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EXCEPTIONAL SETS IN WARING'S PROBLEM: TWO SQUARES, TWO CUBES AND TWO SIXTH POWERS

Xiaodong Lü and Quanwu Mu

Abstract. Let $R(n)$ denote the number of representations of a large positive integer n as the sum of two squares, two cubes and two sixth powers. In this paper, it is proved that the anticipated asymptotic formula of $R(n)$ fails for at most $O((\log X)^{2+\varepsilon})$ positive integers not exceeding X. This is an improvement of T. D. Wooley's result which requires $O((\log X)^{3+\varepsilon})$.

1. Introduction

Let $R(n)$ denote the number of representations of the integer n in the shape

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^6 + x_6^6 = n$$

with $x_i \in \mathbb{N}$ $(1 \le i \le 6)$. Define

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-6} S_2(q,a)^2 S_3(q,a)^2 S_6(q,a)^2 e\left(-\frac{an}{q}\right),$$

where

$$S_k(q,a) = \sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right), \quad e(z) = e^{2\pi i z}.$$

It is worthy to note that $1 \ll \mathfrak{S}(n) \ll 1$ (see Section 2 in [8]). A heuristical application of the Hardy–Littlewood method, based on a major arc analysis only, suggests that $R(n)$ satisfies the asymptotic relation

$$R(n) = \frac{\Gamma(\frac{3}{2})^2 \Gamma(\frac{4}{3})^2 \Gamma(\frac{7}{6})^2}{\Gamma(2)} \mathfrak{S}(n) n (1 + o(1)). \tag{1}$$

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But to prove (1) is beyond the grasp of modern number theory techniques. T. D. Wooley [7] applied Golubeva's method to show, subject to the truth of the generalized Riemann hypothesis, that $R(n) > 0$ for all large integers n. However, his method fails to obtain the anticipated asymptotic formula for $R(n)$.

We refer to a function $\varphi(t)$ as being a *sedately increasing function* when $\varphi(t)$ is a function of a positive variable t, increasing monotonically to infinity, and satisfying the condition that, when t is large, one has $\varphi(t) = O(t^\delta)$ for a positive number δ sufficiently small in the ambient context. We introduce $E(X;\varphi)$ to denote the number of integers n with $1 \le n \le X$ such that

$$\left| R(n) - \frac{\Gamma(\frac{3}{2})^2\Gamma(\frac{4}{3})^2\Gamma(\frac{7}{6})^2}{\Gamma(2)}\mathfrak{S}(n)n \right| > \frac{n}{\varphi(n)}. \tag{2}$$

Wooley [8] established the upper bound

$$E(X;\varphi) \ll \varphi(X)^2(\log X)^3.$$

In this note, we obtain the following result.

Theorem. *When $\varphi(t)$ is a sedately increasing function, one has*

$$E(X;\varphi) \ll \varphi(X)^2(\log X)^2.$$

By taking $\varphi(n) = \log\log n$, it follows that, for each $\varepsilon > 0$, the anticipated asymptotic formula fails for at most $O((\log X)^{2+\varepsilon})$ positive integers not exceeding X.

2. Notation and Some Lemmas

Suppose that X is a large positive number and let $\varphi(x)$ be a sedately increasing function. Whenever ε appears in a statement, either implicity or explicity, we assert that the statement holds for each $\varepsilon > 0$. Note that the value of ε may change from statement to statement. Write $V(X;\varphi)$ to denote the set of integers n with $X/2 < n \le X$ for which (2) holds, and write $V = \mathrm{card}\big(V(X;\varphi)\big)$. Let

$$f_k(\alpha) = \sum_{n \le X^{1/k}} e(\alpha n^k).$$

By orthogonality, we have

$$R(n) = \int_0^1 f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-\alpha n)d\alpha. \tag{3}$$

