TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 5, pp. 1533-1540, October 2015
DOI: 10.11650/tjm.19.2015.5624
This paper is available online at http://journal.taiwanmathsoc.org.tw

CARLEMAN INEQUALITIES FOR FRACTIONAL LAPLACIANS AND UNIQUE CONTINUATION

Ihyeok Seo

Abstract. We obtain a unique continuation result for fractional Schrödinger operators with potential in Morrey spaces. This is based on Carleman inequalities for fractional Laplacians.

1. Introduction

The aim of this paper is to obtain a unique continuation result for the fractional Schrödinger operator $(-\Delta)^{\alpha/2}+V(x)$, $0<\alpha<n$. Recently, this operator has attracted interest from mathematics as well as mathematical physics. This is because Laskin [9] introduced the fractional quantum mechanics governed by the fractional Schrödinger equation

$$i\partial_t\Psi(x,t)=((-\Delta)^{\alpha/2}+V(x))\Psi(x,t),$$

where the fractional Schrödinger operator plays a central role.

More generally, we will consider the following differential inequality

$$|(-\Delta)^{\alpha/2}u(x)|\leq V(x)|u(x)|,\quad x\in\mathbb{R}^n,\quad n\geq 3, \tag{1.1}$$

where $(-\Delta)^{\alpha/2}$ is defined for $0<\alpha<n$ by means of the Fourier transform $\mathcal{F}f$ $(=\widehat{f})$, as follows:

$$\mathcal{F}[(-\Delta)^{\alpha/2}f](\xi)=|\xi|^\alpha\widehat{f}(\xi).$$

The problem is now to find conditions on the potential $V(x)$ that imply the unique continuation property which means that a solution of (1.1) vanishing in an open subset of $\mathbb{R}^n$ must vanish identically.

In the classical case $\alpha=2$, the property was extensively studied in connection with the problem of absence of positive eigenvalues of the Schrödinger operator $-\Delta+V(x)$. Among others, Jerison and Kenig [4] proved the property for $V\in L^{n/2}_{\text{loc}}$, $n\geq 3$. Around

Received December 3, 2014, accepted March 17, 2015.
Communicated by Tai-Peng Tsai.
2010 *Mathematics Subject Classification*: Primary 35B60, 35B45; Secondary 35J10.
Key words and phrases: Unique continuation, Carleman inequalities, Schrödinger operators.

the same time, an extension to $L^{n/2,\infty}_{\text{loc}}$ was obtained by Stein [16] with the smallness assumption that

$$\sup_{a\in\mathbb{R}^n}\lim_{r\to 0}\|\chi_{B(a,r)}V\|_{L^{n/2,\infty}}$$

is sufficiently small. (Here, $\chi_{B(a,r)}$ denotes the characteristic function of the ball with center $a\in\mathbb{R}^n$ and radius $r>0$.) Note that this assumption is trivially satisfied for $V\in L^{n/2}_{\text{loc}}$ because $L^{n/2}_{\text{loc}}\subset L^{n/2,\infty}_{\text{loc}}$. Also, the above-mentioned results later turn out to be optimal in the context of L^p potentials (see [5, 7]).

Recently, there was an attempt [13] to deal with the fractional case where $n-1\le\alpha<n$. After that, the author [14] extended Stein's result completely to $0<\alpha<n$. Namely, it turns out that (1.1) has the unique continuation property for $V\in L^{n/\alpha,\infty}_{\text{loc}}$ with the corresponding smallness assumption that

$$\sup_{a\in\mathbb{R}^n}\lim_{r\to 0}\|\chi_{B(a,r)}V\|_{L^{n/\alpha,\infty}}$$

is sufficiently small. See also [8, 12] for higher orders where $\alpha/2$ are positive integers, and for some fractional elliptic equations see [3, 10, 11].

In this paper we improve the class of potentials to the Morrey class $\mathcal{L}^{\alpha,p}$ which is defined for $\alpha>0$ and $1\le p\le n/\alpha$ by

$$V\in\mathcal{L}^{\alpha,p}\quad\Leftrightarrow\quad\|V\|_{\mathcal{L}^{\alpha,p}}:=\sup_{Q\text{ cubes in }\mathbb{R}^n}|Q|^{\alpha/n}\Big(\frac{1}{|Q|}\int_Q V(y)^p dy\Big)^{\frac{1}{p}}<\infty.$$

In particular, $\mathcal{L}^{\alpha,p}=L^p$ when $p=n/\alpha$, and even $L^{n/\alpha,\infty}\subset\mathcal{L}^{\alpha,p}$ for $p<n/\alpha$. Our result is the following theorem.

