TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 6, pp. 1759-1775, December 2015
DOI: 10.11650/tjm.19.2015.5968
This paper is available online at http://journal.taiwanmathsoc.org.tw

POSITIVE SOLUTIONS FOR ELLIPTIC EQUATIONS IN TWO DIMENSIONS ARISING IN A THEORY OF THERMAL EXPLOSION

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Abstract. In this paper we study a mathematical model of thermal explosion which is described by the boundary value problem

$$\begin{cases} -\Delta u = \lambda e^{u^\alpha}, & x \in \Omega, \\ \mathbf{n} \cdot \nabla u + g(u)u = 0, & x \in \partial\Omega, \end{cases}$$

where the constant $\alpha \in (0, 2]$, $g : [0, \infty) \to (0, \infty)$ is an nondecreasing C^1 function, Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a bifurcation parameter. Using variational methods we show that there exists $0 < \Lambda < \infty$ such that the problem has at least two positive solutions if $0 < \lambda < \Lambda$, no solution if $\lambda > \Lambda$ and at least one positive solution when $\lambda = \Lambda$.

1. Introduction and Main Results

A classical problem in combustion theory is a model of thermal explosion which occurs due to a spontaneous ignition in a rapid combustion process. In this paper, we consider a model involving a nonlinear boundary heat loss which is not a very typical one in classical combustion theory, but is relevant to some more recent applications (see [14] for details). The model reads as:

$$\text{(T)} \qquad \begin{cases} \theta_t - \Delta\theta = f(\theta), & (t, x) \in (0, T) \times \Omega, \\ \mathbf{n} \cdot \nabla\theta + g(\theta)\theta = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ \theta(0, x) = \theta_0, & x \in \Omega. \end{cases}$$

Here θ is the appropriately scaled temperature in a bounded smooth domain Ω in $\mathbb{R}^2$ and $f(\theta)$ is the normalized reaction rate which take the form $f(\theta) = e^\theta$ and is called the Frank-Kamenetskii rate [24]. More generally, throughout this paper, we consider

Received February 25, 2015, accepted May 21, 2015.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification*: 35J66, 35K57.
Key words and phrases: Combustion theory, Semilinear elliptic equations, Exponential nonlinearity.

the reaction term to be of the form $f(\theta) = e^{\theta^\alpha}$ for $\alpha \in (0, 2]$. The initial condition θ_0 is assumed to be bounded and nonnegative so that a classical solution of (T) exists on a maximal interval $(0, T_m)$ (see [7] and Remark 2.1 in [14]). On the C^2 boundary $\partial\Omega$, with the outward unit normal denoted by $\mathbf{n}$, the heat-loss parameter $g(\theta)$ is assumed to satisfy the following hypothesis:

(H1) $g : [0, \infty) \to (0, \infty)$ is a nondecreasing bounded C^1 function.

Physically this assumption means that a heat loss through the boundary always exists and increases linearly with the temperature even in the small temperature regime. We further assume

(H2) there exists a constant $m > 0$ such that $0 \le sg'(s) + g(s) \le m$ for all $s \ge 0$.

A bifurcation (or scaling) parameter $\lambda > 0$ can be associated with the size of domain Ω in (T) which grows linearly as the measure of Ω increases. It is well known that, after normalizing for the size of Ω, the long term behavior of solution of (T) is close to the solution of the time-independent problem:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \lambda e^{u^\alpha}, & x \in \Omega, \\ \mathbf{n} \cdot \nabla u + g(u)u = 0, & x \in \partial\Omega. \end{cases}$$

As a first step in the analysis of thermal explosion described by the dynamic problem (T), we analyze the corresponding stationary problem (P_λ).

In case of Dirichlet boundary condition, existence results for the stationary problem have been established in [1, 11], and for discussion regarding multiplicity of solutions to this problem we refer to [5, 18, 19].

Related existence and multiplicity results for the stationary problem with Neumann boundary condition have been established in [4] and [20]. In these works, the authors have studied the case when $f(u) = u^p - u$ in $\mathbb{R}^{\mathbb{N}}, N \ge 3, 1 < p \le \frac{N+2}{N-2}$ and $f(u) = e^{u^\alpha} - u$ in $\mathbb{R}^2, 0 < \alpha \le 2$ respectively, under the Neumann boundary condition corresponding to the choice $g(u)u = -u^q$ where $0 < q < 1$.

The main difficulty to analyse (P_λ) is that the coercive term like u is not added to the PDE. But coercivity is induced by the boundary condition from the assumption $g(u)u$ is strictly positive. This motivates us to define an equivalent norm in $H^1(\Omega)$ (defined in (2.5)) with respect to which the energy functional corresponding to (P_λ) become easier to analyse.

Finally, we state the theorem we will prove:

Theorem 1.1. *There exists a $\Lambda > 0$ such that (P_λ) has at least two positive solutions for all $\lambda \in (0, \Lambda)$, at least one positive solution for $\lambda = \Lambda$ and no positive solution for any $\lambda > \Lambda$.*

Remark 1.1. Thermal explosion is understood mathematically as the absence of a global (in time) solution for the problem (T) with an arbitrary initial data $\theta_0 \geq 0$.

(i) We note that if u_λ is a classical solution of (P_λ) then the existence of a global solution of (T) follows immediately from the maximum principle [21]. Hence, when $\lambda < \Lambda$, for any $\theta_0 \in L^\infty(\Omega)$ with $0 \leq \theta_0 \leq u_\lambda$ the solution θ of (T) with $\theta(0) = \theta_0$ is global i.e., the phenomenon of thermal explosion is ruled out by the model.

(ii) When $\lambda > \Lambda$, correspondingly, the solution θ of (T) blows up in finite time for any initial data $\theta_0 \geq 0$ resulting in the phenomenon of combustion.

(iii) The result in theorem 1.1 can be seen to be physically consistent in the following sense. When the domain is relatively small ($\lambda \leq \Lambda$), the heat loss through the boundary dominates the chemical reaction inside the domain and hence a stationary equilibrium temperature distribution is possible. However, when the size of domain is large ($\lambda > \Lambda$), the rapid reaction inside the domain dominates and results in the phenomenon of combustion.

(iv) In a general way, the problem (P_λ) may be thought of as an instance of convex-concave type problems whose study was initiated in the influential work of Ambrosetti-Brezis-Cerami [3].