When Q is a positive number, we denote $\mathfrak{M}(Q)$ to be the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in (0,1] : |q\alpha - a| \le QX^{-1}\},$$

with $1 \le a \le q \le Q$ and $(a,q) = 1$. Whenever $Q \le \sqrt{X}/2$, the intervals $\mathfrak{M}(q,a) \subset \mathfrak{M}(Q)$ are pairwise disjoint for $1 \le a \le q \le Q$ and $(a,q) = 1$. Let

$$v = 10^{-4}, \quad Q_0 = X^{30v}, \quad Q_1 = X^{\frac{1}{2}},$$

$$Q_2 = X^{\frac{1}{3}+(\frac{3}{2})^2 v}, \quad Q_3 = X^{\frac{1}{4}+(\frac{3}{2})^4 v}, \quad Q_4 = X^{\frac{1}{8}+(\frac{3}{2})^6 v}.$$

For $\alpha \in \mathfrak{M}(Q_1)$, there may be more than one arc $\mathfrak{M}(q,a) \subset \mathfrak{M}(Q_1)$ for which $\alpha \in \mathfrak{M}(q,a)$. In order to ensure that $\alpha \in (0,1]$ is associated with uniquely defined arc $\mathfrak{M}(q,a)$, we adopt the conversation that α lie in the arc for which q is least.

Lemma 2.1. *For a suitable positive number τ, we have*

$$\int_{\mathfrak{M}(Q_0)} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-\alpha n) d\alpha = \frac{\Gamma(\frac{3}{2})^2 \Gamma(\frac{4}{3})^2 \Gamma(\frac{7}{6})^2}{\Gamma(2)} \mathfrak{S}(n) n + O(n^{1-\tau}).$$

Proof. See (2.1) in [8] and its proof. ∎

Lemma 2.2. *We have*

$$\int_0^1 |f_3(\alpha)^2 f_6(\alpha)^4| d\alpha \ll X^{\frac{2}{3}}.$$

Proof. See Lemma 3.1 in [8]. ∎

The following lemma is partly J. Brüdern's Lemma 2 in [1] (see also Lemma 3.3 in [8]).

Lemma 2.3. *Let D_1, D_2 be positive numbers with $D_1 \le D_2 \le X^{\frac{1}{2}}$. Write $\mathcal{M} = \mathfrak{M}(D_2) \setminus \mathfrak{M}(D_1)$. Let $G : \mathcal{M} \to \mathbb{C}$ be a function which, for $\alpha = \frac{a}{q} + \beta \in \mathcal{M}$, satisfies*

$$G(\alpha) \ll (q + X|q\alpha - a|)^{-1}.$$

Furthermore, let $\Psi : \mathbb{R} \to [0,\infty)$ be a function with a Fourier expansion

$$\Psi(\alpha) = \sum_{|h| \le H} \psi_h e(\alpha h)$$

such that $\log H \ll \log X$. Then

i) $\displaystyle\int_{\mathcal{M}} G(\alpha)\Psi(\alpha) d\alpha \ll |\psi_0| X^{-1} D_2 (\log X) + X^{-1+\varepsilon} \sum_{0<|h|\le H} |\psi_h|,$

ii) $\displaystyle\int_{\mathcal{M}} G(\alpha)^2 \Psi(\alpha) d\alpha \ll |\psi_0| X^{-1} (\log X) + (X D_1)^{-1} X^{\varepsilon} \sum_{0<|h|\le H} |\psi_h|.$

Proof. We will follow the argument of the proof of Lemma 2 in [1]. Note that by Theorem 271 in [2], for $h \neq 0$

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ah}{q}\right) \ll \sum_{d|(h,q)} d. \tag{4}$$

By the definition of $\mathcal{M}$ and (4), we have

$$\begin{aligned}
&\int_{\mathcal{M}} G(\alpha)\Psi(\alpha)d\alpha \\
\leq\ & \sum_{|h|\leq H} \psi_h \sum_{q\leq D_2} \frac{1}{q} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ah}{q}\right) \int_{|\beta|\leq \frac{D_2}{X}} \frac{e(\beta h)}{1+X|\beta|} d\beta \\
\ll\ & |\psi_0| X^{-1}(\log X) \sum_{q\leq D_2} 1 + X^{-1}(\log X) \sum_{0<|h|\leq H} |\psi_h| \sum_{q\leq D_2} \frac{1}{q} \sum_{d|(q,h)} d \\
\ll\ & |\psi_0| X^{-1} D_2(\log X) + X^{-1}(\log X) \sum_{0<|h|\leq H} |\psi_h| \sum_{d|h} \sum_{q_1\leq D_2/d} \frac{1}{q_1} \\
\ll\ & |\psi_0| X^{-1} D_2(\log X) + X^{-1+\varepsilon} \sum_{0<|h|\leq H} |\psi_h|.
\end{aligned}$$

This completes the proof of i).

