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Multiplicity of Solutions for Quasilinear $p(x)$-Laplacian Equations in $\mathbb{R}^N$

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Abstract. This paper investigates the multiplicity of solutions for quasilinear elliptic equations with $p(x)$-Laplacian in $\mathbb{R}^N$ by using the nonsmooth critical point theory. We obtain the existence of critical points for nondifferentiable functionals.

1. Introduction

Quasilinear problem with variable exponents has been one of the most interesting research topics in recent years. Lots of literatures dealing with this problem in various function spaces are published. We refer to [1, 12, 14, 20] for details on those results.

The motivation for the study of the problem is of the applications in elastic mechanics, fluid dynamics, image restoration and continuum mechanics. And the appearance of such physical models is facilitated by the development of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$, which are particular cases of the Orlicz and Orlicz-Sobolev ones.

In [2], Alves and Shibo Liu proved the existence of multiple solutions of the following problem

$$(1.1) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + v(x)\,|u|^{p(x)-2}\,u = f(x,u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases}$$

Also Aouaoui in [5] studied the following quasilinear elliptic equation

$$(1.2) \qquad -\operatorname{div}(A(x,u)\nabla u) + \frac{1}{2}A_s(x,u)\,|\nabla u|^2 + (b(x)-\lambda)u = f(x,u), \text{in } \mathbb{R}^N,$$

and they proved the multiplicity of solutions for problem (1.2) by using the nonsmooth critical point theory.

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In this paper, we investigate the existence of infinitely many solutions for the following problem

$$
\begin{aligned}
&-\operatorname{div}(A(x,u)\,|\nabla u|^{p(x)-2}\,\nabla u) + \frac{1}{p(x)}A_s(x,u)\,|\nabla u|^{p(x)} + (b(x)-\lambda)\,|u|^{p(x)-2}\,u \\
&= f(x,u), \quad \text{in } \mathbb{R}^N,
\end{aligned} \tag{1.3}
$$

where $\lambda \in \mathbb{R}$, $p(x)\colon R^N \to R$ is a Lipschitz continuous function with

$$2 < p_- = \inf_{x\in\mathbb{R}^N} p(x) \le p_+ = \sup_{x\in\mathbb{R}^N} p(x) < N,$$

$A_s(x,u) \equiv \frac{\partial A}{\partial s}(x,s)|_{s=u}$, and $b(x)$ is a given continuous function which satisfies

$$b_- > 0, \quad \mu(b^{-1}(-\infty, M]) < +\infty \quad \text{for all } M \in \mathbb{R},$$

here μ is the Lebesgue measure on $\mathbb{R}^N$.

Our aim is to give multiplicity results of weak solutions for problem (1.3), when $f(x,s)$ is odd with respect to s and $A(x,s)$ is even with respect to s. Such solutions for (1.3) will follow from a version of the symmetric mountain pass theorem due to Ambrosetti and Rabinowitz [3].

Define the subspace

$$E = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + b(x)\,|u|^{p(x)})\,dx < +\infty \right\}$$

and the functional $J\colon E \to \mathbb{R}$:

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(A(x,u)\,|\nabla u|^{p(x)} + (b(x)-\lambda)\,|u|^{p(x)} \right) dx - \int_{\mathbb{R}^N} F(x,u)\,dx, \quad \forall u \in E, \tag{1.4}$$

where $F(x,u) = \int_0^u f(x,t)\,dt$.

Clearly, in order to determine the weak solutions of problem (1.3), we need to find the critical points of functional J. The difficulty we have to face is that we cannot work in the classical framework of critical point theory. Under reasonable assumptions on $A(\cdot,\cdot)$ and $f(\cdot,\cdot)$, the functional J may be continuous but not differentiable in the whole space E. However, the Gateaux-derivative of J exists along directions of $E \cap L^\infty(\mathbb{R}^N)$, and

$$
\begin{aligned}
\langle J'(u), v\rangle &= \lim_{t\to 0} \frac{J(u+tv)-J(u)}{t} \\
&= \int_{\mathbb{R}^N} A(x,u)\,|\nabla u|^{p(x)-2}\,\nabla u \nabla v\,dx + \int_{\mathbb{R}^N} \frac{1}{p(x)} A_s(x,u)\,|\nabla u|^{p(x)}\,v\,dx \\
&\quad + \int_{\mathbb{R}^N} (b(x)-\lambda)\,|u|^{p(x)-2}\,uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx,
\end{aligned}
$$

for all $u \in E$ and $v \in E \cap L^\infty(\mathbb{R}^N)$.

By using the nonsmooth critical point theory developed in [9, 10], we can define the critical points in a general sense.

Definition 1.1. A critical point $u \in E$ of J is defined by

$$\langle J'(u), v\rangle = 0, \quad \forall\, v \in E \cap L^\infty(\mathbb{R}^N),$$

i.e.,

$$\begin{aligned}
&\int_{\mathbb{R}^N} A(x,u)\, |\nabla u|^{p(x)-2}\, \nabla u \nabla v\, dx + \int_{\mathbb{R}^N} \frac{1}{p(x)} A_s(x,u)\, |\nabla u|^{p(x)}\, v\, dx \\
&+ \int_{\mathbb{R}^N} (b(x) - \lambda)\, |u|^{p(x)-2}\, uv\, dx - \int_{\mathbb{R}^N} f(x,u) v\, dx \\
&= 0, \quad \forall\, v \in E \cap L^\infty(\mathbb{R}^N).
\end{aligned} \tag{1.5}$$

Throughout this paper, $\|\cdot\|$ denote the norm of E, $u_n \to u$ ($u_n \rightharpoonup u$) means that u_n converges strongly (weakly) in the corresponding spaces. $\hookrightarrow$ stands for a continuous map, and $\hookrightarrow\hookrightarrow$ means a compact embedding map. Let $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$, $p_+ = \sup_{x\in\mathbb{R}^N} p(x)$ and $p_- = \inf_{x\in\mathbb{R}^N} p(x)$.

This paper is divided into five sections. In the second section, we state some hypotheses and the main results of this paper. In the third section, we introduce some basic properties of the generalized Lebesgue-Sobolev spaces and the nonsmooth critical framework. In the fourth section, we give some lemmas which will be used to prove the main results. The proof of Theorem 2.1 is presented in the fifth section.

2. Basic hypotheses and the main results

To state and prove our main results, we consider the following assumptions. Suppose that $N \geq 3$ and $p(x)^* = \frac{Np(x)}{N-p(x)}$.

(H$_1$) The function $p(x)\colon \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz continuous function and satisfies

$$2 < p_- = \inf_{x\in\mathbb{R}^N} p(x) \leq p_+ = \sup_{x\in\mathbb{R}^N} p(x) < N.$$

(H$_2$) $b(x) \in C(\mathbb{R}^N)$, $b_- > 0$, $\mu(b^{-1}(-\infty, M]) < +\infty$ for all $M \in \mathbb{R}$, here μ is the Lebesgue measure on $\mathbb{R}^N$. Note that if $b(x) \in C(\mathbb{R}^N, (0,+\infty))$ is coercive, namely $\lim_{|x|\to+\infty} b(x) = +\infty$, then (H$_2$) is satisfied.

