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# ON SOLUTIONS TO MATRIX INEQUALITIES WITH APPLICATIONS

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**Abstract.** In this paper, we first study the solution to linear matrix inequality $AXB + (AXB)^* \geqslant (>, \leqslant, <)\ C$ when the matrix $G = \begin{pmatrix} A & B^* \end{pmatrix}$ is full row rank, where $C$ is a Hermitian matrix. Furthermore, for the applications, we derive the representations for the Re-nnd $\{1, 2, i\}$-inverses for $A$, $i = 3, 4$, and the Re-nnd solution to $AXB = C$, some special matrix equations are also considered.

## 1. INTRODUCTION

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the complex field $\mathbb{C}$, $\mathbb{C}_H^m$ denote the set of all $m \times m$ Hermitian matrices, $\mathbb{U}_n$ denote the set of all $n \times n$ unitary matrices. For $A \in \mathbb{C}^{m\times n}$, its range space, rank, Moore-Penrose inverse [1] and conjugate transpose will be denoted by $R(A)$, $r(A)$, $A^\dagger$ and $A^*$ respectively. $i_+(A)$ and $i_-(A)$ denote the numbers of the positive and negative eigenvalues of a Hermitian matrix $A$ counted with multiplicities, respectively. The identity matrix of order $n$ is denoted by $I_n$. Set $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$.

The Hermitian part of $A \in \mathbb{C}^{m\times m}$ is defined as $H(A) = \frac{1}{2}(A+A^*)$. We say that $A$ is Re-nnd (Re-nonnegative definite) if $H(A) \geqslant 0$ and $A$ is Re-pd (Re-positive definite) if $H(A) > 0$. Let $A_{re}^{(i,j,\ldots,k)}$ be the Re-nnd $\{i, j, \ldots, k\}$-inverse of square matrix $A$. Recently, some researches on Re-nnd solution and Re-nnd generalized inverse were done by several authors [2, 3, 4, 6, 12].

In this article, we consider the matrix inequality in the Löwner partial ordering

$$AXB + (AXB)^* \geqslant C, \tag{1.1}$$

$$AXB + (AXB)^* > C, \tag{1.2}$$

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where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ and $C \in \mathbb{C}_H^m$ are given, $X \in \mathbb{C}^{n\times p}$ is variable matrix. Specially, when $C$ is nonnegative definite matrix, this case has been considered by Tian and Rosen [10]. The case $B = I_m$ was studied by Tian [9]; based on Tian's results, Nikolov and Cvetković-Ilić [6] derived general expressions for Re-nnd $\{1,3\}$-inverse and $\{1,4\}$-inverse of $A \in \mathbb{C}^{m\times m}$. Moreover, Tian and Rosen [10] shown that (1.1) can equivalently be written as

$$AXB + (AXB)^* = C + VV^* \tag{1.3}$$

for some $V$. And equation (1.3) is solvable for $X$ if and only if $VV^*$ satisfies

$$E_G VV^* = -E_G C, \;\; E_A VV^* E_A = -E_A C E_A, \;\; F_B VV^* F_B = -F_B C F_B, \tag{1.4}$$

where $G = \begin{pmatrix} A & B^* \end{pmatrix}$. Generally, it is very difficult to establish a common solution to equations (1.4), so it is also difficult to solve (1.1). However, in some special cases, matrix $G$ maybe satisfy some certain conditions. For example, in the study of Re-nnd $\{1,2,3\}$-inverse and $\{1,2,4\}$-inverse of $A \in \mathbb{C}^{m\times m}$, Liu and Fang [4] shown that a necessary condition is $r(A^2) = r(A)$ for the existence of these two Re-nnd generalized inverses. In order to establish a representation of Re-nnd $\{1,2,3\}$-inverse, one needs to solve the following matrix inequality

$$F_A V A A^\dagger + (F_A V A A^\dagger)^* \geqslant -[A^\dagger + (A^\dagger)^*], \tag{1.5}$$

where $V$ is variable matrix. It is easy to verify that $G = \begin{pmatrix} F_A & AA^\dagger \end{pmatrix}$ satisfies $r(G) = m$ under condition $r(A^2) = r(A)$, i.e., $G$ is full row rank.