The paper is organized as follows. In Section 2, we include some preliminaries. In Section 3, we show the existence of local minimum of I_λ for small λ, and in Section 4 we prove the existence of a minimizer u_λ of I_λ in C^1 topology for maximal range of λ and then that $I_\lambda(u_\lambda)$ is in fact a local minimum in $H^1(\Omega)$. In this context, we refer to the work of Brezis-Nirenberg [6]. Section 5 is devoted the existence of second solution and the last section contains the proof of Theorem 1.1.

2. Some Preliminaries

We first extend the functions f, g from $\mathbf{R}^+$ to $\mathbf{R}$ in a continuous manner by defining $f(s) = f(0)$ and $g(s) = g(0)$ for all $s < 0$. Let $H^1(\Omega) = \{u : u \in L^2(\Omega), \nabla u \in (L^2(\Omega))^2\}$ be the standard Sobolev space with the norm $\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u|^2 + |u|^2)$. We then have the following imbedding theorem of the Moser-Trudinger type:

Lemma 2.1. [2] *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a regular boundary. Then, for any $u \in H^1(\Omega)$ and $k > 0$*

$$\int_\Omega e^{k|u|^2} dx < \infty. \tag{2.1}$$

Moreover,

$$\sup_{\|u\|_{H^1(\Omega)} \leq 1} \int_\Omega e^{k|u|^2} dx < \infty \quad \text{if and only if} \quad k \leq 2\pi. \tag{2.2}$$

Let $d\sigma$ denote the surface measure on $\partial\Omega$. We define the energy functional $I_\lambda : H^1(\Omega) \to \mathbb{R}$ associated to the problem (P_λ) as:

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda\int_\Omega F(u) + \int_{\partial\Omega} G(u)\, d\sigma,\ u \in H^1(\Omega) \tag{2.3}$$

where $F(t) := \int_0^t f(s)\, ds,\ f(s) = e^{s^\alpha}$ and $G(t) := \int_0^t g(s)s\, ds.$

Definition 2.1. By a weak solution of (P_λ) we mean $u \in H^1(\Omega)$ satisfying:

$$\int_\Omega \nabla u \cdot \nabla v = \lambda\int_\Omega f(u)v - \int_{\partial\Omega} g(u)uv\, d\sigma, \text{ for all } v \in H^1(\Omega). \tag{2.4}$$

It will be more convenient for our purpose to work with the norm

$$\|u\|_H^2 := \int_\Omega |\nabla u|^2 + m\int_{\partial\Omega} |u|^2\, d\sigma, \tag{2.5}$$

where m is defined in (H2)

Remark 2.2. Thanks to the trace imbedding and the imbedding of Cherrier (see [8, 9, 15]), it follows that $\|\cdot\|_H$ is indeed an equivalent norm in $H^1(\Omega)$. That is, there exists $c_I, c_{II} > 0$ such that

$$c_I\|u\|_{H^1(\Omega)} \le \|u\|_H \le c_{II}\|u\|_{H^1(\Omega)},\ \forall u \in H^1(\Omega). \tag{2.6}$$

We take note also of the following regularity result:

Lemma 2.2. *If u_λ is a weak solution of (P_λ), then $u_\lambda \in C^{2,\gamma}(\Omega)$ for some $\gamma \in (0,1)$.*

Proof. From (2.1), for any $u_\lambda \in H^1(\Omega)$ we obtain that $f(u_\lambda) \in L^p(\Omega), \forall p \ge 1$. It follows by standard elliptic regularity that $u_\lambda \in W^{2,p}(\Omega),\ \forall p \ge 1$, which implies that $u_\lambda \in C^{2,\gamma}(\Omega)$ for some $\gamma \in (0,1)$. Thus, by the Sobolev imbedding theorem $u \in C^{1,\gamma}(\overline{\Omega})$. Consequently, $u_\lambda \in C^{2,\gamma}(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$ is a classical solution of (P_λ). ■

Finally a strong comparison result:

Lemma 2.3. *Let $w_1, w_2 \in C^{2,\gamma}(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$ satisfy $-\Delta w_1 \le -\Delta w_2$ in Ω, $\mathbf{n} \cdot \nabla w_1 + g(w_1)w_1 \le \mathbf{n} \cdot \nabla w_2 + g(w_2)w_2$. Then, $w_1 < w_2$ in $\overline{\Omega}$.*

Proof. Let $w = w_2 - w_1$. Being a super harmonic function, w cannot have a local minimum in Ω. That is, it attains its global minimum in $\overline{\Omega}$ at a point $x_0 \in \partial\Omega$. Note that on the boundary, $\mathbf{n}\cdot\nabla w + a(x)w \ge 0$ where $a(x) := (g(w_2)w_2 - g(w_1)w_1)/(w_2 - w_1) \ge 0$. Therefore we obtain a contradiction by Hopf Lemma if $w(x_0) \le 0$. ■

As a corollary, we have

Lemma 2.4. *Any solution of* (P_λ) *is strictly positive in* $\overline{\Omega}$.

3. Small Norm Solution as Local Minimum

In this section we show the existence of a local minimum for I_λ in a small neighborhood of the origin in $H^1(\Omega)$.

Lemma 3.1. *We may find* $R_0 \in (0, \sqrt{\pi}), \lambda_0 > 0$ *and* $\delta > 0$ *such that* $I_\lambda(u) \geq \delta$ *for all* $\|u\|_{H^1(\Omega)} = R_0$ *and all* $\lambda \in (0, \lambda_0)$.

Proof. From the simple pointwise estimate $F(u) = \int_0^u e^{s^\alpha}\ ds \leq e|u|e^{u^2}$, we obtain that

$$\int_\Omega F(u) \leq \int_\Omega |u|e^{u^2}$$
$$\leq \|u\|_{L^2(\Omega)} \left(\int_\Omega e^{2\|u\|^2_{H^1(\Omega)} \left(u/\|u\|_{H^1(\Omega)}\right)^2} \right)^{1/2}.$$

Now choose $R_0 > 0$ such that $R_0^2 \leq \pi$. Then, by Moser-Trudinger inequality (2.2) and Sobolev imbedding, from the last inequality we get,

$$\int_\Omega F(u) \leq C_1 \|u\|_{H^1(\Omega)}, \quad \forall\, \|u\|_{H^1(\Omega)} \leq R_0, \text{ for some } C_1 > 0. \tag{3.1}$$

Also,

$$\int_{\partial\Omega} G(u)\, d\sigma \geq \frac{g(0)}{2} \int_{\partial\Omega} u^2\, d\sigma.$$