(H$_3$) Let $A(\cdot,\cdot)\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a function such that

- (a) for each $s \in \mathbb{R}$, $A(x,s)$ is measurable with respect to x;
- (b) for a.e. $x \in \mathbb{R}^N$, $A(x,s)$ is of class C^1 with respect to s;
- (c) there exist $0 < \alpha < \beta < +\infty$ such that

$$\alpha \leq A(x,s) \leq \beta, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}, \tag{2.1}$$

$$|A_s(x,s)| \leq \beta, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}. \tag{2.2}$$

(H_4) There exist $r > 0$, $\theta > p_+$, $1 < \gamma < \frac{\theta}{p_+}$ and $\alpha_1 > 0$ such that

$$A_s(x, s)s \geq 0, \quad |s| > r, \tag{2.3}$$

$$\left(\frac{\theta}{p_+} - \gamma\right) A(x, s) - \frac{\gamma}{p_-} A_s(x, s)s \geq \alpha_1, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}. \tag{2.4}$$

(H_5) Let θ be as in (H_4), $f(\cdot, \cdot)\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x, 0) = 0$, a.e. $x \in \mathbb{R}^N$ and

$$0 < \theta F(x, s) \leq f(x, s)s, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \neq 0 \text{ in } \mathbb{R}. \tag{2.5}$$

(H_6) Let r be as in (H_4), the following hold

$$f(x, s) = o(|s|^{p(x)-1}) \text{ uniformly for } x \in \mathbb{R}^N \text{ as } s \to 0 \text{ with } q_- > p_+, \tag{2.6}$$

$$|f(x, s)| \leq C\, |s|^{q(x)-1}\,, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall\, |s| > r, \tag{2.7}$$

for $p(x) \leq q(x) < p^*(x)$.

Set

$$\lambda_* = \inf_{u \in E\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \frac{1}{p(x)} \,|\nabla u|^{p(x)} + b(x)\,|u|^{p(x)} \; dx}{\displaystyle\int_{\mathbb{R}^N} \frac{1}{p(x)} \,|u|^{p(x)} \; dx} > 0.$$

The main result we obtained in the paper is as follows.

Theorem 2.1. *Assume $p(x)$, $b(x)$, $A(x, s)$ and $f(x, s)$ satisfy* (H_1)-(H_6)*. Moreover, let $A(x, -s) = A(x, s)$ and $f(x, -s) = -f(x, s)$, a.e. $x \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$. If there exists a positive number μ such that $\lambda \in (-\infty, \mu\lambda_*)$, then there exists a sequence $\{u_n\} \subset E \cap L^\infty(\mathbb{R}^N)$ of weak solutions of problem* (1.3) *with $J(u_n) \to +\infty$.*

3. Preliminaries

In this section, we recall some results on variable exponent Sobolev spaces and nonsmooth critical point theory which we will use later. For the basic properties of variable exponent spaces and nonsmooth critical point theory, we refer readers to [11, 13, 15, 18] and [8] respectively.

Let $p(x) \in L^\infty(\mathbb{R}^N), p_- > 1$. Define the variable exponent Lebesgue spaces $L^{p(x)}(\mathbb{R}^N)$

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u\colon \mathbb{R}^N \to \mathbb{R} \;\middle|\; u \text{ is a measurable function and } \int_{\mathbb{R}^N} |u|^{p(x)} \; dx < +\infty \right\}.$$

For $u \in L^{p(x)}(\mathbb{R}^N)$, we define the following norms

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 \;\middle|\; \int_{\mathbb{R}^N} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Define the variable exponent Sobolev spaces:

$$W^{1,p(x)}(\mathbb{R}^N) = \left\{ u \colon \mathbb{R}^N \to \mathbb{R} \;\middle|\; u \in L^{p(x)}(\mathbb{R}^N) \text{ and } |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\}$$

which is endowed with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)} .$$

It is easy to check that the spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable reflexive Banach spaces. See [16, 17] for the details.

Proposition 3.1. [14, 20] *Denote*

$$\varphi(u) = \int_{\mathbb{R}^N} |u|^{p(x)} \, dx, \quad u \in L^{p(x)}(\mathbb{R}^N).$$

Then

(1) *For* $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \varphi(\frac{u}{\lambda}) = 1$;

(2) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p_-} \le \varphi(u) \le |u|_{p(x)}^{p_+}$; $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p_+} \le \varphi(u) \le |u|_{p(x)}^{p_-}$;

(3) $|u|_{p(x)} > 1 \ (= 1, < 1) \Leftrightarrow \varphi(u) > 1 \ (= 1, < 1)$;

(4) $|u_n|_{p(x)} \to 0 \Leftrightarrow \varphi(u_n) \to 0$; $|u_n|_{p(x)} \to \infty \Leftrightarrow \varphi(u_n) \to \infty$.

As a consequence of (2), we can obtain

$$|u|_{p(x)} \le \left(\int_{\mathbb{R}^N} |u|^{p(x)} \, dx \right)^{\frac{1}{p_+}} + \left(\int_{\mathbb{R}^N} |u|^{p(x)} \, dx \right)^{\frac{1}{p_-}} . \tag{3.1}$$

For $p(x) \in L^\infty(\mathbb{R}^N)$ with $p_- > 1$, let $q(x) \colon \mathbb{R}^N \to \mathbb{R}$ satisfy

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \text{a.e. } x \in \mathbb{R}^N.$$

We have the following generalized Hölder inequality.

Proposition 3.2. [17, 19] *For any* $u \in L^{p(x)}(\mathbb{R}^N)$ *and* v $inL^{q(x)}(\mathbb{R}^N)$, *we have*

$$\left| \int_{R^N} uv \, dx \right| \le \left(\frac{1}{p_-} + \frac{1}{q_-} \right) |u|_{p(x)} |v|_{q(x)} \le 2 |u|_{p(x)} |v|_{q(x)} .$$

To be concise, we use $a \ll b$ *to denote* $\inf_{x \in \mathbb{R}^N} \{b(x) - a(x)\} > 0$.

Proposition 3.3. [15] *Let $p(x)\colon \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz continuous function which satisfy $1 < p_- \le p_+ < N$, and $q(x)\colon \mathbb{R}^N \to \mathbb{R}$ be a measurable function.*

(1) *If $p(x) \le q(x) \le p^*(x)$, then there is a continuous embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*

(2) *If $p(x) \le q(x) \ll p^*(x)$, then there exists a compact embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{q(x)}_{loc}(\mathbb{R}^N)$.*

Next, we consider the case that $b(x)$ satisfies (H_2). Define the norm

$$\|u\| = \inf\left\{ \lambda > 0 \;\middle|\; \int_{\mathbb{R}^N} \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + b(x)\left|\frac{u}{\lambda}\right|^{p(x)} \right) dx \le 1 \right\}$$

and the subspace

$$E = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + b(x)\,|u|^{p(x)})\, dx < +\infty \right\}.$$

Then $(E, \|\cdot\|)$ is continuously embedding into $W^{1,p(x)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E, \|\cdot\|)$ is also a separable reflexive Banach space.