Theorem 1.1. *Let $n\ge 3$ and $0<\alpha<n$. Assume that $V\in\mathcal{L}^{\alpha,p}$ for $p>(n-1)/\alpha$. Let $u\in L^2\cap L^2(V)$ be a solution of* (1.1) *vanishing in a non-empty open subset of $\mathbb{R}^n$. Then $u\equiv 0$ if*

$$\sup_{a\in\mathbb{R}^n}\lim_{r\to 0}\|\chi_{B(a,r)}V\|_{\mathcal{L}^{\alpha,p}} \tag{1.2}$$

is sufficiently small. Here, $L^2(V)=L^2(V(x)dx)$.

Let us give some remarks about the assumptions in the theorem. First, $L^2\cap L^2(V)$ is the solution space for which we have unique continuation. It should be noted that the space is dense in L^2. In fact, consider $D_n=\{x\in\mathbb{R}^n: V^{1/2}\le n\}$. Then, for $f\in L^2$, $\chi_{D_n}f$ is contained in $L^2\cap L^2(V)$, and $\chi_{D_n}f\to f$ as $n\to\infty$. Now the Lebesgue dominated convergence theorem gives that $\chi_{D_n}f\to f$ in L^2. Thus, the solution space is dense in L^2.

Next, by taking the rescaling $u_\varepsilon(x)=u(\varepsilon x)$, the equation $(-\Delta)^{\alpha/2}u=Vu$ becomes $(-\Delta)^{\alpha/2}u_\varepsilon=V_\varepsilon u_\varepsilon$, where $V_\varepsilon(x)=\varepsilon^\alpha V(\varepsilon x)$. It is also easy to see that $\|V_\varepsilon\|_{\mathcal{L}^{\alpha,p}}=\|V\|_{\mathcal{L}^{\alpha,p}}$. Hence, $\mathcal{L}^{\alpha,p}$ is invariant under the scaling.

The above theorem is a consequence of the following Carleman inequalities which can be seen as natural extensions to the fractional Laplacians $(-\Delta)^{\alpha/2}$ of those in [2] for the case $\alpha = 2$.

Proposition 1.2. *Let* $n \geq 3$ *and* $0 < \alpha < n$. *Assume that* $V \in \mathcal{L}^{\alpha,p}$ *for* $p > (n-1)/\alpha$. *Then there exist sequence* $\{t_m : m = 0, 1, \ldots\}$ *and constants* $C, \beta > 0$ *independent of* m *and* r *such that*

$$\big\|\chi_{B(0,r)}|x|^{-t_m-\frac{n-\alpha}{2}} f\big\|_{L^2(V)} \leq C\|\chi_{B(0,r)}V\|^{\beta}_{\mathcal{L}^{\alpha,p}}\big\||x|^{-t_m-\frac{n-\alpha}{2}}(-\Delta)^{\alpha/2} f\big\|_{L^2(V^{-1})}$$

for $f, (-\Delta)^{\alpha/2} f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. *Here,* $t_m \to \infty$ *as* $m \to \infty$.

Throughout the paper, we will use the letter C to denote positive constants possibly different at each occurrence.

2. Unique Continuation

Here we prove Theorem 1.1 assuming Proposition 1.2 which will be shown in the next section.

Without loss of generality, we may prove that the solution u must vanish identically if it vanishes in a sufficiently small neighborhood of the origin.

Since we are assuming that $u \in L^2 \cap L^2(V)$ vanishes near the origin, by (1.1), $(-\Delta)^{\alpha/2} u \in L^2(V^{-1})$ vanishes also near the origin. Now, from the Carleman inequality in Proposition 1.2 (with a standard limiting argument involving a C_0^∞ approximate identity), one can easily see that

$$\begin{aligned} &\big\|\chi_{B(0,r)}|x|^{-t_m-\frac{n-\alpha}{2}} u\big\|_{L^2(V)} \\ &\leq C\|\chi_{B(0,r)}V\|^{\beta}_{\mathcal{L}^{\alpha,p}}\big\||x|^{-t_m-\frac{n-\alpha}{2}}(-\Delta)^{\alpha/2} u\big\|_{L^2(V^{-1})}. \end{aligned} \tag{2.1}$$