Thus, from (3.1) and Remark 2.2 we have for $R_0^2 \in (0, \pi)$ small enough

$$\begin{aligned} I_\lambda(u) &\geq \tilde{c}\|u\|^2_H - \lambda C_1 \|u\|_{H^1(\Omega)} \\ &\geq \tilde{c}c_I^2 \|u\|^2_{H^1(\Omega)} - \lambda C_1 \|u\|_{H^1(\Omega)}, \ \ \forall\, \|u\|_{H^1(\Omega)} = R_0, \end{aligned} \tag{3.2}$$

where $\tilde{c} = \min\{\frac{1}{2}, \frac{g(0)}{2m}\}$ and c_I is defined in (2.6). We may choose and fix $R_0^2 \in (0, \pi)$ and $\lambda_0 > 0$ small enough so that $\delta := \tilde{c}c_I^2 R_0^2 - \lambda C_1 R_0 > 0$ for all $\lambda \in (0, \lambda_0)$. With this choice of δ, λ_0 and R_0, we get the conclusion of the lemma from (3.2). ■

Lemma 3.2. *Let* λ_0 *be as in the previous lemma. Then,* I_λ *has a local minimum close to the origin for all* $\lambda \in (0, \lambda_0)$.

Proof. Let R_0 be as in the previous lemma. For any $u \in H^1(\Omega), u > 0$ in Ω and a real number $t > 0$,

$$I_\lambda(tu) = \frac{t^2}{2}\int_\Omega |\nabla u|^2 - \lambda \int_\Omega dx \int_0^{tu} e^{s^\alpha} ds + \int_{\partial\Omega} d\sigma \int_0^{tu} g(s)s\ ds$$
$$\le \frac{t^2}{2}\int_\Omega |\nabla u|^2 - \lambda t \int_\Omega u\ dx + \frac{mt^2}{2}\int_{\partial\Omega} u^2 d\sigma.$$

It follows that $\inf I_\lambda(u) < 0$ in a sufficiently small neighborhood of the origin in $H^1(\Omega)$. Hence, if we show the existence of a local minimizer u_λ of I_λ on the set $\{u \in H^1(\Omega) : \|u\|_{H^1(\Omega)} \le R_0\} =: B_{R_0}(0)$, then in view of the last lemma, necessarily $\|u_\lambda\|_{H^1(\Omega)} < R_0$ and hence it is indeed a local minimizer of I_λ in $H^1(\Omega)$. Let $\{u_n\} \subset B_{R_0}(0)$ be a minimizing sequence for I_λ. Since $\{u_n\}$ is bounded in $H^1(\Omega)$, there exists a subsequence $\{u_{n_k}\}$ and a u_λ such that $u_{n_k} \rightharpoonup u_\lambda$ in $H^1(\Omega)$. Clearly, $\int_\Omega |\nabla u_\lambda|^2 \le \liminf_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2$. By Moser-Trudinger's inequality and Vitali's convergence theorem we have $\int_\Omega F(u_{n_k}) \to \int_\Omega F(u_\lambda)$ since $R_0^2 \in (0,\pi)$. By the compactness of the trace imbedding, it also follows that $\int_{\partial\Omega} G(u_{n_k})d\sigma \to \int_{\partial\Omega} G(u_\lambda)d\sigma$. Hence, we have $I_\lambda(u_\lambda) \le \liminf_{k\to\infty} I_\lambda(u_{n_k}) = \inf_{B_{R_0}(0)} I_\lambda$. Since $u_\lambda \in B_{R_0}(0)$, it must be true that $I_\lambda(u_\lambda) = \inf_{B_{R_0}(0)} I_\lambda$. Therefore, u_λ is a local minimizer for I_λ in the set $\{u \in H^1(\Omega) : \|u\|_{H^1(\Omega)} \le R_0\}$. Notice that $u_\lambda \not\equiv 0$ since $I_\lambda(0) = 0 > I_\lambda(u_\lambda)$. ■

4. Local Minimum for Maximal Range of λ

Lemma 4.1. (P_λ) *has no solution when* λ *is large.*

Proof. Let u_λ be a (positive) solution of (P_λ). Thanks to Lemmas 2.2 and 2.4, $1/u_\lambda$ is a $H^1(\Omega)$ function which we can use as a test function in (P_λ). We obtain thus,

$$\lambda \int_\Omega f(u_\lambda)/u_\lambda = \int_{\partial\Omega} g(u_\lambda)\ d\sigma - \int_\Omega |\nabla u_\lambda|^2/u_\lambda^2.$$

Since $f(u_\lambda) \ge c u_\lambda$ in Ω for some fixed constant $c > 0$ and g is a bounded function by (H2) we obtain from the last equation that λ is bounded. ■

Let $\Lambda := \sup\{\lambda > 0 : (\mathrm{P}_\lambda) \text{ has a solution}\}$. Then by Lemmas 3.2 and 4.1, it follows that $0 < \Lambda < \infty$.

Lemma 4.2. I_λ *admits a local minimum for all* $\lambda \in (0,\Lambda)$ *in the* $C^1(\overline{\Omega})$*- topology.*

Proof. For a fixed $\lambda < \Lambda$, there exists $\tilde\lambda$ such that $\lambda < \tilde\lambda < \Lambda$ and $u_{\tilde\lambda}$ a solution of $(\mathrm{P}_{\tilde\lambda})$. By Lemma 2.4, $u_{\tilde\lambda} > 0$ in $\overline{\Omega}$. Let v_λ be the unique (thanks to Lemma 2.3) solution of

$$(\mathrm{S}_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(0), & x \in \Omega \\ \mathbf{n}\cdot\nabla u + g(u)u = 0, & x \in \partial\Omega. \end{cases}$$

Since $\lambda f(0) < \tilde{\lambda} f(u_{\tilde{\lambda}})$, we obtain from Lemma 2.3 that $u_{\tilde{\lambda}} > v_\lambda$ on $\overline{\Omega}$.