Similar to the Proposition 3.1, we have

Proposition 3.4. [2] *The functional $\psi\colon W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ defined by*

$$\psi(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + b(x)\,|u|^{p(x)} \right) dx$$

has the following properties:

(1) *$u \ne 0$, $\|u\| = \lambda \Leftrightarrow \psi(\frac{u}{\lambda}) = 1$;*

(2) *$\|u\| > 1 \Rightarrow \|u\|^{p_-} \le \psi(u) \le \|u\|^{p_+}$, $\|u\| < 1 \Rightarrow \|u\|^{p_+} \le \psi(u) \le \|u\|^{p_-}$;*

(3) *$\|u_n\| \to 0 \Leftrightarrow \psi(u_n) \to 0$.*

Lemma 3.5. [2] *If $b(x)$ satisfies (H_2), then*

(i) *we have a compact embedding $E \hookrightarrow\hookrightarrow L^{p(x)}(\mathbb{R}^N)$;*

(ii) *for any measurable function $q\colon \mathbb{R}^N \to \mathbb{R}$ with $p(x) \le q(x) \ll p^*(x)$, we have a compact embedding $E \hookrightarrow\hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*

Suppose (X, d) is a metric space. Now, we introduce some notions of the nonsmooth critical point theory based on which our results are developed (see [8]).

Definition 3.6. Let $f\colon X \to \mathbb{R}$ be a continuous function and $u \in X$. We denote by $|df|(u)$ the supremum of the σ's in $[0,+\infty)$ such that there exist a $\delta > 0$ and a continuous map $H\colon B(u,\delta)\times[0,\delta] \to X$ satisfying

$$d(H(v,t),v) \le t \quad \text{and} \quad f(H(v,t)) \le f(v) - \sigma t$$

for all $(v,t) \in B(u,\delta)\times[0,\delta]$. The extended real number $|df|(u)$ is called the weak slope of f at u.

Definition 3.7. Let $f\colon X \to \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$. We say that f satisfies (P-S)$_c$, i.e., the Palais-Smale condition at level c, if every sequence $\{u_n\}$ in X with $|df|(u_n) \to 0$ and $f(u_n) \to c$ admits a strongly convergent subsequence.

Definition 3.8. Let c be a real number. We say that J satisfies the concrete Palais-Smale condition at level c (denoted by (C-P-S)$_c$) if every subsequence $\{u_n\} \subset E$ satisfying

$$\lim_{n\to\infty} J(u_n) = c \quad \text{and} \quad \left\langle J'(u_n), v\right\rangle = \langle \omega_n, v\rangle, \quad \forall v \in E \cap L^\infty(\mathbb{R}^N),$$

where $\{\omega_n\}$ is a sequence converging to zero in E^*, it is possible to extract a strongly convergent subsequence in E.

Proposition 3.9. *Under assumptions* (H$_1$)-(H$_6$), *if J satisfies* (1.4), *then J is continuous for every $u \in E$, and we have*

$$|dJ|(u) \ge \sup\left\{\left\langle J'(u), v\right\rangle, v \in E\cap L^\infty(\mathbb{R}^N), \|v\| \le 1\right\},$$

where $|dJ|(u)$ denotes the weak slope of J at u.

Proof. The fact that J is continuous. For every $u \in E$ and every $v \in E\cap L^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
\left\langle J'(u), v\right\rangle &= \lim_{t\to 0}\frac{J(u+tv)-J(u)}{t}\\
&= \int_{\mathbb{R}^N} A(x,u)\,|\nabla u|^{p(x)-2}\,\nabla u\nabla v\,dx + \int_{\mathbb{R}^N}\frac{1}{p(x)}A_s(x,u)\,|\nabla u|^{p(x)}\,v\,dx\\
&\quad + \int_{\mathbb{R}^N}(b(x)-\lambda)\,|u|^{p(x)-2}\,uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx.
\end{aligned}$$

Moreover, for every $v \in E\cap L^\infty(\mathbb{R}^N)$, $u \to \langle J'(u), v\rangle$ is continuous.

If $\sup\left\{\langle J'(u), v\rangle, v \in E\cap L^\infty(\mathbb{R}^N), \|v\|\le 1\right\} = 0$, the assertion is true. Otherwise, consider $\sigma > 0$ such that

$$\sup\left\{\left\langle J'(u), v\right\rangle, v \in E\cap L^\infty(\mathbb{R}^N), \|v\| \le 1\right\} > \sigma.$$

Then there exists $v \in E \cap L^{\infty}(\mathbb{R}^N)$ with $\|v\| \leq 1$ such that

$$\big\langle J'(u), v\big\rangle > \sigma.$$

Hence, there exists $\zeta > 0$ such that $\langle J'(\omega), v\rangle > \sigma$ for every $\omega \in B(u, \zeta)$. Taking $\delta = \frac{\zeta}{2}$ and defining a continuous map $H\colon B(u,\delta) \times [0,\delta] \to E$, by $H(\omega, t) = \omega - tv$, it is trivial that $\|H(\omega,t) - \omega\| = \|\omega - tv - \omega\| \leq t$.

On the other hand, by Lagrange mean value theorem, it is easy to see that

$$J(H(\omega,t)) \leq J(\omega) - \sigma t.$$

Therefore $|dJ|(u) \geq \sigma$. We complete the proof by the arbitrariness of σ. $\square$

Proposition 3.10. *Let c be a real number. If J satisfies* (C-P-S)$_c$*, then J satisfies* (P-S)$_c$*.*

Proof. By Proposition 3.9, it is easy to prove and we omit it here. $\square$

4. Basic lemmas

We introduce a fundamental theorem [8], which is an extension of the well-known result for C^1 functionals (see [21]).

Lemma 4.1. *Let X be an infinite-dimensional Banach space and let $f\colon X \to \mathbb{R}$ be a continuous even functional satisfying* (P-S)$_c$ *for every $c \in \mathbb{R}$. Assume that*

(i) *there exist $\rho > 0$, $\alpha > f(0)$ and a subspace $V \subset X$ of finite codimension such that*

$$\forall u \in \{V : \|u\| = \rho\} \Rightarrow f(u) \geq \alpha.$$

(ii) *for every finite-dimensional subspace $W \subset X$, there exists $R > 0$ such that*

$$\forall u \in \{W : \|u\| = R\} \Rightarrow f(u) \leq f(0).$$

Then there exists a sequence $\{c_h\}$ of critical values of f with $c_h \to \infty$.