Note also that from (1.1)

$$\begin{aligned} \big\||x|^{-t_m-\frac{n-\alpha}{2}}(-\Delta)^{\alpha/2} u\big\|_{L^2(V^{-1})} &\leq C\big\|\chi_{B(0,r)}|x|^{-t_m-\frac{n-\alpha}{2}} u\big\|_{L^2(V)} \\ &\quad + C\big\|(1-\chi_{B(0,r)})|x|^{-t_m-\frac{n-\alpha}{2}}(-\Delta)^{\alpha/2} u\big\|_{L^2(V^{-1})}. \end{aligned}$$

So, if we choose r small enough so that $\|\chi_{B(0,r)}V\|^{\beta}_{\mathcal{L}^{\alpha,p}}$ is sufficiently small (see (1.2)), then the first term on the right-hand side can be absorbed into the left-hand side of (2.1). Thus we get

$$\begin{aligned} \big\|\chi_{B(0,r)}|x|^{-t_m-\frac{n-\alpha}{2}} u\big\|_{L^2(V)} &\leq C\big\|(1-\chi_{B(0,r)})|x|^{-t_m-\frac{n-\alpha}{2}}(-\Delta)^{\alpha/2} u\big\|_{L^2(V^{-1})} \\ &\leq C r^{-t_m-\frac{n-\alpha}{2}}\big\|(-\Delta)^{\alpha/2} u\big\|_{L^2(V^{-1})}, \end{aligned}$$

which in turn implies

$$\left\|\chi_{B(0,r)}\big(\frac{r}{|x|}\big)^{t_m+\frac{n-\alpha}{2}}u\right\|_{L^2(V)} \le C\left\|(-\Delta)^{\alpha/2}u\right\|_{L^2(V^{-1})} < \infty.$$

By letting $m \to \infty$ we conclude that $u = 0$ on $B(0,r)$. Now, $u \equiv 0$ by a standard connectedness argument.

3. Carleman Inequalities

In this section we will obtain the Carleman inequality in Proposition 1.2 by using Stein's complex interpolation [15], as in [2], on an analytic family of operators $S_z^{t,\alpha}$ defined by

$$S_z^{t,\alpha}g(x) = \frac{V(x)^{\frac{z}{2\alpha}}}{\Gamma((n-z)/2)}\int_{\mathbb{R}^n} K_z(x,y)V(y)^{\frac{z}{2\alpha}}g(y)dy,$$

where $0 \le \operatorname{Re} z \le n$ and

$$K_z(x,y) = C_z\left(\frac{|y|}{|x|}\right)^{t+(n-z)/2}\left(|x-y|^{-n+z} - \sum_{j=0}^{m-1}\frac{1}{j!}\Big(\frac{\partial}{\partial s}\Big)^j|sx-y|^{-n+z}\Big|_{s=0}\right).$$

Here, $S_z^{t,2}$ coincides with the analytic family of operators S_z^t in [2]. Note also that

$$S_\alpha^{t,\alpha}\left(\frac{(-\Delta)^{\alpha/2}f(y)}{V(y)^{1/2}|y|^{t+(n-\alpha)/2}}\right)(x) = \frac{f(x)V(x)^{1/2}}{|x|^{t+(n-\alpha)/2}} \tag{3.1}$$

(see Lemma 2.1 in [13]).

Let m be nonnegative integers. Now it is enough to show that there exist constants $C, \beta > 0$ independent of m and r such that

$$\|\chi_{B(0,r)}S_\alpha^{t_m,\alpha}g\|_{L^2} \le C\|\chi_{B(0,r)}V\|_{\mathcal{L}^{\alpha,p}}^\beta\|g\|_{L^2} \tag{3.2}$$

for $q > (n-1)/\alpha$, $0 < \varepsilon < \alpha(q-(n-1)/\alpha)$ and $t_m = m - 1 + (1-\varepsilon)/2$. Indeed, from (3.1) and (3.2),

$$\left\|\chi_{B(0,r)}\frac{f(x)V(x)^{1/2}}{|x|^{t_m+(n-\alpha)/2}}\right\|_{L^2} \le C\|\chi_{B(0,r)}V\|_{\mathcal{L}^{\alpha,p}}^\beta\left\|\frac{(-\Delta)^{\alpha/2}f(y)}{V(y)^{1/2}|y|^{t_m+(n-\alpha)/2}}\right\|_{L^2},$$