For ii), we have

$$\begin{aligned}
&\int_{\mathcal{M}} G(\alpha)^2\Psi(\alpha)d\alpha \\
&= \sum_{|h|\leq H} \psi_h \sum_{q\leq D_1} \frac{1}{q^2} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ah}{q}\right) \int_{\frac{D_1}{qX}<|\beta|\leq \frac{D_2}{qX}} \frac{e(\beta h)}{(1+X|\beta|)^2} d\beta \\
&\quad + \sum_{|h|\leq H} \psi_h \sum_{D_1<q\leq D_2} \frac{1}{q^2} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ah}{q}\right) \int_{|\beta|\leq \frac{D_2}{qX}} \frac{e(\beta h)}{(1+X|\beta|)^2} d\beta.
\end{aligned} \tag{5}$$

By (4), the contribution of the first part on the right-hand side of (5) is

$$\begin{aligned}
&\sum_{|h|\leq H} \psi_h \sum_{2^u\leq 2D_1} \sum_{2^{u-1}<q\leq 2^u} \frac{1}{q^2} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ah}{q}\right) \int_{\frac{D_1}{2^u X}<|\beta|\leq \frac{D_2}{2^u X/2}} \frac{e(\beta h)}{(1+X|\beta|)^2} d\beta \\
&\ll |\psi_0| X^{-1} \sum_{2^u\leq 2D_1} \sum_{2^{u-1}<q\leq 2^u} \frac{1}{q} \frac{1}{1+D_1 2^{-u}} \\
&\quad + X^{-1} \sum_{0<|h|\leq H} |\psi_h| \sum_{2^u\leq 2D_1} \sum_{2^{u-1}<q\leq 2^u} \frac{1}{q^2} \sum_{d|(q,h)} d \frac{1}{1+D_1 2^{-u}} \\
&\ll |\psi_0| X^{-1}(\log X) + X^{-1} \sum_{0<|h|\leq H} |\psi_h| \sum_{2^u\leq 2D_1} \frac{1}{2^u+D_1} \sum_{d|h} d \sum_{2^u/2d<q_1\leq 2^u/d} \frac{1}{dq_1} \\
&\ll |\psi_0| X^{-1}(\log X) + (XD_1)^{-1} X^{\varepsilon} \sum_{0<|h|\leq H} |\psi_h|.
\end{aligned} \tag{6}$$

By (4), the contribution of the second part is

$$\begin{aligned}
&\sum_{|h|\le H}\psi_h\sum_{D_1<q\le D_2}\frac{1}{q^2}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(\frac{ah}{q}\right)\int_{|\beta|\le\frac{D_2}{X}}\frac{e(\beta h)}{(1+X|\beta|)^2}d\beta\\
&\ll|\psi_0|X^{-1}\sum_{D_1<q\le D_2}\frac{1}{q}+X^{-1}\sum_{0<|h|\le H}|\psi_h|\sum_{D_1<q\le D_2}\frac{1}{q^2}\sum_{d|(q,h)}d\\
&\ll|\psi_0|X^{-1}(\log X)+X^{-1}\sum_{0<|h|\le H}|\psi_h|\sum_{d|h}d\sum_{D_1/d<q_1\le D_2/d}\frac{1}{(dq_1)^2}\\
&\ll|\psi_0|X^{-1}(\log X)+(XD_1)^{-1}\sum_{0<|h|\le H}|\psi_h|.
\end{aligned}\tag{7}$$

In view of (5)–(7), we prove ii). ∎

3. Proof of the Theorem

Write $\mathfrak{m}=(0,1]\setminus\mathfrak{M}(Q_0)$. By Lemma 2.1 and (2), for $n\in V(X;\varphi)$, we have

$$\left|\int_{\mathfrak{m}}f_2(\alpha)^2f_3(\alpha)^2f_6(\alpha)^2e(-\alpha n)d\alpha\right|>\frac{n}{\varphi(n)}.$$