Define the following cut-off nonlinearities:

$$(x,t) \in \Omega \times \mathbb{R}; \qquad \tilde{f}_\lambda(x,t) = \begin{cases} f(v_\lambda(x)) & \text{if } t < v_\lambda(x), \\ f(t) & \text{if } v_\lambda(x) \le t \le u_{\tilde{\lambda}}(x), \\ f(u_{\tilde{\lambda}}(x)) & \text{if } t > u_{\tilde{\lambda}}(x). \end{cases}$$

Define the primitive $\tilde{F}_\lambda(x,u) = \int_0^u \tilde{f}_\lambda(x,t)\ dt\ (x \in \Omega)$. Then the functional $\bar{I}_\lambda : H^1(\Omega) \to \mathbb{R}$ given by

$$\tilde{I}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \int_\Omega \tilde{F}_\lambda(x,u) + \int_{\partial\Omega} G(u) d\sigma$$

is coercive and bounded from below. Let u_λ be a global minimizer of $\tilde{I}_\lambda$ on $H^1(\Omega)$. Then u_λ satisfies

$$\begin{cases} -\Delta u_\lambda &= \lambda \tilde{f}_\lambda(x,u_\lambda), \quad \text{in } \Omega \\ \mathbf{n} \cdot \nabla u_\lambda + g(x,u_\lambda) &= 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.2 we have $u_\lambda \in C^{2,\theta}(\Omega)$ for some $\theta \in (0,1)$. Since $\lambda f(0) \le \lambda \tilde{f}_\lambda(x,u_\lambda) \le \tilde{\lambda} f(u_{\tilde{\lambda}})$, from Lemma 2.3 we obtain that $v_\lambda < u_\lambda < u_{\tilde{\lambda}}$ in $\overline{\Omega}$. In particular, u_λ is a solution of (P_λ). Let $\delta := \min\{\min_{x\in\bar{\Omega}} |u_{\tilde{\lambda}}(x) - u_\lambda(x)|, \min_{x\in\bar{\Omega}} |u_\lambda(x) - v_\lambda(x)|\}$. Then $\tilde{I}_\lambda = I_\lambda$ on the set $\{u \in C^1(\overline{\Omega}) : \|u - u_\lambda\|_{C^1(\overline{\Omega})} < \frac{\delta}{2}\}$. Hence u_λ is a local minimizer for I_λ in the $C^1(\overline{\Omega})$ topology. ■

Lemma 4.3. *Let* $\lambda \in (0,\Lambda)$. *Then* u_λ *obtained in Lemma 4.2 is a local minimizer for* I_λ *in* $H^1(\Omega)$.

Proof. Suppose not. Then, for all $\epsilon > 0$ there exists $v_\epsilon \in B_\epsilon(0) := \{\|u\|_{H^1(\Omega)} \le \epsilon\}$ such that $I_\lambda(u_\lambda + v_\epsilon) < I_\lambda(u_\lambda)$. Since I_λ is weakly lower semicontinuous on $H^1(\Omega)$, $I_\lambda(u_\lambda + \cdot)$ achieves its minimum at some point in $B_\epsilon(0)$ which we denote again by v_ϵ. In other words, for every $\epsilon > 0$, we obtain v_ϵ such that $0 < \|v_\epsilon\|_{H^1(\Omega)} \le \epsilon$ and

$$(4.1) \qquad I_\lambda(u_\lambda + v_\epsilon) < I_\lambda(u_\lambda),\ I_\lambda(u_\lambda + v_\epsilon) = \min_{v\in B_\epsilon(0)} I_\lambda(u_\lambda + v).$$

The corresponding Euler-Lagrange equation for v_ϵ involves a Lagrange multiplier $\mu_\epsilon \le 0$, namely, v_ϵ satisfies

$$\int_\Omega \nabla(u_\lambda + v_\epsilon) \cdot \nabla h - \lambda \int_\Omega f(u_\lambda + v_\epsilon)h + \int_{\partial\Omega} g(u_\lambda + v_\epsilon)(u_\lambda + \epsilon)h$$
$$= \mu_\epsilon \int_\Omega (v_\epsilon h + \nabla v_\epsilon \cdot \nabla h),\ \forall h \in H^1(\Omega).$$

This means, in the weak sense,

$$\begin{cases} -(1-\mu_\epsilon)\Delta v_\epsilon - \mu_\epsilon v_\epsilon = \lambda(f(u_\lambda + v_\epsilon) - f(u_\lambda)) \text{ in } \Omega, \\ (1-\mu_\epsilon)\ \mathbf{n}\cdot\nabla v_\epsilon + g(u_\lambda+v_\epsilon)(u_\lambda+v_\epsilon) - g(u_\lambda)u_\lambda = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.2}$$

Since $\mu_\epsilon \le 0$, by Moser iteration technique (see Theorem 15.7 in [13]) we conclude that $\{v_\epsilon\}$ is uniformly bounded, as $\epsilon \to 0$, in a Holder space. From standard elliptic regularity ([10] and [22]) it follows that $\overline{\lim}_{\epsilon\to 0}\|v_\epsilon\|_{C^{1,\theta}(\bar\Omega)} < \infty$ for some $\theta \in (0,1)$. By Arzela-Ascoli and the fact that $\|v_\epsilon\|_{H^1(\Omega)} \to 0$ as $\epsilon \to 0$, it follows that $v_\epsilon \to 0$ in $C^1(\overline{\Omega})$. This gives a contradiction to the fact that u_λ is a local minimizer for I_λ in $C^1(\overline{\Omega})$ topology. ■

5. The Second Solution is a Saddle-point

We fix $\lambda \in (0,\Lambda)$ and recall that u_λ was obtained as the local minimizer for I_λ in Lemma 4.3. We now show that I_λ possesses a second solution of mountain-pass or saddle-point type. For the easy computations, it will be better to translate the functional I_λ by u_λ and consider the resulting functional which will have the origin as the local minimum.

Define $\bar f_\lambda : \Omega\times\mathbb{R}\to\mathbb{R}$ by

$$\bar f_\lambda(x,s) = \begin{cases} f(s+u_\lambda) - f(u_\lambda) & \text{if } s \ge 0, \\ 0 & \text{if } s < 0, \end{cases}$$

and $\bar g_\lambda : \partial\Omega\times\mathbb{R}\to\mathbb{R}$ by

$$\bar g_\lambda(x,s) = \begin{cases} g(s+u_\lambda)(s+u_\lambda) - g(u_\lambda)u_\lambda & \text{if } s \ge 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Now we define the translated functional $\bar I_\lambda : H^1(\Omega)\to\mathbb{R}$ by

$$\bar I_\lambda(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 - \lambda\int_\Omega \bar F_\lambda(x,w) + \int_{\partial\Omega}\bar G_\lambda(x,w), \tag{5.1}$$

where $\bar F_\lambda(x,t) = \int_0^t \bar f_\lambda(x,s)\,ds$ and $\bar G_\lambda(x,t) = \int_0^t \bar g_\lambda(x,s)\,ds$.