which is equivalent to

$$\left\|\chi_{B(0,r)}\frac{f(x)}{|x|^{t_m+(n-\alpha)/2}}\right\|_{L^2(V)} \le C\|\chi_{B(0,r)}V\|_{\mathcal{L}^{\alpha,p}}^\beta\left\|\frac{(-\Delta)^{\alpha/2}f(y)}{|y|^{t_m+(n-\alpha)/2}}\right\|_{L^2(V^{-1})}$$

as desired.

To show (3.2), we use Stein's complex interpolation between the following two estimates for the cases of $\mathrm{Re}\, z = 0$ and $n-1 < \mathrm{Re}\, z < \alpha q$:

$$\|\chi_{B(0,r)} S_{i\gamma}^{t_m,\alpha} g\|_{L^2} \le Ce^{c|\gamma|}\|g\|_{L^2} \tag{3.3}$$

and

$$\|\chi_{B(0,r)} S_{n-1+\varepsilon+i\gamma}^{t_m,\alpha} g\|_{L^2} \le Ce^{c|\gamma|}\|\chi_{B(0,r)}V\|_{\mathcal{L}^{\alpha,p}}^{(n-1+\varepsilon)/2\alpha}\|g\|_{L^2}, \tag{3.4}$$

where $\gamma \in \mathbb{R}$, $p > (n-1)/\alpha$, $0 < \varepsilon < \alpha(p-(n-1)/\alpha)$ and $t_m = m-1+(1-\varepsilon)/2$. Indeed, since $n-1 < n-1+\varepsilon < \alpha p \le n$ and $p > 1$, we can easily get (3.2) using the complex interpolation between (3.3) and (3.4).

It remains to show (3.3) and (3.4). The first estimate (3.3) follows immediately from Lemma 2.3 in [4]. Indeed, consider the family of operators T_z^t given by

$$T_z^t g(x) = \frac{1}{\Gamma((n-z)/2)}\int_{\mathbb{R}^n} H_z(x,y)g(y)|y|^{-n}dy,$$

where

$$H_z(x,y) = C_z|x|^{-t}|y|^{n+t-z}\left(|x-y|^{-n+z} - \sum_{j=0}^{m-1}\frac{1}{j!}\Big(\frac{\partial}{\partial s}\Big)^j|sx-y|^{-n+z}\bigg|_{s=0}\right).$$

Then it is clear that (3.3) follows from

$$\|T_{i\gamma}^{t_m} g\|_{L^2(dx/|x|^n)} \le Ce^{c|\gamma|}\|g\|_{L^2(dx/|x|^n)}$$

which is Lemma 2.3 of [4].

For the second one, we first recall from [2] (see (3.9) there) that

$$\begin{aligned}&\left||x-y|^{-n+z} - \sum_{j=0}^{m-1}\frac{1}{j!}\Big(\frac{\partial}{\partial s}\Big)^j|sx-y|^{-n+z}\bigg|_{s=0}\right| \\ &\le Ce^{c|\mathrm{Im}\, z|}\left(\frac{|x|}{|y|}\right)^{m-1+n-\mathrm{Re}\, z}|x-y|^{-n+\mathrm{Re}\, z}\end{aligned}$$

for $n-1 < \mathrm{Re}\, z < n$. From this, we then get

$$|S_{n-1+\varepsilon+i\gamma}^{t_m,\alpha} g(x)| \le Ce^{c|\gamma|}V(x)^{(n-1+\varepsilon)/2\alpha}\int_{\mathbb{R}^n}|x-y|^{n-1+\varepsilon-n}V(y)^{(n-1+\varepsilon)/2\alpha}|g(y)|dy$$

if $0 < \varepsilon < 1$. Hence it follows that

$$\begin{aligned}&\|\chi_{B(0,r)} S_{n-1+\varepsilon+i\gamma}^{t_m,\alpha} g\|_{L^2} \\ &\le Ce^{c|\gamma|}\|\chi_{B(0,r)} I_{n-1+\varepsilon}(V(y)^{(n-1+\varepsilon)/2\alpha}|g(y)|)\|_{L^2(V^{(n-1+\varepsilon)/\alpha})},\end{aligned} \tag{3.5}$$

where I_α denotes the fractional integral operator defined for $0 < \alpha < n$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Here we will use the following lemma to show (3.4), which characterizes weighted L^2 inequalities for I_α, due to Kerman and Sawyer [6] (see Theorem 2.3 there and also Lemma 2.1 in [1]):