Hence

$$\sum_{n\in V(X;\varphi)}\left|\int_{\mathfrak{m}}f_2(\alpha)^2f_3(\alpha)^2f_6(\alpha)^2e(-\alpha n)d\alpha\right|\gg\frac{VX}{\varphi(X)}.\tag{8}$$

There is a sequence of complex numbers $\eta(n)$ satisfying $|\eta(n)|=1$ such that

$$\begin{aligned}
&\sum_{n\in V(X;\varphi)}\left|\int_{\mathfrak{m}}f_2(\alpha)^2f_3(\alpha)^2f_6(\alpha)^2e(-\alpha n)d\alpha\right|\\
&=\sum_{n\in V(X;\varphi)}\eta(n)\int_{\mathfrak{m}}f_2(\alpha)^2f_3(\alpha)^2f_6(\alpha)^2e(-\alpha n)d\alpha\\
&=\int_{\mathfrak{m}}f_2(\alpha)^2f_3(\alpha)^2f_6(\alpha)^2K(\alpha)d\alpha,
\end{aligned}\tag{9}$$

where

$$K(\alpha)=\sum_{n\in V(X;\varphi)}\eta(n)e(-\alpha n).\tag{10}$$

From (8) and (9), we have

$$V\ll\frac{\varphi(X)}{X}\int_{\mathfrak{m}}\left|f_2(\alpha)^2f_3(\alpha)^2f_6(\alpha)^2K(\alpha)\right|d\alpha.\tag{11}$$

Write

$$\mathfrak{N}_1 = \mathfrak{M}(Q_1) \setminus \mathfrak{M}(Q_2), \quad \mathfrak{N}_2 = \mathfrak{M}(Q_2) \setminus \mathfrak{M}(Q_0), \quad f_2^*(\alpha) = X^{\frac{1}{2}}(q+X|q\alpha-a|)^{-\frac{1}{2}}.$$

According to Theorem 4 in [5], when $\alpha \in \mathfrak{M}(q,a) \subset \mathfrak{M}(Q)$ and $1 \le Q \le 2X^{\frac{1}{2}}$, one has

$$\begin{aligned} f_2(\alpha) &\ll X^{\frac{1}{2}}(q+X|q\alpha-a|)^{-\frac{1}{2}} + (q+X|q\alpha-a|)^{\frac{1}{2}} \\ &\ll X^{\frac{1}{2}}(q+X|q\alpha-a|)^{-\frac{1}{2}} = f_2^*(\alpha). \end{aligned} \tag{12}$$

As a consequence of Dirichlet's theorem on Diophantine approximation and (12), we obtain

$$\begin{aligned} &\int_{\mathfrak{m}} \Big| f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha) \Big| d\alpha \\ \le\ &\Big(\int_{\mathfrak{N}_1} + \int_{\mathfrak{N}_2}\Big) \big| f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha) \big| d\alpha \\ \ll\ &\Big(\int_{\mathfrak{N}_1} + \int_{\mathfrak{N}_2}\Big) f_2^*(\alpha)^2 \big| f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha) \big| d\alpha \\ :=\ &\int_{\mathfrak{N}_1} + \int_{\mathfrak{N}_2}. \end{aligned} \tag{13}$$

Lemma 3.1. *We have*

$$\int_0^1 |f_3(\alpha)^2 K(\alpha)^2| d\alpha \ll X^{\frac{1}{3}} V + X^{\varepsilon} V^2.$$

Proof. See Lemma 2.1 in [6] and its proof. ■

Lemma 3.2. *We have*

$$\int_{\mathfrak{N}_1} \ll X V^{\frac{1}{2}} (\log X)^{\frac{1}{2}} + X^{1-v} V.$$

