omega (f_1'(z, 0) - f_2'(z, 0))\phi_1^3 \, dx.$$

Due to

$$\frac{d\theta_{m_1}}{ds} = \frac{d\theta_{m_1}}{dm_1} \cdot \frac{dm_1}{ds},$$

we obtain

$$(3.8) \qquad \frac{d\theta_{m_1}}{dm_1}\bigg|_{m_1 = \lambda_1} = \frac{\frac{d\theta_{m_1}}{ds}\big|_{s=0}}{\frac{dm_1}{ds}\big|_{s=0}} = \frac{\phi_1}{m_1'(0)}.$$

Hence, it follows from (3.4), (3.5), (3.6), (3.7) and (3.8) that the assertion (3) holds. □

The stability for $(0, \Theta)$ can be obtained in the same way as that for $(\theta, 0)$. The linearized operator of (1.3) at $(0, \Theta)$ is given by

$$T_2(m_1, m_2) = \begin{pmatrix} \Delta + m_1 f(z - \Theta, 0) & 0 \\ -m_2\Theta g_1'(z - \Theta, \Theta) & L_2 \end{pmatrix},$$

where $L_2 = \Delta + m_2(g(z - \Theta, \Theta) - \Theta g_1'(z - \Theta, \Theta) + \Theta g_2'(z - \Theta, \Theta))$. Moreover,

$$\sigma(T_2(m_1, m_2)) = \sigma(\Delta + m_1 f(z - \Theta, 0)) \cup \sigma(L_2).$$

Denote $\overline{m}_1(m_2)$ be the principal eigenvalue of the following problem

$$\Delta\phi + m_1 f(z - \Theta, 0)\phi = 0, \ x \in \Omega, \quad \frac{\partial\phi}{\partial n} + d(x)\phi = 0, \ x \in \partial\Omega.$$

By Lemma 2.1, we see that

$$(3.9) \qquad \overline{m}_1(m_2) = \inf_{\phi} \frac{\int_{\Omega} |\nabla \phi|^2 \, dx + \int_{\partial \Omega} d(x) \phi^2 \, ds}{\int_{\Omega} f(z - \Theta, 0) \phi^2 \, dx}.$$

Hence, the stability of $(0, \Theta)$ is stated as follows.

Theorem 3.4. *Let $m_2 > \sigma_1$. Then the semitrivial solution $(0, \Theta)$ is asymptotically stable if $m_1 < \overline{m}_1(m_2)$ and unstable if $m_1 > \overline{m}_1(m_2)$.*

Finally, we can prove some properties of the function $\overline{m}_1(m_2)$ in the same way as Lemma 3.3.

Lemma 3.5. *The function $\overline{m}_1(m_2)$ defined by (3.9) satisfies*

(1) $\overline{m}_1(m_2) \in C([\sigma_1, +\infty))$ *and* $\overline{m}_1(\sigma_1) = \lambda_1$;

(2) $\overline{m}_1(m_2) \in C^1((\sigma_1, +\infty))$ *and* $\overline{m}_1'(m_2) > 0$;

(3) $\lim_{m_2 \to \sigma_1} \overline{m}_1'(m_2) = \frac{\lambda_1}{\sigma_1} \frac{\int_{\Omega} f_1'(z,0) \psi_1 \phi_1^2 \, dx}{\int_{\Omega} f(z,0) \phi_1^2 \, dx} \frac{\int_{\Omega} g(z,0) \psi_1^2 \, dx}{\int_{\Omega} (g_1'(z,0) - g_2'(z,0)) \psi_1^3 \, dx}$.

4. Existence and stability of positive solutions

In this section, we establish the existence and stability of positive solutions to (1.3) bifurcating from a simple eigenvalue and a double eigenvalue, respectively.

4.1. Bifurcation from a simple eigenvalue

Wang et al. [25] obtained the local bifurcation result from a simple eigenvalue, and established the global structure of positive solutions [27]. Unfortunately, Dancer [4] pointed out the existence of some gaps in the proofs of Rabinowitz's Theorems 1.27 and 1.40 in [21]. Hence, we reconstruct the proof of global structure of positive solutions by the revised global bifurcation theory developed by López-Gómez in [13].

We first introduce the local bifurcation result from [25]:

Theorem 4.1. *Suppose that $m_1 > \lambda_1$. Then $(\overline{m}_2(m_1); \theta, 0)$ is a bifurcation point for (1.3). Moreover, there exists a curve of non-constant positive solutions $(m_2(s); u(s), v(s))$ for sufficiently small $s > 0$, where $m_2(s) = \overline{m}_2(m_1) + o(s)$, $u(s) = \theta - s(\omega_1 + o(s))$, $v(s) = s(\overline{\psi}(m_1) + o(s))$. Here, ω_1 satisfies $L_1 \omega_1 = -m_1 \theta f_1'(z - \theta, \theta) \overline{\psi}(m_1)$. Moreover, let*

$$I = \int_{\Omega} \left[g_1'(z - \theta, 0)(\overline{\psi}(m_1) - \omega_1) - g_2'(z - \theta, 0) \overline{\psi}(m_1) \right] \overline{\psi}^2(m_1) \, dx.$$

Then the bifurcating solution $(u(s), v(s))$ is local asymptotically stable if $I > 0$.