If we show the existence of a non-trivial critical point w_λ of $\bar I_\lambda$, then w_λ will be a positive solution of the problem

$$(\mathrm{Q}_\lambda)\qquad \begin{cases} -\Delta w_\lambda = \lambda \bar f_\lambda(x,w_\lambda), & x\in\Omega \\ \mathbf{n}\cdot\nabla w_\lambda + \bar g_\lambda(x,w_\lambda) = 0, & x\in\partial\Omega \end{cases}$$

and $w_\lambda + u_\lambda$ will be a second solution for (P_λ).

First note that $\bar{I}_\lambda(0) = 0$ and $w \equiv 0$ is a local minimizer for $\bar{I}_\lambda$. Choose $R_1 > 0$ so that

$$0 = \bar{I}_\lambda(0) \le \bar{I}_\lambda(u) \text{ for all } \|u\|_{H^1(\Omega)} \le R_1.$$

Since $\lim_{t\to\infty} \bar{I}_\lambda(tw) = -\infty$ for any $w \in H^1(\Omega) \setminus \{0\}$, we can fix $e \in H^1(\Omega) \setminus \{0\}$ such that $\bar{I}_\lambda(e) < 0$. Necessarily, $\|e\|_{H^1(\Omega)} > R_1$. Set

$$\Gamma = \{\gamma : [0,1] \to H^1(\Omega) : \gamma \text{ is continuous }, \gamma(0) = 0, \gamma(1) = e\}$$

and define the mountain-pass level

$$\rho = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \bar{I}_\lambda(\gamma(t)). \tag{5.2}$$

Clearly, $\rho \ge 0$ since $\bar{I}_\lambda(0) = 0$. We distinguish the following two cases:

$(\mathrm{P1})$ (Zero altitude case)

$$\inf\{\bar{I}_\lambda(w) : w \in H^1(\Omega) \text{ and } \|w\|_{H^1(\Omega)} = l\} = 0 \text{ for all } l < R_1;$$

$(\mathrm{P2})$ (Mountain-Pass case) there exists $0 < l_1 < R_1$ such that

$$\inf\{\bar{I}_\lambda(w) : w \in H^1(\Omega) \text{ and } \|w\|_{H^1(\Omega)} = l_1\} > 0.$$

Note that $(\mathrm{P2})$ implies $\rho > 0$. That is, $\rho = 0$ implies that $(\mathrm{P1})$ holds. We recall the definition of the Palais-Smale sequence around the closed set F:

Definition 5.1. By a Palais-Smale sequence for $\bar{I}_\lambda$ at the level $\beta \in \mathbb{R}$ around F ($(\mathrm{PS})_{F,\beta}$ for short) we mean a sequence $\{w_n\} \subset H^1(\Omega)$ such that

$$\lim_{n\to\infty} \operatorname{dist}(w_n, F) = 0, \ \lim_{n\to\infty} \bar{I}_\lambda(w_n) = \beta \text{ and } \lim_{n\to\infty} \|\bar{I}'_\lambda(w_n)\|_{(H^1(\Omega))^*} = 0.$$

Definition 5.2. We define the closed set $F = \{w \in H^1(\Omega) : \|w\|_{H^1(\Omega)} = \frac{R_1}{2}\}$ if $\rho = 0$, and $F = H^1(\Omega)$ if $\rho > 0$.

In the case when $F = \{w \in H^1(\Omega) : \|w\|_{H^1(\Omega)} = \frac{R_1}{2}\}$, Ghoussoub and Preiss (Theorem (1) [12]) proved the existence of such a Palais-Smale sequence around F. They further showed in Theorem $(1.bis)$ in the same work that there exists a critical point of $\bar{I}_\lambda$ on F with critical value β provided this $(\mathrm{PS})_{F,\beta}$ sequence has a convergent subsequence. We also remark that when $F = H^1(\Omega)$ the above definition is same as the usual definition of Palais-Smale squence at the level β.

In the next lemma, we show convergence properties of a $(\mathrm{PS})_{F,\rho}$ sequence for $\bar{I}_\lambda$ with the above choice of F and ρ defined as in (5.2).

Lemma 5.1. *Let F be as in the Definition 5.2 and $\{w_n\} \subset H^1(\Omega)$ be a $(\mathrm{PS})_{F,\rho}$ sequence for $\bar{I}_\lambda$. Then, $w_n \rightharpoonup w_\lambda$ in $H^1(\Omega)$. Moreover, as $n \to \infty$,*

$$\int_\Omega \bar{f}_\lambda(x,w_n) \to \int_\Omega \bar{f}_\lambda(x,w_\lambda),\ \int_{\partial\Omega} \bar{g}_\lambda(x,w_n) \to \int_{\partial\Omega} \bar{g}_\lambda(x,w_\lambda), \tag{5.3}$$

$$\int_\Omega \bar{F}_\lambda(x,w_n) \to \int_\Omega \bar{F}_\lambda(x,w_\lambda),\ \int_{\partial\Omega} \bar{G}_\lambda(x,w_n) \to \int_{\partial\Omega} \bar{G}_\lambda(x,w_\lambda). \tag{5.4}$$

Proof. Since $\{w_n\}$ is a $(\mathrm{PS})_{F,\rho}$ sequence for $\bar{I}_\lambda$, we get

$$\frac{1}{2}\int_\Omega |\nabla w_n|^2 - \lambda \int_\Omega \bar{F}_\lambda(x,w_n) + \int_{\partial\Omega} \bar{G}_\lambda(x,w_n) = \rho + o_n(1) \tag{5.5}$$

and

$$\begin{aligned} &\left|\int_\Omega \nabla w_n \cdot \nabla\phi - \lambda \int_\Omega \bar{f}_\lambda(x,w_n)\phi + \int_{\partial\Omega} \bar{g}_\lambda(x,w_n)\phi\right| \\ &= o_n(1)\|\phi\|_{H^1(\Omega)}, \quad \forall \phi \in H^1(\Omega). \end{aligned} \tag{5.6}$$

Note that (5.5) implies

$$\tilde{c}\|w_n\|_H^2 \le \rho + o_n(1) + \lambda \int_\Omega \bar{F}_\lambda(x,w_n) \ \text{ for some } \tilde{c} > 0. \tag{5.7}$$