Cvetković-Ilić [2] provided a condition for the existence of Re-nnd solution to $AXB = C$, where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$ and $C \in \mathbb{C}^{m\times p}$, and the structure of the general solution was provided. Since the general solution to $AXB = C$ is given by $X = A^\dagger C B^\dagger + F_A W_1 + W_2 E_B$, thus establishing a expression of the Re-nnd solution is equivalent to solve the following matrix inequality

$$\begin{pmatrix} F_A & E_B \end{pmatrix} \begin{pmatrix} W_1 \\ W_2^* \end{pmatrix} + \begin{pmatrix} W_1^* & W_2 \end{pmatrix} \begin{pmatrix} F_A \\ E_B \end{pmatrix} \geqslant -[A^\dagger C B^\dagger + (A^\dagger C B^\dagger)^*].$$

Here, $G = \begin{pmatrix} F_A & E_B & I_n \end{pmatrix}$ is also full row rank.

Indeed, when matrix $G$ is full row rank, then (1.4) reduces to

$$E_A VV^* E_A = -E_A C E_A, \;\; F_B VV^* F_B = -F_B C F_B. \tag{1.6}$$

This paper is organized as follows. In section 2, we first present a general solution to the linear matrix inequality $AXB + (AXB)^* \geqslant (>, \leqslant, <)\ C$ when the matrix $G = \begin{pmatrix} A & B^* \end{pmatrix}$ is full row rank. In section 3, for the applications, we derive the representations for the Re-nnd $\{1,2,i\}$-inverses, $i = 3,4$, and the Re-nnd solution to $AXB = C$.

Before giving the main results, we first introduce several lemmas as follows.

**Lemma 1.1.** [5] *Let* $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$ *and* $C \in \mathbb{C}^{p\times m}$ *be given. Then*

$$\max_{X\in\mathbb{C}^{n\times p}} i_{\pm}[A - BXC - (BXC)^*] = \min\{i_{\pm}(M_1),\ \ i_{\pm}(M_2)\},$$

$$\min_{X\in\mathbb{C}^{n\times p}} i_{\pm}[A - BXC - (BXC)^*] = r\begin{pmatrix} A & B & C^* \end{pmatrix} + \max\{i_{\pm}(M_1) - r(N_1),\ \ i_{\pm}(M_2) - r(N_2)\},$$

*where*

$$M_1 = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix},\ \ M_2 = \begin{pmatrix} A & C^* \\ C & 0 \end{pmatrix},$$
$$N_1 = \begin{pmatrix} A & B & C^* \\ B^* & 0 & 0 \end{pmatrix},\ \ N_2 = \begin{pmatrix} A & B & C^* \\ C & 0 & 0 \end{pmatrix}.$$

**Lemma 1.2.** [5] *Let* $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$, *and denote* $M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$. *Then*

$$i_{\pm}(M) = r(B) + i_{\pm}(E_B A E_B).$$

**Lemma 1.3.** [10] *Let* $A \in \mathbb{C}^{m\times p}$ *and* $B \in \mathbb{C}^{q\times m}$ *and* $C \in \mathbb{C}_H^m$ *are given. Then the matrix equation* $AXB + (AXB)^* = C$ *has a solution* $X \in \mathbb{C}^{p\times q}$ *if and only if*

$$\begin{pmatrix} A & B^* \end{pmatrix}\begin{pmatrix} A & B^* \end{pmatrix}^{\dagger} C = C,\ \ E_A C E_A = 0,\ \ F_B C F_B = 0.$$

*In this case, the general solution can be written as*

$$X = \frac{1}{2}(X_1 + X_2^*),$$

*where* $X_1$ *and* $X_2$ *are general solutions of the equation* $AX_1B + B^*X_2A^* = C$.