Lemma 4.2. *Let $\{u_n\}$ be a bounded sequence in E satisfying*

$$\big\langle J'(u_n), v\big\rangle = \langle \omega_n, v\rangle, \quad \forall v \in E \cap L^{\infty}(\mathbb{R}^N), \tag{4.1}$$

with $\{\omega_n\}$ being a sequence converging to zero in E^*. Then there exists $u \in E$ such that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$, and up to a subsequence, $\{u_n\}$ is weakly convergent to u in E. Moreover, we have

$$\big\langle J'(u), v\big\rangle = 0, \quad \forall v \in E \cap L^{\infty}(\mathbb{R}^N). \tag{4.2}$$

Proof. Since $\{u_n\}$ is bounded in E and there is a $u \in E$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } E; \quad u_n \to u \text{ in } L^{q(x)}(\mathbb{R}^N), \quad p(x) \le q(x) < p^*(x); \quad u_n \to u \text{ a.e. in } \mathbb{R}^N.$$

Moreover, since $\{u_n\}$ satisfies (4.1), by Theorem 2.1 of [6], we have, up to a further subsequence,

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N.$$

Consider test functions

$$v_n = \varphi \exp\left\{-Mu_n^+\right\},$$

where $\varphi \in E \cap L^\infty(\mathbb{R}^N)$, $\varphi \ge 0$, and $M \ge 0$. According to (2.1) and (2.2), we have

$$MA(x, u_n) \ge \frac{1}{p_-} |A_s(x, u_n)|.$$

Then (4.1) gives

$$\left\langle J'(u_n), \varphi \exp\left\{-Mu_n^+\right\}\right\rangle = \left\langle \omega_n, \varphi \exp\left\{-Mu_n^+\right\}\right\rangle,$$

i.e.,

(4.3)
$$\begin{aligned}
&\int_{\mathbb{R}^N} A(x, u_n) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi \exp\left\{-Mu_n^+\right\} dx \\
&+ \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} A_s(x, u_n) |\nabla u_n|^{p(x)} - MA(x, u_n) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla u_n^+\right) \\
&\quad \times \varphi \exp\left\{-Mu_n^+\right\} dx \\
&+ \int_{\mathbb{R}^N} (b(x) - \lambda) |u_n|^{p(x)-2} u_n \varphi \exp\left\{-Mu_n^+\right\} dx - \int_{\mathbb{R}^N} f(x, u_n) \varphi \exp\left\{-Mu_n^+\right\} dx \\
= &\left\langle \omega_n, \varphi \exp\left\{-Mu_n^+\right\}\right\rangle.
\end{aligned}$$

By the convergence of $\{u_n\}$, $\omega_n \to 0$ in E^*, recalling (2.6) and (2.7), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} A(x, u_n) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi \exp\left\{-Mu_n^+\right\} dx \\
\to &\int_{\mathbb{R}^N} A(x, u) |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \exp\left\{-Mu^+\right\} dx, \\
&\int_{\mathbb{R}^N} (b(x) - \lambda) |u_n|^{p(x)-2} u_n \varphi \exp\left\{-Mu_n^+\right\} dx \\
\to &\int_{\mathbb{R}^N} (b(x) - \lambda) |u|^{p(x)-2} u \varphi \exp\left\{-Mu^+\right\} dx,
\end{aligned}$$

$$\int_{\mathbb{R}^N} f(x, u_n) \varphi \exp\left\{-Mu_n^+\right\} dx \to \int_{\mathbb{R}^N} f(x, u) \varphi \exp\left\{-Mu^+\right\} dx,$$
$$\left\langle \omega_n, \varphi \exp\left\{-Mu_n^+\right\}\right\rangle \to 0 \text{ as } n \to \infty.$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\frac{1}{p(x)}A_s(x,u_n)\left|\nabla u_n\right|^{p(x)}-MA(x,u_n)\left|\nabla u_n\right|^{p(x)-2}\nabla u_n\nabla u_n^+\right)\\&\quad\times\varphi\exp\left\{-Mu_n^+\right\}\,dx\\&\le 0.\end{aligned}$$

Fatou's Lemma shows that

$$\begin{aligned}&\int_{\mathbb{R}^N}A(x,u)\left|\nabla u\right|^{p(x)-2}\nabla u\nabla\varphi\exp\left\{-Mu^+\right\}\,dx\\&+\int_{\mathbb{R}^N}\left(\frac{1}{p(x)}A_s(x,u)\left|\nabla u\right|^{p(x)}-MA(x,u)\left|\nabla u\right|^{p(x)-2}\nabla u\nabla u^+\right)\varphi\exp\left\{-Mu^+\right\}\,dx\\&+\int_{\mathbb{R}^N}(b(x)-\lambda)\left|u\right|^{p(x)-2}u\varphi\exp\left\{-Mu^+\right\}\,dx-\int_{\mathbb{R}^N}f(x,u)\varphi\exp\left\{-Mu^+\right\}\,dx\\&\ge 0.\end{aligned}\tag{4.4}$$

Next, let $\varphi=\psi g(\frac{u}{n})\exp\{Mu^+\}$ with $\psi\in E\cap L^\infty(\mathbb{R}^N)$, $\psi\ge 0$ and

$$\begin{gathered}g\colon\mathbb{R}\to\mathbb{R},\quad g\in C^1(\mathbb{R}),\ 0\le g\le 1,\\ g=1\text{ on }\left[-\frac{1}{2},\frac{1}{2}\right],\quad g=0\text{ on }(-\infty,-1]\cup[1,+\infty).\end{gathered}$$

Together with (4.4), we have

$$\begin{aligned}&\int_{\mathbb{R}^N}A(x,u)\left|\nabla u\right|^{p(x)-2}\nabla u\nabla\left(\psi g\left(\frac{u}{n}\right)\right)\,dx+\int_{\mathbb{R}^N}\frac{1}{p(x)}A_s(x,u)\left|\nabla u\right|^{p(x)}\psi g\left(\frac{u}{n}\right)\,dx\\&+\int_{\mathbb{R}^N}(b(x)-\lambda)\left|u\right|^{p(x)-2}u\psi g\left(\frac{u}{n}\right)\,dx-\int_{\mathbb{R}^N}f(x,u)\psi g\left(\frac{u}{n}\right)\,dx\\&\ge 0,\quad\forall\psi\in E\cap L^\infty(\mathbb{R}^N).\end{aligned}\tag{4.5}$$

Letting $n\to\infty$, we can obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}A(x,u)\left|\nabla u\right|^{p(x)-2}\nabla u\nabla\psi\,dx+\int_{\mathbb{R}^N}\frac{1}{p(x)}A_s(x,u)\left|\nabla u\right|^{p(x)}\psi\,dx\\&+\int_{\mathbb{R}^N}(b(x)-\lambda)\left|u\right|^{p(x)-2}u\varphi\,dx-\int_{\mathbb{R}^N}f(x,u)\psi\,dx\\&\ge 0,\quad\forall\psi\in E\cap L^\infty(\mathbb{R}^N).\end{aligned}\tag{4.6}$$

Similarly, consider the test functions $v_n=\varphi\exp\{-Mu_n^-\}$, it is easy to prove the opposite inequality. So we have

$$\left\langle J'(u),\psi\right\rangle=0\quad\text{for }\psi\ge 0,\,\psi\in E\cap L^\infty(\mathbb{R}^N).$$

Hence

$$\left\langle J'(u), v\right\rangle = 0, \quad v \in E \cap L^\infty(\mathbb{R}^N).$$

The proof has been completed. □

In (4.2) we only select test functions in $E \cap L^\infty(\mathbb{R}^N)$. The following lemma enlarges the class of admissible test functions. We employ the methods which were introduced in [7].