Let Γ_1 be the positive solution branch bifurcating from $(\overline{m}_2(m_1); \theta, 0)$ (note that the detail definition of Γ_1 can be found in p. 829 of [29]). Then

Theorem 4.2. *Suppose that $m_1 > \lambda_1$. Then Γ_1 joins with the semitrivial branch $\{(m_2; 0, \Theta_{m_2}) \mid m_2 > \sigma_1\}$ at the point $(\widehat{m}_2; 0, \Theta_{\widehat{m}_2})$, where $\widehat{m}_2$ is given uniquely by $m_1 = \overline{m}_1(m_2)$.*

Proof. As the argument of Theorem 1 in [27], one finds that the conditions of Theorem 6.4.3 in [13] hold. We define $Z = \{(u, v) \in X : \int_\Omega -\Delta \overline{\psi}(m_1) \cdot v \, dx = 0\}$, that is, Z is the complement of span $\{(\omega_1, \overline{\psi}(m_1))\}$ in X. Therefore, the continuum $\Gamma_1 - \{(\overline{m}_2(m_1); \theta, 0)\}$ must satisfy one of the three alternatives

(i) joining up with a bifurcation point of the form $(\widetilde{m}_2; \theta, 0)$ where $\widetilde{m}_2 \neq \overline{m}_2(m_1)$;

(ii) joining up with ∞;

(iii) containing points of the form $(m_2, (u, v)) \in \mathbb{R} \times (Z \setminus \{(0,0)\})$.

If $\Gamma_1 - \{(\overline{m}_2(m_1); \theta, 0)\} \subset P$, where $P = \{(u, v) \in X : u > 0, v > 0 \text{ on } \overline{\Omega}\}$, then it is clear that (i) is impossible. By Lemmas 2 and 5 in [27], we see that $0 < u, v, u + v < z$ and $0 < m_2 < A(m_1, k_1, k_2, \beta_1, \beta_2)$. Thanks to L^p estimates, we find $\|u\|_{W^{2,p}(\Omega)}, \|v\|_{W^{2,p}(\Omega)} < K$ for some constant K, which is independent of m_2. So Γ_1 is bounded in $\mathbb{R} \times X$ and (ii) is impossible. Furthermore, if $\Gamma_1 - \{(\overline{m}_2(m_1); \theta, 0)\} \subset P$ holds, then $\int_\Omega -\Delta \overline{\psi}(m_1) \cdot v \, dx = \int_\Omega \overline{m}_2(m_1) g(z - \theta, 0) \overline{\psi}(m_1) \cdot v \, dx > 0$, which implies that (iii) is also impossible. This proves that $\Gamma_1 - \{(\overline{m}_2(m_1); \theta, 0)\} \not\subset P$.

The remaining analysis is similar to the proof of Theorem 1 in [27]. Hence, we complete the whole proof. $\square$

Remark 4.3. Suppose that $m_2 > \sigma_1$. Then $(\overline{m}_1(m_2); 0, \Theta_{m_2})$ is also a bifurcation point for (1.3). Let Γ_2 be the positive solution branch bifurcating from $(\overline{m}_1(m_2); 0, \Theta_{m_2})$. Then Γ_2 joins with the semitrivial branch $\{(m_1; \theta_{m_1}, 0) \mid m_1 > \lambda_1\}$ at the point $(\widehat{m}_1; \theta_{\widehat{m}_1}, 0)$, where $\widehat{m}_1$ is given uniquely by $m_2 = \overline{m}_2(m_1)$.

4.2. Bifurcation from a double eigenvalue

We begin to establish the existence and stability of positive solutions to (1.3) when (m_1, m_2) lies in a neighborhood of (λ_1, σ_1).