**Lemma 1.4.** [11] *Let* $A_1 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{p\times k}$, $A_2 \in \mathbb{C}^{m\times l}$, $B_2 \in \mathbb{C}^{q\times k}$ *and* $C \in \mathbb{C}^{m\times k}$ *be known and* $X_1 \in \mathbb{C}^{n\times p}$, $X_2 \in \mathbb{C}^{l\times q}$ *unknown;* $M = E_{A_1}A_2$, $N = B_2F_{B_1}$, $S = A_2F_M$. *Then the following statements are equivalent:*

(i) *The system* $A_1X_1B_1 + A_2X_2B_2 = C$ *is solvable;*

(ii) *The following rank equalities are satisfied,*

$$r\begin{pmatrix} A_1 & C \\ 0 & B_2 \end{pmatrix} = r\begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix},\ \ r\begin{pmatrix} A_2 & C \\ 0 & B_1 \end{pmatrix} = r\begin{pmatrix} A_2 & 0 \\ 0 & B_1 \end{pmatrix},$$
$$r\begin{pmatrix} C & A_1 & A_2 \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix},\ \ r\begin{pmatrix} B_1 \\ B_2 \\ C \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

*In this case, the general solution can be expressed as*

$$\begin{aligned} X_1 &= A_1^\dagger CB_1^\dagger - A_1^\dagger A_2 M^\dagger E_{A_1} CB_1^\dagger - A_1^\dagger SA_2^\dagger CF_{B_1} N^\dagger B_2 B_1^\dagger \\ &\quad - A_1^\dagger SVE_N B_2 B_1^\dagger + F_{A_1} U + ZE_{B_1}, \\ X_2 &= M^\dagger E_{A_1} CB_2^\dagger + F_M S^\dagger SA_2^\dagger CF_{B_1} N^\dagger + F_M(V - S^\dagger SVNN^\dagger) + WE_{B_2}, \end{aligned}$$

*where $U$, $V$, $W$ and $Z$ are arbitrary matrices over complex field with appropriate sizes.*

**Lemma 1.5.** [13] *Given matrix $A, B, C, D \in \mathbb{C}^{p\times n}$. The matrix equations $AXX^*A^* = BB^*$ and $CXX^*C^* = DD^*$ have a common Hermitian nonnegative-definite solution if and only if $AA^\dagger B = B$ and there exists $T \in \mathbb{U}_n$ such that*

$$E_{CF_A}(DT - CA^\dagger B) = 0. \tag{1.7}$$

*If a common Hermitian nonnegative-definite solution exists, then a representation of the general common Hermitian nonnegative-definite solution is $XX^*$ with*

$$X = A^\dagger B + F_A(CF_A)^\dagger(DT - CA^\dagger B) + F_A F_{CF_A} Z,$$

*where $Z \in \mathbb{C}^{n\times n}$ is arbitrary and $T \in \mathbb{U}_n$ is a parameter matrix satisfying (1.7).*

**Lemma 1.6.** [10] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then*

$$r\begin{pmatrix} A & B \end{pmatrix} = r(A) + r[(I - AA^\dagger)B],$$
$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r[C(I - A^\dagger A)].$$

**Lemma 1.7.** [8] *Let $A, B \in \mathbb{C}^{m\times n}$, if $R(A) = R(B)$, then $AA^\dagger = BB^\dagger$; similarly if $R(A^*) = R(B^*)$, then $A^\dagger A = B^\dagger B$.*

**Lemma 1.8.** [7] *Let $A \in \mathbb{C}^{p\times q}$, $B \in \mathbb{C}^{q\times p}$ and $C \in \mathbb{C}^{p\times p}$ be given matrices. Then the matrix equation $AXB = C$ is consistent if and only if*

$$AA^\dagger C = C, \;\; CB^\dagger B = C.$$

*In this case, the general solution can be expressed as*

$$X = A^\dagger CB^\dagger + F_A Y_1 + Y_2 E_B,$$

*where $Y_1, Y_2$ are arbitrary with proper sizes.*

## 2. Main Results

In this section, our purpose is to derive some necessary and sufficient conditions for the solvability for linear matrix inequality $AXB+(AXB)^* \geqslant (>, \leqslant, <)\ C$ when the matrix $G = \begin{pmatrix} A & B^* \end{pmatrix}$ is full row rank, and establish general expressions..

Firstly, we give a explicit formula for the common solution to (1.6).