Lemma 4.3. *Suppose that $u \in E$ satisfies $\langle J'(u), \varphi\rangle = \langle \omega, \varphi\rangle$, $\varphi \in E \cap L^\infty(\mathbb{R}^N)$, where $\omega \in E^*$. If for $v \in E$, there exists $\eta(x) \in L^1(\mathbb{R}^N)$ with the estimate*

$$\frac{1}{p(x)} A_s(x,u)\, |\nabla u|^{p(x)}\, v \geq \eta(x), \quad a.e.\ x \in \mathbb{R}^N, \tag{4.7}$$

then $\frac{1}{p(x)} A_s(x,u)\, |\nabla u|^{p(x)}\, v \in L^1(\mathbb{R}^N)$ and $\langle J'(u), v\rangle = \langle \omega, v\rangle$.

Proof. Let

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

It is clear that for every $v \in E$, $T_k(v) \in E \cap L^\infty(\mathbb{R}^N)$. Then, we have

$$\left\langle J'(u), T_k(v)\right\rangle = \langle \omega, T_k(v)\rangle\,,$$

i.e.,

$$\begin{aligned} &\int_{\mathbb{R}^N} A(x,u)\, |\nabla u|^{p(x)-2}\, \nabla u \nabla T_k(v)\, dx + \int_{\mathbb{R}^N} \frac{1}{p(x)} A_s(x,u)\, |\nabla u|^{p(x)}\, T_k(v)\, dx \\ &\quad + \int_{\mathbb{R}^N} (b(x)-\lambda)\, |u|^{p(x)-2}\, u T_k(v)\, dx - \int_{\mathbb{R}^N} f(x,u) T_k(v)\, dx \\ &= \langle \omega, T_k(v)\rangle\,. \end{aligned} \tag{4.8}$$

Note that

$$\left| A(x,u)\, |\nabla u|^{p(x)-2}\, \nabla u \nabla T_k(v)\right| \leq \left| A(x,u)\, |\nabla u|^{p(x)-2}\, \nabla u \nabla v\right|, \quad \text{a.e. } x \in \mathbb{R}^N,$$

and $A(x,u)\, |\nabla u|^{p(x)-2}\, \nabla u \nabla v \in L^1(\mathbb{R}^N)$. Owing to Lebesgue's dominated convergence theorem, we have

$$\begin{gathered} \int_{\mathbb{R}^N} A(x,u)\, |\nabla u|^{p(x)-2}\, \nabla u \nabla T_k(v)\, dx \to \int_{\mathbb{R}^N} A(x,u)\, |\nabla u|^{p(x)-2}\, \nabla u \nabla v\, dx, \\ \int_{\mathbb{R}^N} (b(x)-\lambda)\, |u|^{p(x)-2}\, u T_k(v)\, dx \to \int_{\mathbb{R}^N} (b(x)-\lambda)\, |u|^{p(x)-2}\, uv\, dx, \\ \int_{\mathbb{R}^N} f(x,u) T_k(v)\, dx \to \int_{\mathbb{R}^N} f(x,u) v\, dx, \\ \langle \omega, T_k(v)\rangle \to \langle \omega, v\rangle\,, \end{gathered}$$

as $k \to \infty$. From (4.7), we have

$$\frac{1}{p(x)} A_s(x,u) \left|\nabla u\right|^{p(x)} T_k(v) \geq -\eta^-(x).$$

Taking inferior limit in (4.8) and applying Fatou's Lemma, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} A(x,u) \left|\nabla u\right|^{p(x)-2} \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} \frac{1}{p(x)} A_s(x,u) \left|\nabla u\right|^{p(x)} v \, dx \\ &+ \int_{\mathbb{R}^N} (b(x)-\lambda) \left|u\right|^{p(x)-2} uv \, dx - \int_{\mathbb{R}^N} f(x,u) v \, dx \\ \leq{}& \langle \omega, v \rangle ,\end{aligned}$$

and $\frac{1}{p(x)} A_s(x,u) \left|\nabla u\right|^{p(x)} v \in L^1(\mathbb{R}^N)$. Finally, Lebesgue's dominated convergence theorem in (4.8) gives

$$\begin{aligned}&\int_{\mathbb{R}^N} A(x,u) \left|\nabla u\right|^{p(x)-2} \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} \frac{1}{p(x)} A_s(x,u) \left|\nabla u\right|^{p(x)} v \, dx \\ &+ \int_{\mathbb{R}^N} (b(x)-\lambda) \left|u\right|^{p(x)-2} uv \, dx - \int_{\mathbb{R}^N} f(x,u) v \, dx \\ ={}& \langle \omega, v \rangle .\end{aligned}$$

The lemma has been proved. □

Remark 4.4. Let $\{u_n\}$ be a sequence in E satisfying (4.1). Then we have

$$\left\langle J'(u_n), u_n \right\rangle = \langle \omega_n, u_n \rangle . \tag{4.9}$$

In the following lemma, we will prove the boundedness of a (C-P-S)$_c$ sequence $\{u_n\} \subset E$ under (2.1), (2.4) and (2.5).

Lemma 4.5. *Let $c \in \mathbb{R}$ and $\{u_n\}$ be a sequence in E satisfying* (4.1) *and*

$$\lim_{n \to +\infty} J(u_n) = c. \tag{4.10}$$

Then $\{u_n\}$ is bounded in E.