For this, we define a nonlinear mapping $H \colon \mathbb{R} \times \mathbb{R} \times X \to Y$ by

$$H(m_1, m_2; U) = \begin{pmatrix} \Delta u + m_1 u f(z - u - v, u) \\ \Delta v + m_2 v g(z - u - v, v) \end{pmatrix} \quad \text{for } U = \begin{pmatrix} u \\ v \end{pmatrix} \in X.$$

It is clear that $H(m_1, m_2; U) = 0$ for all m_1, m_2 and $H_U(\lambda_1, \sigma_1; 0) = T_0(\lambda_1, \sigma_1)$. For convenience, we simply write $T_0(\lambda_1, \sigma_1)$ by T_0 in the following. Clearly, 0 is a double

eigenvalue of T_0 with the corresponding eigenfunctions $\Phi = (\phi_1, 0)^\top$ and $\Psi = (0, \psi_1)^\top$. Hence, we see that $N(T_0) = \text{span}\{\Phi, \Psi\}$ and $\dim N(T_0) = \text{codim } R(T_0) = 2$. Moreover, $(h_1, h_2)^\top \in R(T_0)$ if and only if $(h_1, \phi_1)_2 = (h_2, \psi_1)_2 = 0$.

Obviously, the Crandall-Rabinowitz bifurcation theorem [3] does not work. We now resort to the techniques of space decomposition and implicit function theorem to deal with this case.

First, for $U = (u, v)^\top \in X$, we define the operator P by

$$PU = (u, \phi_1)_2 \Phi + (v, \psi_1)_2 \Psi$$

and decompose X as $X_1 \oplus X_2$ with $X_1 = PX$ and $X_2 = (I-P)X$. Similarly, we decompose Y as $Y = Y_1 \oplus Y_2$ with $Y_1 = PY$ and $Y_2 = (I - P)Y$. Hence, $X_1 = Y_1 = N(T_0)$, $X_2 = R(T_0) \cap X$ and $Y_2 = R(T_0)$.

Next, we apply the implicit function theorem to look for the solutions of $H(m_1, m_2; U) = 0$ in the form:

$$U = s(\cos\omega\Phi + \sin\omega\Psi + W(s)), \ W(s) = (w_1(s), w_2(s))^\top \in X_2,$$

where $s, \omega \in \mathbb{R}$ are parameters. Since we are only concerned with positive solutions, we may restrict ω to $(0, \pi/2)$. For fixed $\omega \in (0, \pi/2)$, we define a nonlinear mapping $K(\alpha, \beta, W; s) \colon \mathbb{R} \times \mathbb{R} \times X_2 \times \mathbb{R} \to Y$ by

$$K(\alpha, \beta, W; s) = s^{-1}H(\lambda_1 + \alpha(s), \sigma_1 + \beta(s), s(\cos\omega\Phi + \sin\omega\Psi + W(s)))$$

$$= T_0 W + \begin{pmatrix} \alpha(\cos\omega\phi_1 + w_1)f(z - u - v, u) \\ \beta(\sin\omega\psi_1 + w_2)g(z - u - v, v) \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda_1(\cos\omega\phi_1 + w_1)(f(z - u - v, u) - f(z, 0)) \\ \sigma_1(\sin\omega\psi_1 + w_2)(g(z - u - v, v) - g(z, 0)) \end{pmatrix},$$

where $u = s(\cos\omega\phi_1 + w_1)$ and $v = s(\sin\omega\psi_1 + w_2)$. It is clear that $K \colon \mathbb{R} \times \mathbb{R} \times X_2 \times \mathbb{R} \to Y$ is a C^1 mapping, which satisfies $K(0, 0, 0; 0) = 0$. By some calculations, we know that the Fréchet derivative of K with respect to (α, β, W) at $(0, 0, 0; 0)$ is the linear mapping

$$K_{(\alpha, \beta, W)}(0, 0, 0; 0)(\widehat{\alpha}, \widehat{\beta}, \widehat{W}) = T_0\widehat{W} + (\widehat{\alpha}\cos\omega f(z, 0))\Phi + \left(\widehat{\beta}\sin\omega g(z, 0)\right)\Psi,$$

where $(\widehat{\alpha}, \widehat{\beta}, \widehat{W}) \in \mathbb{R} \times \mathbb{R} \times X_2$.

To apply the implicit function theorem, we have to verify that $K_{(\alpha, \beta, W)}(0, 0, 0; 0) \colon \mathbb{R} \times \mathbb{R} \times X_2 \times \mathbb{R} \to Y$ is an isomorphism. Suppose that $K_{(\alpha, \beta, W)}(0, 0, 0; 0)(\widehat{\alpha}, \widehat{\beta}, \widehat{W}) = 0$. Then we have

$$T_0\widehat{W} + (\widehat{\alpha}\cos\omega f(z, 0))\Phi + (\widehat{\beta}\sin\omega g(z, 0))\Psi = 0.$$

By the decomposition of Y, we obtain

$$T_0 \widehat{W} = 0 \quad \text{and} \quad (\widehat{\alpha} \cos \omega f(z,0))\Phi + (\widehat{\beta} \sin \omega g(z,0))\Psi = 0.$$