Observing that given $\epsilon > 0$ there exists $t_\epsilon > 0$ such that $\bar{F}_\lambda(x,t) \le \epsilon t \bar{f}_\lambda(x,t)$ for all $t \ge t_\epsilon$, we have

$$\begin{aligned} \tilde{c}\|w_n\|_H^2 &\le \rho + o_n(1) + \lambda \int_{\Omega\cap\{x:|w_n|\le t_\epsilon\}} \bar{F}_\lambda(x,w_n) + \epsilon\lambda \int_{\Omega\cap\{x:|w_n|\ge t_\epsilon\}} \bar{f}_\lambda(x,w_n)w_n \\ &\le \rho + o_n(1) + C_\epsilon + \epsilon\lambda \int_\Omega \bar{f}_\lambda(x,w_n)w_n, \end{aligned}$$

where $C_\epsilon \to 0$ as $\epsilon \to 0$. Now, substituting w_n for ϕ in (5.6), since $\bar{g}_\lambda(x,s) \le ms,\ \forall s \ge 0$, we get

$$\begin{aligned} \lambda \int_\Omega \bar{f}_\lambda(x,w_n)w_n &\le \int_\Omega |\nabla w_n|^2 + \int_{\partial\Omega} \bar{g}_\lambda(x,w_n)w_n + o_n(1)\|w_n\|_{H^1(\Omega)} \\ &\le C\left(\int_\Omega |\nabla w_n|^2 + m\int_{\partial\Omega} w_n^2\right) + o_n(1)\|w_n\|_{H^1(\Omega)} \\ &\le C\|w_n\|_H^2 + o_n(1)\|w_n\|_{H^1(\Omega)}. \end{aligned} \tag{5.8}$$

Hence we obtain

$$\begin{aligned} \tilde{c}\|w_n\|_H^2 &\le \rho + o_n(1) + C_\epsilon + \epsilon C\|w_n\|_H^2 + \epsilon o_n(1)\|w_n\|_{H^1(\Omega)} \\ &\le \rho + o_n(1) + C_\epsilon + \epsilon C\|w_n\|_H^2 + c_{II}\epsilon o_n(1)\|w_n\|_{H(\Omega)}. \end{aligned}$$

If we choose ϵ small so that $\tilde{c} - \epsilon C > 0$, then from the last inequality we obtain that $\sup_n \|w_n\|_H \leq \underline{c} < \infty$ for some $\underline{c} > 0$, which implies that $\sup_n \|w_n\|_{H^1(\Omega)} \leq c_{II}\underline{c}$ by (2.6). Therefore, there exists $w_\lambda \in H^1(\Omega)$ such that $w_n \rightharpoonup w_\lambda$ in $H^1(\Omega)$.

Next, we show that $\int_\Omega \bar{f}_\lambda(x, w_n) \to \int_\Omega \bar{f}_\lambda(x, w_\lambda)$ as $n \to \infty$. Notice that

$$\bar{C} := \sup_n \int_\Omega \bar{f}_\lambda(x, w_n)w_n < \infty$$

from (5.8) and the fact that $\sup_n \|w_n\|_H < \infty$. Given $\epsilon > 0$ we define $\delta_\epsilon := \max\limits_{x\in\bar{\Omega}, |s|\leq \frac{\bar{C}}{\epsilon}} \bar{f}_\lambda(x, s)$. Then, for any subset $E \subset \Omega$ with $|E| \leq \frac{\epsilon}{\delta_\epsilon}$, we have

$$\begin{aligned}\int_E |\bar{f}_\lambda(x, w_n)| &= \int_{E\cap\{|w_n|\geq \frac{\bar{C}}{\epsilon}\}} \left|\frac{\bar{f}_\lambda(x, w_n)w_n}{w_n}\right| + \int_{E\cap\{|w_n|\leq \frac{\bar{C}}{\epsilon}\}} |\bar{f}_\lambda(x, w_n)| \\ &\leq \frac{\epsilon}{\bar{C}} \int_E \left|\bar{f}_\lambda(x, w_n)w_n\right| + \delta_\epsilon |E| \leq 2\epsilon.\end{aligned}$$

This shows that $\{\bar{f}_\lambda(x, w_n)\}$ is equi-absolutely continuous. By Vitali's convergence theorem, we get $\int_\Omega \bar{f}_\lambda(x, w_n) \to \int_\Omega \bar{f}_\lambda(x, w_\lambda)$ as $n \to \infty$.

Notice that for all $(x, s) \in \Omega \times \mathbb{R}^+$, we can find $C > 0$ such that

$$\bar{F}_\lambda(x, s) \leq C\bar{f}_\lambda(x, s).$$

Hence, by the generalized Lebesgue dominated convergence theorem we conclude that

$$\int_\Omega \bar{F}_\lambda(x, w_n) \to \int_\Omega \bar{F}_\lambda(x, w_\lambda).$$

By the compactness of the trace imbedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$, we obtain $\int_{\partial\Omega} \bar{G}_\lambda(x, w_n) \to \int_{\partial\Omega} \bar{G}_\lambda(x, w_\lambda)$ as well as $\int_{\partial\Omega} \bar{g}_\lambda(x, w_n) \to \int_{\partial\Omega} \bar{g}_\lambda(x, w_\lambda)$. ■

Next we show that $\bar{I}_\lambda$ has a critical point $w_\lambda > 0$ of mountain-pass type. However, due to the lack of compactness when $\alpha = 2$, we need the following strict upper bound of ρ.

Lemma 5.2. *Let $\alpha = 2$. Then $\rho < \pi$.*

Proof. Without loss of generality we may assume that $0 \in \partial\Omega$. Let m_n be the Moser function given by

$$m_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \dfrac{\log Ln}{\sqrt{\log n}} & 0 \leq |x| < \frac{1}{n}, \\ \dfrac{\log \frac{L}{|x|}}{\sqrt{\log n}} & \frac{1}{n} \leq |x| < L, \\ 0 & |x| > L. \end{cases} \tag{5.9}$$

We take n large so that $nL > 1$. It is easy to see that $\|\nabla m_n\|_{L^2(\mathbb{R}^2)} = 1$ and $\|m_n\|_{L^2(\mathbb{R}^2)} = O(\frac{1}{\log nL})$. Let $\overline{m}_n$ be the restriction of m_n to Ω and define $\psi_n = \frac{\overline{m}_n}{\|\overline{m}_n\|_H}$. Then ψ_n is constant in $B_{\frac{1}{n}}(0) \cap \overline{\Omega}$ and supp $\psi_n \subset B_L(0) \cap \overline{\Omega}$. Observing carefully the proof of Lemma 3.3 in [2], we also get $\int_\Omega |\nabla \overline{m}_n|^2 + m \int_{\partial\Omega} \overline{m}_n^2 = \frac{1}{2} + O(\frac{1}{\log nL})$, and hence $\psi_n^2(x) = \frac{1}{2\pi} \frac{\log nL}{[\frac{1}{2} + O(\frac{1}{\log nL})]} = \frac{1}{\pi} \log nL + O(1)$ as $n \to \infty$ on $B_{\frac{1}{n}}(0) \cap \overline{\Omega}$.