Proof. According to (4.9) and (4.10), we have

$$\theta J(u_n) - \gamma \left\langle J'(u_n), u_n \right\rangle \leq C(1 + \|u_n\|),$$

i.e.,

$$\begin{aligned}&\int_{\mathbb{R}^N} \left[\left(\frac{\theta}{p(x)} - \gamma \right) A(x,u_n) \left|\nabla u_n\right|^{p(x)} - \frac{\gamma}{p(x)} A_s(x,u_n) \left|\nabla u_n\right|^{p(x)} u_n \right] dx \\ &+ \int_{\mathbb{R}^N} \left(\frac{\theta}{p(x)} - \gamma \right) (b(x)-\lambda) \left|u_n\right|^{p(x)} \, dx + \int_{\mathbb{R}^N} \gamma f(x,u_n) u_n - \theta F(x,u_n) \, dx \\ \leq{}& C(1 + \|u_n\|).\end{aligned}$$

The combination of (2.4) and (2.5) gives

(4.11)
$$\alpha_1 \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)}\, dx + \left(\frac{\theta}{p_+} - \gamma\right) \int_{\mathbb{R}^N} (b(x)-\lambda)\, |u_n|^{p(x)}\, dx + \theta(\gamma-1) \int_{\mathbb{R}^N} F(x,u_n)\, dx$$
$$\leq C(1+\|u_n\|).$$

Moreover, there exists $C_1(\lambda) > 0$ such that

$$\begin{aligned} &\left(\frac{\theta}{p_+} - \gamma\right) \int_{\mathbb{R}^N} (b(x)-\lambda)\, |u_n|^{p(x)}\, dx \\ &\geq \left(\frac{\theta}{p_+} - \gamma\right) \int_{\mathbb{R}^N} \frac{b(x)}{2}\, |u_n|^{p(x)}\, dx - C_1(\lambda) \int_{\{x|b(x)<2\lambda\}} |u_n|^{p(x)}\, dx. \end{aligned}$$

Denoting $B_\lambda = \left\{x \in \mathbb{R}^N, b(x) < 2\lambda\right\}$ and assuming $\|u_n\| > 1$, we obtain from (4.11) that

$$\begin{aligned} &\min\left\{\alpha_1, \frac{\theta}{2p_+} - \frac{\gamma}{2}\right\} \|u_n\|^{p_-} + \theta(\gamma-1) \int_{\mathbb{R}^N} F(x,u_n)\, dx \\ &\leq C(1+\|u_n\|) + C_1 \int_{B_\lambda} |u_n|^{p(x)}\, dx. \end{aligned} \tag{4.12}$$

By virtue of hypotheses (H_5), we know that there exist $a_0 > 0$ and $b_0 > 0$ such that

$$F(x,s) \geq a_0\, |s|^\theta - b_0, \quad \text{a.e. } x \in B_\lambda \text{ and } \forall s \in \mathbb{R}. \tag{4.13}$$

Then (4.12) and (4.13) yield

$$\begin{aligned} &\min\left(\alpha_1, \frac{\theta}{2p_+} - \frac{\gamma}{2}\right) \|u_n\|^{p_-} + \theta(\gamma-1) a_0\, \|u_n\|^\theta_{L^\theta(B_\lambda)} \\ &\leq C(1+\|u_n\|) + C_1 \int_{B_\lambda} |u_n|^{p(x)}\, dx + b_0\theta(\gamma-1)\, \mathrm{mes}(B_\lambda). \end{aligned} \tag{4.14}$$

On the other hand, by Hölder inequality, we have

$$C_1 \int_{B_\lambda} |u_n|^{p(x)}\, dx \leq C_2\, \|u_n\|^{p_+}_{L^\theta(B_\lambda)},$$

which implies that, for all $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$C_1 \int_{B_\lambda} |u_n|^{p(x)}\, dx \leq C_2\, \|u_n\|^{p_+}_{L^\theta(B_\lambda)} \leq \varepsilon\, \|u_n\|^\theta_{L^\theta(B_\lambda)} + C(\varepsilon). \tag{4.15}$$

Combining (4.14) and (4.15), we get

$$\begin{aligned} &\min\left(\alpha_1, \frac{\theta}{2p_+} - \frac{\gamma}{2}\right) \|u_n\|^{p_-} + (\theta(\gamma-1)a_0 - \varepsilon)\, \|u_n\|^\theta_{L^\theta(B_\lambda)} \\ &\leq C(1+\|u_n\|) + b_0\theta(\gamma-1)\, \mathrm{mes}(B_\lambda) + C_\varepsilon. \end{aligned}$$

So for $0 < \varepsilon < \theta(\gamma-1)a_0$, $\{u_n\}$ is bounded in E. □

Lemma 4.6. *Let $\{u_n\}$ be a sequence as in Lemma 4.5. Then $\{u_n\}$, up to a subsequence, converges strongly to u in E.*

Proof. Consider the function

$$\zeta(s) = \begin{cases} M\,|s|\,, & \text{if } |s| \le r, \\ MR, & \text{if } |s| > r, \end{cases}$$

where $M = \frac{\beta}{\alpha p_-}$. It is easy to prove $\left\{u_n e^{\zeta(u_n)}\right\}$ is bounded in E, up to a subsequence, having

$$\begin{aligned} &u_n e^{\zeta(u_n)} \rightharpoonup u e^{\zeta(u)} \text{ in } E, \\ &u_n e^{\zeta(u_n)} \to u e^{\zeta(u)} \text{ in } L^{q(x)}(\mathbb{R}^N), \quad p(x) \le q(x) < p^*(x), \\ &u_n e^{\zeta(u_n)} \to u e^{\zeta(u)} \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

From Lemma 4.3, we obtain $\left\langle J'(u_n), u_n e^{\zeta(u_n)}\right\rangle = \left\langle \omega_n, u_n e^{\zeta(u_n)}\right\rangle$, i.e.,

$$\begin{aligned} &\int_{\mathbb{R}^N} A(x,u_n)\,|\nabla u_n|^{p(x)}\, e^{\zeta(u_n)}\,dx \\ &+ \int_{\mathbb{R}^N} \left(A(x,u_n)\zeta'(u_n) + \frac{1}{p(x)} A_s(x,u_n)\right) |\nabla u_n|^{p(x)}\, u_n e^{\zeta(u_n)}\,dx \\ &+ \int_{\mathbb{R}^N} (b(x)-\lambda)\,|u_n|^{p(x)}\, e^{\zeta(u_n)}\,dx - \int_{\mathbb{R}^N} f(x,u_n) u_n e^{\zeta(u_n)}\,dx \\ = &\left\langle \omega_n, u_n e^{\zeta(u_n)}\right\rangle. \end{aligned} \tag{4.16}$$

We claim that

$$\left(A(x,u_n)\zeta'(u_n) + \frac{1}{p(x)} A_s(x,u_n)\right) |\nabla u_n|^{p(x)}\, u_n e^{\zeta(u_n)} \ge 0.$$

In fact, when $|u_n| \ge r$, we have

$$\begin{aligned} &\left(A(x,u_n)\zeta'(u_n) + \frac{1}{p(x)} A_s(x,u_n)\right) |\nabla u_n|^{p(x)}\, u_n e^{\zeta(u_n)} \\ &= \frac{1}{p(x)} A_s(x,u_n)\,|\nabla u_n|^{p(x)}\, u_n e^{\zeta(u_n)} \\ &\ge 0. \end{aligned}$$

When $0 \le u_n < r$, we get

$$\begin{aligned} &\left(A(x,u_n)\zeta'(u_n) + \frac{1}{p(x)} A_s(x,u_n)\right) |\nabla u_n|^{p(x)}\, u_n e^{\zeta(u_n)} \\ &\ge \left(M\alpha - \frac{\beta}{p_+}\right) |\nabla u_n|^{p(x)}\, u_n e^{\zeta(u_n)} \\ &\ge 0. \end{aligned}$$