Note that T_0 is an isomorphism from X_2 to Y_2. Then we get $\widehat{W} = 0$. Moreover, since $\omega \in (0, \pi/2)$ and Φ and Ψ are the linearly independent, we see that $\widehat{\alpha} = \widehat{\beta} = 0$. This implies that $K_{(\alpha,\beta,W)}(0,0,0;0)$ is injective. On the other hand, for any $(h,k)^\top \in Y$, we need to look for $(\widehat{\alpha}, \widehat{\beta}, \widehat{W}) \in \mathbb{R} \times \mathbb{R} \times X_2$ such that $K_{(\alpha,\beta,W)}(0,0,0;0)(\widehat{\alpha}, \widehat{\beta}, \widehat{W}) = (h,k)^\top$. Making use of the decomposition of Y again, we find that

$$(h,k)^\top = (h_1, k_1)^\top + (h_2, k_2)^\top,$$

where $(h_1, k_1)^\top \in Y_1$ and $(h_2, k_2)^\top \in Y_2$. Hence, we obtain

$$T_0 \widehat{W} = (h_2, k_2)^\top \quad \text{and} \quad (\widehat{\alpha} \cos \omega f(z,0))\Phi + (\widehat{\beta} \sin \omega g(z,0))\Psi = (h_1, k_1)^\top.$$

Since T_0 is an isomorphism from X_2 to Y_2, we get $\widehat{W} = T_0^{-1}(h_2, k_2)^\top$. Also

$$(\widehat{\alpha} \cos \omega f(z,0))\Phi + (\widehat{\beta} \sin \omega g(z,0))\Psi = (h_1, k_1)^\top$$

is equivalent to $\widehat{\alpha} \cos \omega f(z,0)\phi_1 = h_1$ and $\widehat{\beta} \sin \omega g(z,0)\psi_1 = k_1$, we have

$$\widehat{\alpha} = (\cos \omega f(z,0)\phi_1)^{-1} h_1 \quad \text{and} \quad \widehat{\beta} = (\sin \omega g(z,0)\psi_1)^{-1} k_1.$$

Hence, we can find $(\widehat{\alpha}, \widehat{\beta}, \widehat{W}) \in \mathbb{R} \times \mathbb{R} \times X_2$ such that

$$K_{(\alpha,\beta,W)}(0,0,0;0)(\widehat{\alpha}, \widehat{\beta}, \widehat{W}) = (h,k)^\top,$$

which suggests that $K_{(\alpha,\beta,W)}(0,0,0;0)$ is surjective. Hence, we show that $K_{(\alpha,\beta,W)}(0,0,0;0)\colon \mathbb{R} \times \mathbb{R} \times X_2 \times \mathbb{R} \to Y$ is an isomorphism.

Therefore, by the implicit function theorem, we derive the existence of continuously differentiable functions $(\widehat{\alpha}(s), \widehat{\beta}(s), \widehat{W}(s))$ for sufficiently small $s > 0$ satisfying $\widehat{\alpha}(0) = \widehat{\beta}(0) = 0$, $\widehat{W}(0) = (0,0)$ and $K(\widehat{\alpha}(s), \widehat{\beta}(s), \widehat{W}(s)) = 0$, where $\widehat{W}(s) = (\widehat{w}_1(s), \widehat{w}_1(s))^\top$ satisfies $(\widehat{w}_1(s), \phi_1)_2 = 0$ and $(\widehat{w}_2(s), \psi_1)_2 = 0$. Let

$$(4.1) \qquad \widehat{m}_1(s) = \lambda_1 + \widehat{\alpha}(s), \quad \widehat{m}_2(s) = \sigma_1 + \widehat{\beta}(s),$$

$$(4.2) \qquad \widehat{u}(s) = s(\cos \omega \phi_1 + \widehat{w}_1(s)), \quad \widehat{v}(s) = s(\sin \omega \psi_1 + \widehat{w}_2(s)).$$

Then we get a curve of non-constant positive solutions of $H(m_1, m_2; U) = 0$, which we denote by $(\widehat{m}_1(s), \widehat{m}_2(s); \widehat{U}(s))$ with $\widehat{U}(s) = (\widehat{u}(s), \widehat{v}(s))^\top$.

Therefore, we have the following theorem about the existence of positive solutions to system (1.3).

Theorem 4.4. $(\lambda_1, \sigma_1; (0,0))$ *is a bifurcation point of* $H(m_1, m_2; U) = 0$. *Moreover, there exists a curve of non-constant positive solutions* $(\widehat{m}_1(s), \widehat{m}_2(s); \widehat{U}(s))$ *of* $H(m_1, m_2; U) = 0$ *for sufficiently small* $s > 0$, *where* $\widehat{m}_1(s), \widehat{m}_2(s), \widehat{u}(s), \widehat{v}(s)$ *are given in* (4.1) *and* (4.2).