We now suppose $\rho_0 \geq \pi$ and derive a contradiction. It follows from Lemma 3.1 in [17] that we can find some $t_n > 0$ such that $\bar{I}_\lambda(t_n\psi_n) = \sup_{t>0} \bar{I}_\lambda(t\psi_n) \geq \pi, \ \forall n$. That is, we have

$$\bar{I}_\lambda(t_n\psi_n) = \frac{1}{2}\int_\Omega |\nabla(t_n\psi_n)|^2 - \lambda \int_\Omega \bar{F}_\lambda(x, t_n\psi_n) + \int_{\partial\Omega} \bar{G}_\lambda(x, t_n\psi_n) \geq \pi, \ \forall n. \tag{5.10}$$

Since $\bar{g}_\lambda(x,s) \leq ms, \forall s \geq 0$, we have

$$\int_{\partial\Omega} \bar{G}_\lambda(x, t_n\psi_n) = \int_{\partial\Omega}\int_0^{t_n\psi_n} \bar{g}_\lambda(x,s) \leq \frac{m}{2} t_n^2 \int_{\partial\Omega} \psi_n^2.$$

Hence, from (5.10), we obtain

$$t_n^2 = t_n^2 \|\psi_n\|_H^2 \geq 2\bar{I}_\lambda(t_n\psi_n) \geq 2\pi, \ \forall n. \tag{5.11}$$

Since the maximum of the map $t \mapsto \bar{I}_\lambda(t\psi_n)$ on $(0,\infty)$ is attained at $t = t_n$, its derivative must be 0 at this point. That is,

$$\int_\Omega |\nabla(t_n\psi_n)|^2 - \lambda \int_\Omega \bar{f}_\lambda(x, t_n\psi_n) t_n\psi_n + \int_{\partial\Omega} \bar{g}_\lambda(x, t_n\psi_n) t_n\psi_n = 0. \tag{5.12}$$

Note that $\inf_{x\in\overline{\Omega}} \bar{f}_\lambda(x,s) \geq e^{s^2}$ for s large and $t_n\psi_n \to \infty$ on $B_{\frac{1}{n}}(0)$ as $n \to \infty$. Since $\int_{\partial\Omega} \bar{g}_\lambda(x, t_n\psi_n) t_n\psi_n \leq m t_n^2 \int_{\partial\Omega} \psi_n^2$, we obtain from (5.12)

$$t_n^2 = t_n^2 \|\psi_n\|_H^2 \geq \lambda \int_{\{|x|<\frac{1}{n}\}} \bar{f}_\lambda(x, t_n\psi_n) t_n\psi_n \geq \lambda \int_{\{|x|<\frac{1}{n}\}} e^{t_n^2\psi_n^2} t_n\psi_n. \tag{5.13}$$

Using the explicit value of ψ_n at 0, we get

$$\begin{aligned} t_n^2 &\geq \lambda\sqrt{\pi} e^{\left(\frac{t_n^2}{\pi} - 2\right)\log nL + 2\log L + t_n^2 O(1)} t_n (\log nL + O(1))^{\frac{1}{2}}, \\ &= \lambda\sqrt{\pi} e^{\left((\frac{1}{\pi} + \frac{O(1)}{\log nL}) t_n^2 - 2\right)\log nL + 2\log L} t_n (\log nL + O(1))^{\frac{1}{2}}, \end{aligned} \tag{5.14}$$

which implies that $\{t_n\}$ is bounded sequence since $t_n^2 \geq 2\pi$. Now using (5.11), we obtain from (5.14)

$$t_n^2 \geq \lambda\sqrt{\pi}e^{t_n^2 O(1)+2\log L}t_n(\log nL + O(1))^{\frac{1}{2}}.$$

Since $\{t_n\}$ is bounded, we note that $e^{t_n^2 O(1)} \geq C > 0$, $\forall n$. Hence we have

$$t_n \geq \lambda\sqrt{\pi}Ce^{2\log L}(\log nL + O(1))^{\frac{1}{2}},$$

which implies that $t_n \to \infty$ as $n \to \infty$. This contradiction shows that $\rho < \pi$ when $\alpha = 2$. ■

Lemma 5.3. *$\bar{I}_\lambda$ possesses a critical point $w_\lambda > 0$ of mountain-pass type.*

Proof. Let $\{w_n\} \subset H^1(\Omega)$ be $(\mathrm{PS})_{F,\rho}$ sequence for $\bar{I}_\lambda$. From Lemma 5.1, $\{w_n\}$ is a bounded sequence in $H^1(\Omega)$. Let $w_\lambda \in H^1(\Omega)$ such that

$$w_n \rightharpoonup w_\lambda \ \text{ in } H^1(\Omega). \tag{5.15}$$

Hence, from (5.3) and (5.15) and the fact that $\{w_n\}$ is a Palais-Smale sequence, we obtain

$$\int_\Omega \nabla w_\lambda \cdot \nabla\phi - \lambda\int_\Omega \bar{f}_\lambda(x, w_\lambda)\phi + \int_{\partial\Omega} \bar{g}_\lambda(x, w_\lambda)\phi = 0,\ \forall\phi \in H^1(\Omega), \tag{5.16}$$

which implies that w_λ is a weak solution for (Q_λ).