Lemma 3.1. *Let $0 < \alpha < n$. Assume that w is a nonnegative measurable function on $\mathbb{R}^n$. Then there exists a constant C_w depending on w such that the following two equivalent estimates*

$$\|I_{\alpha/2} f\|_{L^2(w)} \le C_w \|f\|_{L^2}$$

and

$$\|I_{\alpha/2} f\|_{L^2} \le C_w \|f\|_{L^2(w^{-1})}$$

are valid for all measurable functions f if and only if

$$\sup_Q \left(\int_Q w(x) dx \right)^{-1} \int_Q \int_Q \frac{w(x)w(y)}{|x-y|^{n-\alpha}} dx dy \tag{3.6}$$

is finite. Here the sup is taken over all dyadic cubes Q in $\mathbb{R}^n$, and the constant C_w may be taken to be a constant multiple of the square root of (3.6).

Indeed, it is known that $\|w\|_{\mathcal{L}^{\alpha,p}} < \infty$ for $p > 1$ is a sufficient condition for the finiteness of (3.6) (see Subsection 2.2 in [1]). Namely, (3.6) $\le C\|w\|_{\mathcal{L}^{\alpha,p}}$ for $p > 1$. Using this fact and applying the above lemma with $\alpha = n - 1 + \varepsilon$, from (3.5) and $I_{\alpha/2} I_{\alpha/2} = I_\alpha$, we see that for $1 < q \le n/(n-1+\varepsilon)$

$$\begin{aligned}
&\|\chi_{B(0,r)} S^{t_m,\alpha}_{n-1+\varepsilon+i\gamma} g\|_{L^2} \\
&\le Ce^{c|\gamma|} \|\chi_{B(0,r)} V^{(n-1+\varepsilon)/\alpha}\|^{1/2}_{\mathcal{L}^{n-1+\varepsilon,q}} \|I_{(n-1+\varepsilon)/2}(V(y)^{(n-1+\varepsilon)/2\alpha}|g(y)|)\|_{L^2} \\
&\le Ce^{c|\gamma|} \|\chi_{B(0,r)} V^{(n-1+\varepsilon)/\alpha}\|^{1/2}_{\mathcal{L}^{n-1+\varepsilon,q}} \\
&\quad \times \|V^{(n-1+\varepsilon)/\alpha}\|^{1/2}_{\mathcal{L}^{n-1+\varepsilon,q}} \|V^{(n-1+\varepsilon)/2\alpha} g\|_{L^2(V^{-(n-1+\varepsilon)/\alpha})} \\
&= Ce^{c|\gamma|} \|\chi_{B(0,r)} V^{(n-1+\varepsilon)/\alpha}\|^{1/2}_{\mathcal{L}^{n-1+\varepsilon,q}} \|V^{(n-1+\varepsilon)/\alpha}\|^{1/2}_{\mathcal{L}^{n-1+\varepsilon,q}} \|g\|_{L^2} \\
&= Ce^{c|\gamma|} \|\chi_{B(0,r)} V\|^{(n-1+\varepsilon)/2\alpha}_{\mathcal{L}^{\alpha,q(n-1+\varepsilon)/\alpha}} \|V\|^{(n-1+\varepsilon)/2\alpha}_{\mathcal{L}^{\alpha,q(n-1+\varepsilon)/\alpha}} \|g\|_{L^2}.
\end{aligned}$$

Since $(n-1)/\alpha < (n-1+\varepsilon)/\alpha < q(n-1+\varepsilon)/\alpha \le n/\alpha$ and $V \in \mathcal{L}^{\alpha,p}$, by choosing q, ε so that $p = q(n-1+\varepsilon)/\alpha$, we now get the desired estimate (3.4).

ACKNOWLEDGMENT

The author would like to thank Luis Escauriaza for bringing his attention to Carleman inequalities for fractional Laplacians.

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Ihyeok Seo
Department of Mathematics
Sungkyunkwan University
Suwon 440-746
Republic of Korea
E-mail: ihseo@skku.edu