Now, we investigate the global structure of bifurcating solutions obtained in Theorem 4.4. Substituting $(\widehat{m}_1(s), \widehat{m}_1(s), \widehat{U}(s))$ into (1.3) and dividing by s, we have

$$(4.3) \quad \Delta(\cos\omega\phi_1 + \widehat{w}_1(s)) + (\lambda_1 + \widehat{\alpha}(s))(\cos\omega\phi_1 + \widehat{w}_1(s))f(z - \widehat{u}(s) - \widehat{v}(s), \widehat{u}(s)) = 0,$$

$$(4.4) \quad \Delta(\sin\omega\psi_1 + \widehat{w}_2(s)) + (\sigma_1 + \widehat{\beta}(s))(\sin\omega\psi_1 + \widehat{w}_2(s))g(z - \widehat{u}(s) - \widehat{v}(s), \widehat{v}(s)) = 0.$$

We differentiate (4.3) with respect to s and set $s = 0$ to obtain

$$0 = \Delta\widehat{w}_1'(0) + \widehat{\alpha}'(0)\cos\omega\phi_1 f(z,0) + \lambda_1\widehat{w}_1'(0)f(z,0)$$
$$- \lambda_1\cos\omega\phi_1\left(f_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - f_2'(z,0)\cos\omega\phi_1\right).$$

Multiplying by ϕ_1 and integrating by parts, we get

$$(4.5) \quad \widehat{\alpha}'(0)\int_\Omega f(z,0)\phi_1^2\,dx = \lambda_1\int_\Omega\left(f_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - f_2'(z,0)\cos\omega\phi_1\right)\phi_1^2\,dx.$$

For (4.4), we differentiate with respect to s and set $s = 0$ to derive

$$0 = \Delta\widehat{w}_2'(0) + \widehat{\beta}'(0)\sin\omega\psi_1 g(z,0) + \sigma_1\widehat{w}_2'(0)g(z,0)$$
$$- \sigma_1\sin\omega\psi_1\left(g_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - g_2'(z,0)\sin\omega\psi_1\right).$$

Multiplying by ψ_1 and integrating by parts, we have

$$(4.6) \quad \widehat{\beta}'(0)\int_\Omega g(z,0)\psi_1^2\,dx = \sigma_1\int_\Omega\left(g_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - g_2'(z,0)\sin\omega\psi_1\right)\psi_1^2\,dx.$$

Hence, it follows from (4.5) and (4.6) that

$$(4.7) \quad \begin{aligned}\lim_{s\to0}\frac{\widehat{m}_2(s) - \sigma_1}{\widehat{m}_1(s) - \lambda_1} &= \lim_{s\to0}\frac{\widehat{\beta}(s)}{\widehat{\alpha}(s)} = \lim_{s\to0}\frac{\widehat{\beta}'(s)}{\widehat{\alpha}'(s)} = \frac{\widehat{\beta}'(0)}{\widehat{\alpha}'(0)}\\ &= \frac{\sigma_1}{\lambda_1}\cdot\frac{\int_\Omega(g_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - g_2'(z,0)\sin\omega\psi_1)\psi_1^2\,dx}{\int_\Omega(f_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - f_2'(z,0)\cos\omega\phi_1)\phi_1^2\,dx}\\ &\quad\times\frac{\int_\Omega f(z,0)\phi_1^2\,dx}{\int_\Omega g(z,0)\psi_1^2\,dx}.\end{aligned}$$

Denote the right hand of (4.7) by $l_1(\omega)$ and let $l_2(\omega) = 1/l_1(\omega)$. Then

$$\lim_{\omega\to0}l_1(\omega) = \frac{\sigma_1}{\lambda_1}\cdot\frac{\int_\Omega g_1'(z,0)\phi_1\psi_1^2\,dx}{\int_\Omega(f_1'(z,0) - f_2'(z,0))\phi_1^3\,dx}\cdot\frac{\int_\Omega f(z,0)\phi_1^2\,dx}{\int_\Omega g(z,0)\psi_1^2\,dx}$$

and

$$\lim_{\omega \to \pi/2} l_2(\omega) = \frac{\lambda_1}{\sigma_1} \cdot \frac{\int_\Omega f_1'(z,0)\phi_1^2\psi_1 dx}{\int_\Omega (g_1'(z,0) - g_2'(z,0))\psi_1^3 \, dx} \cdot \frac{\int_\Omega g(z,0)\psi_1^2 \, dx}{\int_\Omega f(z,0)\phi_1^2 \, dx}.$$

By Lemma 3.3, we find that Γ_1 curve, defined by $m_2 = \overline{m}_2(m_1)$, satisfies

$$\tag{4.8} \lim_{m_1 \to \lambda_1} \overline{m}_2'(m_1) = \lim_{\omega \to 0} l_1(\omega).$$