Now we claim that $w_\lambda \not\equiv 0$. Note that $w_n(x) \to w_\lambda(x)$ pointwise a.e. in Ω. By the compactness of the trace imbedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$, we obtain

$$\int_{\partial\Omega} \bar{g}_\lambda(x, w_n)w_n \to \int_{\partial\Omega} \bar{g}_\lambda(x, w_\lambda)w_\lambda \text{ as } n \to \infty. \tag{5.17}$$

First we consider the compact case when $\alpha < 2$. Note that there exists $\tilde{C} > 0$ such that $e^{pt^\alpha} \leq \tilde{C}e^{t^2}$ for all $p \geq 1$ and $s^2 \leq \tilde{C}e^{(u_\lambda+s)^\alpha}$ for all $s \geq 0$. Hence, taking $p := \sup_n 3\|u_\lambda + w_n^+\|_{H^1(\Omega)}^\alpha$, by Moser-Trudinger inequality,

$$\begin{aligned}\int_\Omega |\bar{f}_\lambda(x, w_n)w_n|^2 &= \int_{\Omega\cap\{w_n\geq 0\}} |e^{(u_\lambda+w_n)^\alpha} - e^{u_\lambda^\alpha}|^2 w_n^2 \\ &\leq \tilde{C}\int_\Omega e^{3\|u_\lambda+w_n^+\|^\alpha\left(\frac{u_\lambda+w_n^+}{\|u_\lambda+w_n^+\|}\right)^\alpha} \\ &\leq \tilde{C}^2\int_\Omega e^{\left(\frac{u_\lambda+w_n^+}{\|u_\lambda+w_n^+\|}\right)^2} < \infty.\end{aligned}$$

Again applying Vitali's convergence theorem, we have

$$\int_\Omega \bar{f}_\lambda(x, w_n)w_n \to \int_\Omega \bar{f}_\lambda(x, w_\lambda)w_\lambda \text{ as } n \to \infty. \tag{5.18}$$

Substituting ϕ by w_n in (5.6) and using (5.17) and (5.18), we obtain $\int_\Omega |\nabla w_n|^2 \to \int_\Omega |\nabla w_\lambda|^2$ as $n \to \infty$, which implies that $w_n \to w_\lambda$ in $H^1(\Omega)$ as well as $\bar{I}_\lambda(w_\lambda) = \rho$.

In case $\rho > 0$, necessarily this means $w_\lambda \not\equiv 0$ and we are done. Consider the case $\rho = 0$. Since $w_n \to w_\lambda$ in $H^1(\Omega)$, from Theorem $(1.bis)$ in [12] we have $w_\lambda \in F = \{\|u\|_{H^1(\Omega)} = \frac{R_1}{2}\}$ and hence $w_\lambda \not\equiv 0$.

We now handle the case $\alpha = 2$ with a contradiction argument. Suppose that $w_\lambda \equiv 0$ on $\overline{\Omega}$. Note that $\rho < \pi$ from Lemma 5.2. Since $w_n \rightharpoonup 0$ as $n \to \infty$, from (5.4)-(5.5) and the compactness of the trace imbedding we have $\|w_n\|_{H^1(\Omega)} < 2\pi - \epsilon$ for some $\epsilon > 0$ small and for n large. Let us choose $0 < \delta < \frac{\epsilon}{2\pi}$ and fix $p = \frac{2\pi}{(1+\delta)(2\pi-\epsilon)}$. Then $p > 1$. Observing that $\int_\Omega \bar{f}_\lambda(x, s)s \leq C \int_\Omega e^{(1+\delta)s^2}$ $\forall s \in \mathbb{R}$ for some $C > 0$, we have

$$\int_\Omega |\bar{f}_\lambda(x, w_n)w_n|^p \leq C \int_\Omega e^{(1+\delta)pw_n^2} \leq C \int_\Omega e^{(1+\delta)p\|w_n\|^2\left(\frac{w_n}{\|w_n\|_{H^1(\Omega)}}\right)}. \tag{5.19}$$

Since $(1 + \delta)p\|w_n\|_{H^1(\Omega)} < 2\pi$, by the Moser-Trudinger inequality we have $\sup_n \int_\Omega |\bar{f}_\lambda(x, w_n)w_n|^p < \infty$. Hence again by the Vitali's convergence theorem we obtain $\int_\Omega \bar{f}_\lambda(x, w_n)w_n \to 0$. Clearly, $\int_{\partial\Omega} \bar{g}_\lambda(x, w_n)w_n \to 0$ as $n \to \infty$ from similar argument leading to (5.17). Hence, taking $\phi = w_n$ in (5.6) we get,

$$o_n(1)\|w_n\|_{H^1(\Omega)} = \int_\Omega |\nabla w_n|^2 + o_n(1). \tag{5.20}$$

However, since $\int_\Omega \bar{F}(x, w_n) \to \int_\Omega \bar{F}(x, w_\lambda) = 0$ and $\int_{\partial\Omega} \bar{G}(x, w_n) \to \int_{\partial\Omega} \bar{G}(x, w_\lambda) = 0$, from (5.5) we obtain

$$\int_\Omega |\nabla w_n|^2 \to 2\rho. \tag{5.21}$$

From (5.20)-(5.21) we get $\rho = 0$. That is, $w_n \to 0$ in $H^1(\Omega)$ which is a contradiction to the fact that $\{w_n\}$ is a $(\mathrm{PS})_{F,\rho}$ sequence. Therefore, $w_\lambda \not\equiv 0$. We obtain from Lemma 2.3 that $w_\lambda > 0$ in Ω. ■

6. Proof of Theorem 1.1

By the definition of Λ, there is no solution if $\lambda > \Lambda$. When $\lambda \in (0, \Lambda)$, from Lemma 4.3 we obtain the solution u_λ which is a local minimizer of $I_\lambda(u_\lambda)$. By

Lemma 5.3 we have a mountain pass type solution of the form $w_\lambda + u_\lambda$ where w_λ is a positive solution of the translated problem (Q_λ). Therefore, this solution is different from u_λ.

Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \uparrow \Lambda$. Then from Lemma 4.3 there exists sequence of solutions $\{u_{\lambda_n}\} \subset H^1(\Omega)$ to (P_{λ_n}) satisfying

$$\limsup_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) < +\infty, \;\; I'_{\lambda_n}(u_{\lambda_n}) = 0.$$

The first bound can be seen from the arguments in the proof of Lemma 4.2 where we show that $I_\lambda(u_\lambda) \le I_\lambda(v_\lambda)$ and noting the fact that $\{v_\lambda\}_{0\le\lambda\le\Lambda}$ is uniformly bounded in $C^1(\overline{\Omega})$. This implies (by an argument similar to the one in the proof of Lemma 5.1) that $\{u_{\lambda_n}\}$ is bounded in $H^1(\Omega)$, and hence there exists u_Λ such that $u_{\lambda_n} \rightharpoonup u_\Lambda$ in $H^1(\Omega)$. It is easy to see that u_Λ is a weak solution of (P_Λ).

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