**Theorem 2.1.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$, $C \in \mathbb{C}_H^m$ satisfy $E_ACE_A \leqslant 0$, $F_BCF_B \leqslant 0$, and denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. If $r(G) = m$, then there always exists a common solution $VV^*$ to equations (1.6), and a representation of the general common solution is $VV^*$ with*

$$V = (-E_ACE_A)^{\frac{1}{2}} + (F_BAA^\dagger)^\dagger[(-F_BCF_B)^{\frac{1}{2}}T - (-E_ACE_A)^{\frac{1}{2}}] + AA^\dagger F_{F_BAA^\dagger}Z, \tag{2.1}$$

*where $Z \in \mathbb{C}^{m\times m}$ and $T \in \mathbb{U}_m$ are arbitrary matrices.*

**Proof.** In view of Lemma 1.5, there exists a common solution to (1.6) if and only if there exists $T \in \mathbb{U}_m$ such that

$$E_{F_BAA^\dagger}[(-F_BCF_B)^{\frac{1}{2}}T - F_BE_A(-E_ACE_A)^{\frac{1}{2}}] = 0. \tag{2.2}$$

Since $r(G) = m$, it follows from the first equality in Lemma 1.6 that

$$r(F_BAA^\dagger) = r(F_BA) = r(G) - r(B) = m - r(B) = r(F_B),$$

means that $R(F_BAA^\dagger) = R(F_B)$. It follows from Lemma 1.7 that $E_{F_BAA^\dagger} = B^\dagger B$. Therefore, (2.2) holds for any $T \in \mathbb{U}_m$.

Finally, by Lemma 1.5, we have

$$V = (-E_ACE_A)^{\frac{1}{2}} + AA^\dagger(F_BAA^\dagger)^\dagger[(-F_BCF_B)^{\frac{1}{2}}T - F_BE_A(-E_ACE_A)^{\frac{1}{2}}] + AA^\dagger F_{F_BAA^\dagger}Z.$$

Hence, (2.1) is evident. ∎

According to Theorem 2.1, next, we give a solution to (1.1) under condition $r\begin{pmatrix} A & B^* \end{pmatrix} = m$.

**Theorem 2.2.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$, $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix, and denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. If $r(G) = m$, then the following statements are equivalent:*

(1) *Matrix inequality (1.1) is solvable;*

(2) $E_ACE_A \leqslant 0$, $F_BCF_B \leqslant 0$;

(3) $i_+\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} = r(A)$, $i_+\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} = r(B)$.

*In this case, a general solution can be expressed as*

$$X = \frac{1}{2}(X_1 + X_2^*), \tag{2.3}$$

*where*

$$\begin{aligned} X_1 = \ & A^\dagger(C+VV^*)B^\dagger - A^\dagger B^* M^\dagger (C+VV^*)B^\dagger \\ & - A^\dagger S(B^*)^\dagger (C+VV^*)N^\dagger A^* B^\dagger - A^\dagger S Y_1 E_N A^* B^\dagger + F_A Y_2 + Y_3 E_B, \end{aligned} \tag{2.4}$$

$$\begin{aligned} X_2 = \ & M^\dagger(C+VV^*)(A^*)^\dagger + S^\dagger S(B^*)^\dagger (C+VV^*)N^\dagger \\ & + F_M(Y_1 - S^\dagger S Y_1 N N^\dagger) + Y_4 F_A, \end{aligned} \tag{2.5}$$

*with* $V$ *is given by* $(2.1)$, $M = E_AB^*$, $N = A^*F_B$, $S = B^*F_M$, *and* $Y_i$ $(i = 1, 2, 3, 4)$ *are arbitrary matrices over complex field with appropriate sizes.*

*Proof.* Note that $(1.1)$ can be rewritten as $C - AXB - (AXB)^* \leqslant 0$. So, $(1.1)$ is solvable if and only if

$$\min_X i_+[C - AXB - (AXB)^*] = 0.$$

Applying Lemma 1.1, we get

$$\begin{aligned} & \min_X i_+[C - AXB - (AXB)^*] \\ = \ & r\begin{pmatrix} C & A & B^* \end{pmatrix} \\ & + \max\left\{ i_+\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} - r\begin{pmatrix} C & A & B^* \\ A^* & 0 & 0 \end{pmatrix}, i_+\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} - r\begin{pmatrix} C & A & B^* \\ B & 0 & 0 \end{pmatrix} \right\} \\ = \ & r(G) + \max\left\{ i_+(E_ACE_A) - r(G), \ \ i_+(F_BCF_B) - r(G) \right\}. \end{aligned} \tag{2.6}$$

Letting the right hand side of (2.6) be zero yields

$$i_+(E_ACE_A) = 0, \ \ i_+(F_BCF_B) = 0,$$

which are equivalent to $E_ACE_A \leqslant 0, \ F_BCF_B \leqslant 0$, while statement (3) is followed by Lemma 1.2.