Moreover, by Lemma 3.5, the Γ_2 curve, defined by $m_1 = \overline{m}_1(m_2)$, satisfies

$$\tag{4.9} \lim_{m_2 \to \sigma_1} \overline{m}_1'(m_2) = \lim_{\omega \to \pi/2} l_2(\omega).$$

In view of (4.8) and (4.9), we see that for any $\omega \in (0, \pi/2)$ sufficiently close to zero, $(\widehat{u}(s), \widehat{v}(s))$ derived in Theorem 4.4 coincides with positive solutions which bifurcating from $(\theta_{m_1}, 0)$ at the Γ_1 curve; while, for any $\omega \in (0, \pi/2)$ sufficiently close to $\pi/2$, $(\widehat{u}(s), \widehat{v}(s))$ coincides with the positive solution which bifurcates from $(0, \Theta_{m_2})$ at the Γ_2 curve.

Therefore, we have the following result.

Theorem 4.5. *If (m_1, m_2) lies in the neighborhood of (λ_1, σ_1), then $(\widehat{u}(s), \widehat{v}(s))$ connects the bifurcation solution from the Γ_1 curve with that from the Γ_2 curve.*

Finally, we discuss the asymptotic stability of $(\widehat{u}(s), \widehat{v}(s))$ by the spectral analysis. Consider the following eigenvalue problem

$$\tag{4.10} H_U(\widehat{m}_1(s), \widehat{m}_2(s); \widehat{U}(s))\chi = \gamma(s)\chi,$$

where $\gamma(0) = 0$. We will look for the eigenfunction χ:

$$\tag{4.11} \chi = \Phi + p\Psi + \overline{W}(s), \quad \overline{W}(s) = (\overline{w}_1(s), \overline{w}_2(s))^\top \in X_2,$$

where $p \in \mathbb{R}$ will be provided later and $\overline{W}(s)$ satisfying $\overline{W}(0) = (0, 0)$ can be determined by the implicit function theorem. Substituting (4.11) into (4.10) to get

(4.12)
$$\gamma(s)(\phi_1 + \overline{w}_1) = \Delta(\phi_1 + \overline{w}_1) + (\lambda_1 + \widehat{\alpha})$$
$$\times \left(f(z - \widehat{u} - \widehat{v}, \widehat{u}) - \widehat{u}f_1'(z - \widehat{u} - \widehat{v}, \widehat{u}) + \widehat{u}f_2'(z - \widehat{u} - \widehat{v}, \widehat{u}) \right)(\phi_1 + \overline{w}_1)$$
$$- (\lambda_1 + \widehat{\alpha})\widehat{u}f_1'(z - \widehat{u} - \widehat{v}, \widehat{u})(p\psi_1 + \overline{w}_2),$$

(4.13)
$$\gamma(s)(p\psi_1 + \overline{w}_2) = \Delta(p\psi_1 + \overline{w}_2) + (\sigma_1 + \widehat{\beta})$$
$$\times \left(g(z - \widehat{u} - \widehat{v}, \widehat{v}) - \widehat{v}g_1'(z - \widehat{u} - \widehat{v}, \widehat{v}) + \widehat{v}g_2'(z - \widehat{u} - \widehat{v}, \widehat{v}) \right)(p\psi_1 + \overline{w}_2)$$
$$- (\sigma_1 + \widehat{\beta})\widehat{v}g_1'(z - \widehat{u} - \widehat{v}, \widehat{v})(\phi_1 + \overline{w}_1).$$

Multiplying (4.12) by ϕ_1, integrating by parts and using Green's formula, we derive

$$(4.14) \quad \gamma(s) = \widehat{\alpha}(s) \int_\Omega f(z,0)(\phi_1 + \overline{w}_1)\phi_1 \, dx + (\lambda_1 + \widehat{\alpha}(s)) \int_\Omega \left(N(s)(\Phi + p\Psi + \overline{W}), \Phi \right)_2.$$

For (4.13), we get

$$(4.15) \quad p\gamma(s) = \widehat{\beta}(s) \int_\Omega g(z,0)(p\psi_1 + \overline{w}_2)\psi_1 \, dx + (\sigma_1 + \widehat{\beta}(s)) \int_\Omega \left(N(s)(\Phi + p\Psi + \overline{W}), \Psi \right)_2.$$