When $-r < u_n < 0$, we obtain

$$\begin{aligned}&\left(A(x,u_n)\zeta'(u_n)+\frac{1}{p(x)}A_s(x,u_n)\right)|\nabla u_n|^{p(x)}u_ne^{\zeta(u_n)}\\ &\geq\left(\frac{\beta}{p_-}-M\alpha\right)|\nabla u_n|^{p(x)}u_ne^{\zeta(u_n)}\\ &\geq 0.\end{aligned}$$

Thanks to Lemma 4.2, $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$. Then by Fatou's Lemma, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(A(x,u)\zeta'(u)+\frac{1}{p(x)}A_s(x,u)\right)|\nabla u|^{p(x)}ue^{\zeta(u)}\,dx\\ &\leq\varliminf_{n\to\infty}\left(\int_{\mathbb{R}^N}\left(A(x,u_n)\zeta'(u_n)+\frac{1}{p(x)}A_s(x,u_n)\right)|\nabla u_n|^{p(x)}u_ne^{\zeta(u_n)}\,dx\right).\end{aligned}\tag{4.17}$$

Moreover, recalling (2.6) and (2.7), by the convergence of $\left\{u_ne^{\zeta(u_n)}\right\}$, and Fatou's Lemma, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p(x)}e^{\zeta(u_n)}\,dx=\int_{\mathbb{R}^N}|u|^{p(x)}e^{\zeta(u)}\,dx,\tag{4.18}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u_n)u_ne^{\zeta(u_n)}\,dx=\int_{\mathbb{R}^N}f(x,u)ue^{\zeta(u)}\,dx.\tag{4.19}$$

By Lemma 4.2, we know that u is a critical point of the functional J. Then owing to Lemma 4.3, we obtain $\left\langle J'(u), ue^{\zeta(u)}\right\rangle = 0$, i.e.,

(4.20)
$$\begin{aligned}&\int_{\mathbb{R}^N}A(x,u)|\nabla u|^{p(x)}e^{\zeta(u)}\,dx+\int_{\mathbb{R}^N}\left(A(x,u)\zeta'(u)+\frac{1}{p(x)}A_s(x,u)\right)|\nabla u|^{p(x)}ue^{\zeta(u)}\,dx\\ &+\int_{\mathbb{R}^N}(b(x)-\lambda)|u|^{p(x)}e^{\zeta(u)}\,dx-\int_{\mathbb{R}^N}f(x,u)ue^{\zeta(u)}\,dx\\ &=0.\end{aligned}$$

Using (4.17)-(4.20) and taking superior limit in (4.16), we have

$$\begin{aligned}&\varlimsup_{n\to\infty}\left(\int_{\mathbb{R}^N}A(x,u_n)|\nabla u_n|^{p(x)}e^{\zeta(u_n)}\,dx+\int_{\mathbb{R}^N}b(x)|u_n|^{p(x)}e^{\zeta(u_n)}\,dx\right)\\ &=\varlimsup_{n\to\infty}\left(\int_{\mathbb{R}^N}-\left(A(x,u_n)\zeta'(u_n)+\frac{1}{p(x)}A_s(x,u_n)\right)|\nabla u_n|^{p(x)}u_ne^{\zeta(u_n)}\,dx\right.\\ &\qquad\left.+\int_{\mathbb{R}^N}\lambda|u_n|^{p(x)}e^{\zeta(u_n)}\,dx+\int_{\mathbb{R}^N}f(x,u_n)u_ne^{\zeta(u_n)}\,dx\right)\\ &\leq-\int_{\mathbb{R}^N}\left(A(x,u)\zeta'(u)+\frac{1}{p(x)}A_s(x,u)\right)|\nabla u|^{p(x)}ue^{\zeta(u)}\,dx\\ &\quad+\int_{\mathbb{R}^N}\lambda|u|^{p(x)}e^{\zeta(u)}\,dx+\int_{\mathbb{R}^N}f(x,u)ue^{\zeta(u)}\,dx\\ &=\int_{\mathbb{R}^N}A(x,u)|\nabla u|^{p(x)}e^{\zeta(u)}\,dx+\int_{\mathbb{R}^N}b(x)|u|^{p(x)}e^{\zeta(u)}\,dx.\end{aligned}$$

Then, Fatou's Lemma indicates that

$$\begin{aligned}
&\int_{\mathbb{R}^N} A(x,u)\,|\nabla u|^{p(x)}\,e^{\zeta(u)}\,dx + \int_{\mathbb{R}^N} b(x)\,|u|^{p(x)}\,e^{\zeta(u)}\,dx \\
&\le \varliminf_{n\to\infty}\left(\int_{\mathbb{R}^N} A(x,u_n)\,|\nabla u_n|^{p(x)}\,e^{\zeta(u_n)}\,dx + \int_{\mathbb{R}^N} b(x)\,|u_n|^{p(x)}\,e^{\zeta(u_n)}\,dx\right) \\
&\le \varlimsup_{n\to\infty}\left(\int_{\mathbb{R}^N} A(x,u_n)\,|\nabla u_n|^{p(x)}\,e^{\zeta(u_n)}\,dx + \int_{\mathbb{R}^N} b(x)\,|u_n|^{p(x)}\,e^{\zeta(u_n)}\,dx\right) \\
&\le \int_{\mathbb{R}^N} A(x,u)\,|\nabla u|^{p(x)}\,e^{\zeta(u)}\,dx + \int_{\mathbb{R}^N} b(x)\,|u|^{p(x)}\,e^{\zeta(u)}\,dx.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} A(x,u_n)\,|\nabla u_n|^{p(x)}\,e^{\zeta(u_n)}\,dx + \int_{\mathbb{R}^N} b(x)\,|u_n|^{p(x)}\,e^{\zeta(u_n)}\,dx\right) \\
&= \int_{\mathbb{R}^N} A(x,u)\,|\nabla u|^{p(x)}\,e^{\zeta(u)}\,dx + \int_{\mathbb{R}^N} b(x)\,|u|^{p(x)}\,e^{\zeta(u)}\,dx.
\end{aligned}$$

Since

$$\frac{A(x,u_n)\,|\nabla u_n|^{p(x)}\,e^{\zeta(u_n)}}{\alpha} + b(x)\,|u_n|^{p(x)}\,e^{\zeta(u_n)} \ge |\nabla u_n|^{p(x)} + b(x)\,|u_n|^{p(x)},$$

it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} + b(x)\,|u_n|^{p(x)}\,dx = \int_{\mathbb{R}^N} |\nabla u|^{p(x)} + b(x)\,|u|^{p(x)}\,dx. \tag{4.21}$$

On the other hand, Lebesgue's dominated convergence theorem and the weak convergence of $\{u_n\}$ to u in E show