Proof. Applying Cauchy's inequality and Lemma 2.2, we have

$$\begin{aligned} \textstyle\int_{\mathfrak{N}_1} &\ll \Big(\int_0^1 \big| f_3(\alpha)^2 f_6(\alpha)^4 \big| d\alpha\Big)^{\frac{1}{2}} \Big(\int_{\mathfrak{N}_1} f_2^*(\alpha)^4 \big| f_3(\alpha)^2 K(\alpha)^2 \big| d\alpha\Big)^{\frac{1}{2}} \\ &\ll X^{\frac{1}{3}} \Big(\int_{\mathfrak{N}_1} f_2^*(\alpha)^4 \big| f_3(\alpha)^2 K(\alpha)^2 \big| d\alpha\Big)^{\frac{1}{2}}. \end{aligned} \tag{14}$$

One has

$$\begin{aligned}
&\left|f_3(\alpha)^2K(\alpha)^2\right| = f_3(\alpha)K(\alpha)\overline{f_3(\alpha)K(\alpha)} \\
=& \sum_{x_1,x_2\le X^{1/3}} \sum_{y_1,y_2\in V(X;\varphi)} \eta(y_1)\overline{\eta(y_2)}e\big(\alpha(x_1^3-x_2^3+y_1-y_2)\big) \\
=& \sum_{|h|\le 2X} e(\alpha h) \sum_{\substack{x_1^3-x_2^3+y_1-y_2=h \\ x_1,x_2\le X^{1/3},\ y_1,y_2\in V(X;\varphi)}} \eta(y_1)\overline{\eta(y_2)} \\
=& \sum_{|h|\le 2X} s(h)e(\alpha h),
\end{aligned}$$

where

$$s(h) = \sum_{\substack{x_1^3-x_2^3+y_1-y_2=h \\ x_1,x_2\le X^{1/3},\ y_1,y_2\in V(X;\varphi)}} \eta(y_1)\overline{\eta(y_2)}.$$

Then according to Lemma 2.3 ii), we have

$$\begin{aligned}
&\int_{\mathfrak{N}_1} f_2^*(\alpha)^4\left|f_3(\alpha)^2K(\alpha)^2\right|d\alpha = \int_{\mathfrak{N}_1} f_2^*(\alpha)^4 \sum_{|h|\le 2X} s(h)e(\alpha h)d\alpha \\
&\ll X^2X^{-1}|s(0)|(\log X) + X^2(XQ_2)^{-1}X^\varepsilon \sum_{0<|h|\le 2X} |s(h)| \\
&\ll X(X^{\frac13}V + X^\varepsilon V^2)(\log X) + XQ_2^{-1}X^\varepsilon X^{\frac23}V^2 \\
&\ll X^{\frac43}V(\log X) + X^{\frac43-(\frac32)^2v+\varepsilon}V^2,
\end{aligned} \tag{15}$$

where we used Lemma 3.1 and

$$\begin{aligned}
|s(0)| &\le \int_0^1 \left|f_3(\alpha)^2K(\alpha)^2\right|d\alpha, \\
\sum_{0<|h|\le 2X} |s(h)| &\ll \sum_{0<|h|\le 2X} \sum_{\substack{x_1^3-x_2^3+y_1-y_2=h \\ x_1,x_2\le X^{1/3},y_1,y_2\in V(X;\varphi)}} 1 \ll X^{\frac23}V^2.
\end{aligned}$$

As a consequence of (14) and (15), we obtain

$$\begin{aligned}
\int_{\mathfrak{N}_1} &\ll X^{\frac13}X^{\frac23}V^{\frac12}(\log X)^{\frac12} + X^{\frac13}X^{\frac23-\frac12(\frac32)^2v+\varepsilon}V \\
&\ll XV^{\frac12}(\log X)^{\frac12} + X^{1-v}V.
\end{aligned}$$

∎

Lemma 3.3. *We have*

$$\int_{\mathfrak{N}_2} \ll X^{1-v}V.$$

Proof. For $\alpha \in \mathfrak{M}(q,a) \subset \mathfrak{N}_2$, by Theorem 4.1 in [4] and Lemma 4.2 in [3], we have

$$\begin{aligned} f_3(\alpha) &\ll X^{\frac{1}{3}} q^{-\frac{1}{3}} \big(1 + X|\beta|\big)^{-\frac{1}{3}} + q^{\frac{1}{2}+\varepsilon} \big(1 + X|\beta|\big)^{\frac{1}{2}} \\ &\ll X^{\frac{1}{3}} q^{-\frac{1}{3}} \big(1 + X|\beta|\big)^{-\frac{1}{3}} + Q_2^{\frac{1}{2}+\varepsilon}. \end{aligned} \tag{16}$$