In (4.14) and (4.15), $N(s)$ is given by

$$N(s) = \begin{pmatrix} f - \widehat{u}f_1' + \widehat{u}f_2' - f(z,0) & -\widehat{u}f_1' \\ -\widehat{v}g_1' & g - \widehat{v}g_1' + \widehat{v}g_2' - g(z,0) \end{pmatrix},$$

where $f(z - \widehat{u} - \widehat{v}, \widehat{u})$, $f_1'(z - \widehat{u} - \widehat{v}, \widehat{u})$, $f_2'(z - \widehat{u} - \widehat{v}, \widehat{u})$ are replace by f, f_1', f_2' respectively and g, g_1', g_2' are similar. By (4.5) and (4.6), for sufficiently small $s > 0$, we have

$$\widehat{\alpha}(s) = s \frac{\lambda_1 \int_\Omega \left(f_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - f_2'(z,0)\cos\omega\phi_1 \right) \phi_1^2 \, dx}{\int_\Omega f(z,0)\phi_1^2 \, dx} + o(s)$$

and

$$\widehat{\beta}(s) = s \frac{\sigma_1 \int_\Omega \left(g_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - g_2'(z,0)\sin\omega\psi_1 \right) \psi_1^2 \, dx}{\int_\Omega g(z,0)\psi_1^2 \, dx} + o(s).$$

Moreover, straightforward but tedious calculations show that

$$(N(s)\Phi, \Phi)_2 = -s \int_\Omega (2\cos\omega\phi_1 f_1'(z,0) - 2\cos\omega\phi_1 f_2'(z,0) + \sin\omega\psi_1 f_1'(z,0))\phi_1^2 \, dx + o(s),$$

$$(N(s)\Phi, \Psi)_2 = -s \int_\Omega \sin\omega\phi_1\psi_1^2 g_1'(z,0) \, dx + o(s),$$

$$(N(s)\Psi, \Phi)_2 = -s \int_\Omega \cos\omega\phi_1^2\psi_1 f_1'(z,0) \, dx + o(s),$$

$$(N(s)\Psi, \Psi)_2 = -s \int_\Omega (2\sin\omega\psi_1 g_1'(z,0) - 2\sin\omega\psi_1 g_2'(z,0) + \cos\omega\phi_1 g_1'(z,0))\psi_1^2 \, dx + o(s).$$

Set

$$(N(s)\Phi, \Phi)_2 = -sa_{11} + o(s), \qquad (N(s)\Phi, \Psi)_2 = -sa_{12} + o(s),$$
$$(N(s)\Psi, \Phi)_2 = -sa_{21} + o(s), \qquad (N(s)\Psi, \Psi)_2 = -sa_{22} + o(s).$$

Then

$$(4.16)$$
$$\gamma(s) = s \left(\lambda_1 \int_\Omega (f_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - f_2'(z,0)\cos\omega\phi_1)\phi_1^2 \, dx - \lambda_1 a_{11} - \lambda_1 p a_{21} \right)$$
$$+ o(s),$$

(4.17)

$$p\gamma(s) = s\left(p\sigma_1\int_\Omega (g_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - g_2'(z,0)\sin\omega\psi_1)\psi_1^2\,dx - \sigma_1 a_{12} - \sigma_1 p a_{22}\right)$$
$$+ o(s).$$

Substituting (4.16) into (4.17), we get the following equation about p,

(4.18)
$$\lambda_1 a_{21} p^2 + \delta p - \sigma_1 a_{12} + o(1) = 0,$$

where

$$\delta = -\lambda_1\int_\Omega \left(f_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - f_2'(z,0)\cos\omega\phi_1\right)\phi_1^2\,dx + \lambda_1 a_{11}$$
$$+ \sigma_1\int_\Omega \left(g_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - g_2'(z,0)\sin\omega\psi_1\right)\psi_1^2\,dx - \sigma_1 a_{22}.$$

By the expression of a_{11} and a_{22}, we have

$$-\lambda_1\int_\Omega \left(f_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - f_2'(z,0)\cos\omega\phi_1\right)\phi_1^2\,dx + \lambda_1 a_{11}$$
$$= \lambda_1\int_\Omega \left(f_1'(z,0)\cos\omega\phi_1 - f_2'(z,0)\cos\omega\phi_1\right)\phi_1^2\,dx$$

and

$$\sigma_1\int_\Omega \left(g_1'(z,0)(\cos\omega\phi_1 + \sin\omega\psi_1) - g_2'(z,0)\sin\omega\psi_1\right)\psi_1^2\,dx - \sigma_1 a_{22}$$
$$= \sigma_1\int_\Omega \left(-g_1'(z,0)\sin\omega\psi_1 + g_2'(z,0)\sin\omega\psi_1\right)\psi_1^2\,dx.$$

Hence, by a simple rearrangement, we derive

$$\delta = \lambda_1\int_\Omega (f_1'(z,0) - f_2'(z,0))\cos\omega\phi_1^3\,dx + \sigma_1\int_\Omega (g_2'(z,0) - g_1'(z,0))\sin\omega\psi_1^3\,dx$$
$$= \lambda_1\int_\Omega \cos\omega\phi_1^3\frac{1 + \beta_1 z}{(1 + k_1 z)^2}\,dx - \sigma_1\int_\Omega \sin\omega\psi_1^3\frac{1 + \beta_2 z}{(1 + k_2 z)^2}\,dx.$$