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\,\nabla u_n\nabla u\,dx = \int_{\mathbb{R}^N} |\nabla u|^{p(x)}\,dx, \tag{4.22}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\,\nabla u\nabla u_n\,dx = \int_{\mathbb{R}^N} |\nabla u|^{p(x)}\,dx, \tag{4.23}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} b(x)\,|u_n|^{p(x)-2}\,u_n u\,dx = \int_{\mathbb{R}^N} b(x)\,|u|^{p(x)}\,dx, \tag{4.24}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} b(x)\,|u|^{p(x)-2}\,u u_n\,dx = \int_{\mathbb{R}^N} b(x)\,|u|^{p(x)}\,dx. \tag{4.25}$$

The combination of (4.21)-(4.25) gives

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n-\nabla u)\,dx\right.\\
&\qquad\left.+\int_{\mathbb{R}^N}b(x)\left(|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right)(u_n-u)\,dx\right)\\
&=\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}-|\nabla u_n|^{p(x)-2}\nabla u_n\nabla u-|\nabla u|^{p(x)-2}\nabla u\nabla u_n\right)\,dx\right.\\
&\qquad\left.+\int_{\mathbb{R}^N}\left(b(x)\,|u_n|^{p(x)}+b(x)\,|u|^{p(x)}-b(x)\,|u_n|^{p(x)-2}u_nu-b(x)\,|u|^{p(x)-2}uu_n\right)\,dx\right)\\
&=0.
\end{aligned}
$$

It is observed now that (see [4,14]) for ξ and η in $\mathbb{R}^N$, we have the following estimates

$$
\left[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\right]^{\frac{p}{2}}(|\xi|^p+|\eta|^p)^{\frac{2-p}{2}}\geq(p-1)\,|\xi-\eta|^p\quad\text{for }1<p<2;
$$

$$
\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta)\geq 2^{-p}\,|\xi-\eta|^p\quad\text{for }p>2.
$$

Therefore

$$
\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n-\nabla u|^{p(x)}\,dx+\int_{\mathbb{R}^N}b(x)\,|u_n-u|^{p(x)}\,dx\right)=0,
$$

which implies that $\{u_n\}$ converges strongly to u in E. □

Lemma 4.7. *For every real number c, the functional J satisfies* (C-P-S)$_c$.

Proof. Let $\{u_n\}$ be a sequence on E satisfying (4.1) and (4.10). By Lemma 4.5, $\{u_n\}$ is bounded in E. The conclusion can be deduced from Lemma 4.6. □

5. Proof of Theorem 2.1

The functional J is continuous and even. Moreover, by Proposition 3.10 and Lemma 4.7, for every $c\in\mathbb{R}$, J satisfies (P-S)$_c$. On the other hand, by (2.1), (2.5), (2.6) and (2.7), for $u\in E$, we get $\|u\|<1$. Moreover, we have

$$
\begin{aligned}
J(u)&=\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(A(x,u)\,|\nabla u|^{p(x)}+(b(x)-\lambda)\,|u|^{p(x)}\right)\,dx-\int_{\mathbb{R}^N}F(x,u)\,dx\\
&\geq\frac{\min\{1,\alpha\}}{p_+}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+b(x)\,|u|^{p(x)}\right)\,dx-\int_{\mathbb{R}^N}\frac{\lambda}{p(x)}\,|u|^{p(x)}\,dx\\
&\quad-\frac{1}{\theta}\int_{\mathbb{R}^N}f(x,u)u\,dx\\
&\geq\frac{\min\{1,\alpha\}}{p_+}\,\|u\|^{p_+}-\int_{\mathbb{R}^N}\frac{\lambda}{p(x)}\,|u|^{p(x)}\,dx-\varepsilon\,\|u\|^{p_-}-C(\varepsilon)\,\|u\|^{q_-}\,.
\end{aligned}\tag{5.1}
$$

In the case of $\lambda \le 0$, we have

$$J(u) \ge \frac{\min\{1,\alpha\}}{p_+} \|u\|^{p_+} - \varepsilon \|u\|^{p_-} - C(\varepsilon) \|u\|^{q_-} . \tag{5.2}$$

For $\lambda > 0$, by the definition of λ_*, we have

$$J(u) \ge \left(\min\{1,\alpha\} - \frac{\lambda}{\lambda_*}\right) \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\right) dx - \varepsilon \|u\|^{p_-} - C(\varepsilon) \|u\|^{q_-} .$$

If $0 < \lambda < \min\{1,\alpha\}\lambda_*$ (i.e., $\mu = \min\{1,\alpha\}$), then

$$J(u) \ge \frac{\min\{1,\alpha\} - \frac{\lambda}{\lambda_*}}{p_+} \|u\|^{p_+} - \varepsilon \|u\|^{p_-} - C(\varepsilon) \|u\|^{q_-} . \tag{5.3}$$

Therefore, if λ satisfies $0 < \lambda < \min\{1,\alpha\}\lambda_*$, then there exist $\rho > 0$ small enough and $\delta > 0$ such that

$$J(u) \ge \delta \quad \text{for } \|u\| = \rho > 0.$$

Hence, the condition (i) of Lemma 4.1 holds with $V = E$.

Now, we consider a finite-dimensional subspace W of E. For any $u \in W$ with $\|u\| > 1$, from (2.1), we have

$$J(u) \le \frac{\max\{1,\beta\}}{p_-} \|u\|^{p_+} - \int_{\mathbb{R}^N} \frac{\lambda}{p(x)} |u|^{p(x)} - \int_{\mathbb{R}^N} F(x,u)\, dx. \tag{5.4}$$

By (2.5) and (2.6), we know that there exist $z(x) \in L^\infty(\mathbb{R}^N)$ satisfying $z(x) > 0$, a.e. $x \in \mathbb{R}^N$ and a positive constant C_2 such that

$$F(x,s) \ge z(x)|s|^\theta - C_2 |s|^{p(x)}, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall s \in \mathbb{R}. \tag{5.5}$$

Combining (5.4) and (5.5), we obtain

$$J(u) \le \frac{\max\{1,\beta\} + |\lambda|}{p^-} \|u\|^{p_+} - \int_{\mathbb{R}^N} z(x)|u|^\theta \, dx + C_2 \int_{\mathbb{R}^N} |u|^{p(x)}\, dx. \tag{5.6}$$

Observe that $u \mapsto \left(\int_{\mathbb{R}^N} z(x)|u|^\theta\right)^{\frac{1}{\theta}}$ is a norm on W. Since W is finite-dimensional, it follows that all norms of W are equivalent. There exists $C_3 > 0$ such that

$$C_3 \|u\|^\theta \le \int_{\mathbb{R}^N} z(x)|u|^\theta \, dx.$$

From (5.6) we have

$$J(u) \le \frac{\max\{1,\beta\} + |\lambda|}{p^-} \|u\|^{p_+} - C_3 \|u\|^\theta_{L^\theta(\mathbb{R}^N)} + C_2 |u|^{p_+}_{L^{p(x)}(\mathbb{R}^N)} .$$

Since $\theta > p_+$, then there exists $R > 1$ such that $J(u) < 0$ when $\|u\| = R$. So the condition (ii) of Lemma 4.1 holds. Applying Lemma 4.1, the conclusion follows.

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