Write

$$f_3^*(\alpha) = X^{\frac{1}{3}} q^{-\frac{1}{3}} \big(1 + X|\beta|\big)^{-\frac{1}{3}}.$$

In view of (16), we have

$$\begin{aligned} \textstyle\int_{\mathfrak{N}_2} \ll& \sup_{\alpha\in\mathfrak{N}_2} \big|K(\alpha)\big| \int_{\mathfrak{N}_2} f_2^*(\alpha)^2 f_3^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha \\ &+ Q_2^{1+2\varepsilon} \sup_{\alpha\in\mathfrak{N}_2} \big|K(\alpha)\big| \int_{\mathfrak{N}_2} f_2^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha. \end{aligned} \tag{17}$$

Define

$$\mathfrak{N}_3 = \mathfrak{M}(Q_2) \setminus \mathfrak{M}(Q_3), \quad \mathfrak{N}_4 = \mathfrak{M}(Q_3) \setminus \mathfrak{M}(Q_4), \quad \mathfrak{N}_5 = \mathfrak{M}(Q_4) \setminus \mathfrak{M}(Q_0).$$

Whence

$$\begin{aligned} &\int_{\mathfrak{N}_2} f_2^*(\alpha)^2 f_3^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha \\ &= \Big(\int_{\mathfrak{N}_3} + \int_{\mathfrak{N}_4} + \int_{\mathfrak{N}_5}\Big) f_2^*(\alpha)^2 f_3^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha. \end{aligned} \tag{18}$$

For $\alpha \in \mathfrak{N}_3$, we have

$$f_3^*(\alpha) \ll X^{\frac{1}{3}} Q_3^{-\frac{1}{3}}. \tag{19}$$

Following the argument of the proof of (15), by (19) and Lemma 2.3 i), we obtain

$$\begin{aligned} &\int_{\mathfrak{N}_3} f_2^*(\alpha)^2 f_3^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha \\ &\ll X^{\frac{2}{3}} Q_3^{-\frac{2}{3}} \int_{\mathfrak{N}_3} f_2^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha \\ &\ll X^{\frac{2}{3}} Q_3^{-\frac{2}{3}} X \Big(X^{\frac{1}{6}} X^{-1} Q_2 (\log X) + X^{-1+\varepsilon} X^{\frac{1}{3}}\Big) \\ &\ll X^{1-v}. \end{aligned} \tag{20}$$

By the same reason, we have

$$\int_{\mathfrak{N}_2} f_2^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha \ll X^{\frac{1}{2}+(\frac{3}{2})^2 v+\varepsilon}, \tag{21}$$

$$\int_{\mathfrak{N}_4} f_2^*(\alpha)^2 f_3^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha \ll X^{1-v} \tag{22}$$

and

$$\int_{\mathfrak{N}_5} f_2^*(\alpha)^2 f_3^*(\alpha)^2 \big|f_6(\alpha)^2\big| d\alpha \ll X^{1-v}. \tag{23}$$

Hence by (17), (18) and (20)–(23), we prove the lemma. ■

As a consequence of Lemmas 3.2 and 3.3, we have

$$V \ll \frac{\varphi(X)}{X}\Big(XV^{\frac{1}{2}}(\log X)^{\frac{1}{2}} + X^{1-v}V\Big),$$

whence

$$V \ll \varphi(X)^2\big(\log X\big).$$

Summing over dyadic intervals to cover the set of integers $[1, X] \cap \mathbb{Z}$, we conclude the estimate

$$E(X;\varphi) \leq \sum_{2^j \leq X} \text{card}\big(V(2^{j+1};\varphi)\big) \ll \varphi(X)^2\big(\log X\big)^2.$$

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Xiaodong Lü
Department of Mathematics
Tongji University
Shanghai 200092
P. R. China
E-mail: xidolv@gmail.com

Quanwu Mu
School of Science
Xi'an Polytechnic University
Xi'an, Shaanxi Province 710048
P. R. China
E-mail: muquanwu@163.com