It follows from (4.18) that

(4.19)
$$p_\pm = \frac{1}{2\lambda_1 a_{21}}\left(-\delta \pm \sqrt{\delta^2 + 4\lambda_1\sigma_1 a_{21} a_{12}}\right).$$

Substituting (4.19) into (4.16), we get

(4.20)
$$\gamma_\mp = \frac{s}{2}\left(\lambda_1\int_\Omega (f_2'(z,0) - f_1'(z,0))\cos\omega\phi_1^3\,dx + \sigma_1\int_\Omega (g_2'(z,0) - g_1'(z,0))\sin\omega\psi_1^3\,dx\right.$$
$$\left.\mp \sqrt{\delta^2 + 4\lambda_1\sigma_1 a_{21} a_{12}}\right) + o(s)$$
$$= \frac{s}{2}\left(-\lambda_1\int_\Omega \cos\omega\phi_1^3\frac{1 + \beta_1 z}{(1 + k_1 z)^2}\,dx - \sigma_1\int_\Omega \sin\omega\psi_1^3\frac{1 + \beta_2 z}{(1 + k_2 z)^2}\,dx\right.$$
$$\left.\mp \sqrt{\delta^2 + 4\lambda_1\sigma_1 a_{21} a_{12}}\right) + o(s).$$

It is noted that $\lambda_1 \sigma_1 a_{21} a_{12} > 0$ by the expression of a_{21} and a_{12}, and so $\gamma_{\mp}$ is real. Clearly, $\gamma_- < 0$. Thus, we have show the following result.

Theorem 4.6. *The non-constant positive solution* $(\widehat{u}(s), \widehat{v}(s))$ *of* $H(m_1, m_2; U) = 0$ *for sufficiently small* $s > 0$ *is asymptotically stable if* $\gamma_+ < 0$*, where* γ_+ *is given in* (4.20)*.*

Remark 4.7. Note that ϕ_1, ψ_1 are independent of β_1 and β_2. Suppose that β_1 and β_2 are both large. Then

$$a_{12} = \int_\Omega \sin \omega \phi_1 \psi_1^2 \frac{1}{(1 + k_2 z)^2} \, dx < \int_\Omega \sin \omega \psi_1^3 \frac{1 + \beta_2 z}{(1 + k_2 z)^2} \, dx,$$

$$a_{21} = \int_\Omega \cos \omega \phi_1^2 \psi_1 \frac{1}{(1 + k_1 z)^2} \, dx < \int_\Omega \cos \omega \phi_1^3 \frac{1 + \beta_1 z}{(1 + k_1 z)^2} \, dx.$$

Thus,

$$\delta^2 + 4\lambda_1 \sigma_1 a_{21} a_{12} < \left(\lambda_1 \int_\Omega \cos \omega \phi_1^3 \frac{1 + \beta_1 z}{(1 + k_1 z)^2} \, dx + \sigma_1 \int_\Omega \sin \omega \psi_1^3 \frac{1 + \beta_2 z}{(1 + k_2 z)^2} \, dx \right)^2.$$

Consequently, we have

$$
\begin{aligned}
&\lambda_1 \int_\Omega (f_2'(z, 0) - f_1'(z, 0)) \cos \omega \phi_1^3 \, dx + \sigma_1 \int_\Omega (g_2'(z, 0) - g_1'(z, 0)) \sin \omega \psi_1^3 \, dx \\
&\quad + \sqrt{\delta^2 + 4\lambda_1 \sigma_1 a_{21} a_{12}} \\
&< -\lambda_1 \int_\Omega \cos \omega \phi_1^3 \frac{1 + \beta_1 z}{(1 + k_1 z)^2} \, dx - \sigma_1 \int_\Omega \sin \omega \psi_1^3 \frac{1 + \beta_2 z}{(1 + k_2 z)^2} \, dx \\
&\quad + \left(\lambda_1 \int_\Omega \cos \omega \phi_1^3 \frac{1 + \beta_1 z}{(1 + k_1 z)^2} \, dx + \sigma_1 \int_\Omega \sin \omega \psi_1^3 \frac{1 + \beta_2 z}{(1 + k_2 z)^2} \, dx \right) \\
&= 0,
\end{aligned}
$$

and so $\gamma_+ < 0$. Therefore, we see that $\gamma_+ < 0$ if β_1 and β_2 are both large, that is, the non-constant positive solution $(\widehat{u}(s), \widehat{v}(s))$ of $H(m_1, m_2; U) = 0$ for sufficiently small $s > 0$ is asymptotically stable when β_1 and β_2 are both large.

Acknowledgments

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