Next, we come to solve the matrix inequality $(1.1)$, which can be written as

$$AXB + (AXB)^* = C + VV^*, \tag{2.7}$$

where $V$ is given by (2.1). In view of Lemma 1.3, the general solution to (2.7) can be written as

$$X = \frac{1}{2}(X_1 + X_2^*),$$

where $X_1$ and $X_2$ are general solutions of the equation

$$AX_1B + B^*X_2A^* = C + VV^*. \tag{2.8}$$

According to (1.6) and $r(G) = m$, it follows from Lemma 1.4 that (2.8) is solvable, and

$$\begin{aligned} X_1 &= A^\dagger(C+VV^*)B^\dagger - A^\dagger B^* M^\dagger E_A(C+VV^*)B^\dagger \\ &\quad -A^\dagger S(B^*)^\dagger(C+VV^*)F_B N^\dagger A^* B^\dagger - A^\dagger S Y_1 E_N A^* B^\dagger + F_A Y_2 + Y_3 E_B, \end{aligned}$$

$$\begin{aligned} X_2 &= M^\dagger E_A(C+VV^*)(A^*)^\dagger + F_M S^\dagger S(B^*)^\dagger(C+VV^*)F_B N^\dagger \\ &\quad +F_M(Y_1 - S^\dagger S Y_1 N N^\dagger) + Y_4 F_A, \end{aligned}$$

where $M = E_AB^*$, $N = A^*F_B$, $S = B^*F_M$, and $Y_i, (i = 1, 2, 3, 4)$ are arbitrary matrices over complex field with appropriate sizes. Together with $M^\dagger E_A = M^\dagger$, $F_BN^\dagger = N^\dagger$ and $F_MS^\dagger = S^\dagger$, then (2.4) and (2.5) are followed. ■

A special case of Theorem 2.2 for $B = I_m$ is given below.

**Corollary 2.1.** *Let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times m}$ be variable matrix. Then the following statements are equivalent:*

(1) *Matrix inequality $AX + (AX)^* \geqslant C$ is solvable;*

(2) $E_ACE_A \leqslant 0$*;*

(3) $i_+\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} = r(A)$.

*In this case, the general solution can be expressed as*

$$X = \frac{1}{2}\left(A^\dagger(C+VV^*)(I_m + E_A) + F_AY_1 + [(A^\dagger Y_2)^* - A^\dagger Y_2]A^*\right),$$

*where $V = (-E_ACE_A)^{\frac{1}{2}} + AA^\dagger Z$, and $Z, Y_1, Y_2$ are arbitrary matrices over complex field with appropriate sizes.*

**Theorem 2.3.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$, $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix, and denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. If $r(G) = m$, then the following statements are equivalent:*

(1) *Matrix inequality* $(1.2)$ *is solvable;*

(2) *$i_-(E_ACE_A) = m - r(A)$ and $i_-(F_BCF_B) = m - r(B)$;*

(3) *$E_ACE_A < 0$, $F_BCF_B < 0$, $r(E_ACE_A) = m - r(A)$ and $r(F_BCF_B) = m - r(B)$;*

(4) $i_-\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} = m$, $i_-\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} = m$.

*In this case, the general solution of $X$ can be expressed by (2.3) with $Z$ in $V$ given by (2.1) such that $r(V) = m$.*

**Proof.** Note that $(1.2)$ can be rewritten as $C - AXB - (AXB)^* < 0$. So, $(1.2)$ is solvable if and only if

$$\max_X i_-[C - AXB - (AXB)^*] = m.$$

Applying Lemma 1.1, we get

$$\begin{aligned} \max_X i_-[C - AXB - (AXB)^*] &= \min\left\{ i_-\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix}, i_-\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} \right\} \\ &= \min\{i_-(E_ACE_A) + r(A),\ \ i_-(F_BCF_B) + r(B)\} \\ &= m. \end{aligned} \tag{2.9}$$

On the other hand, $i_-(E_ACE_A) + r(A) \leqslant r(E_ACE_A) + r(A) \leqslant r(E_A) + r(A) = m$, similarly, $i_-(F_BCF_B) + r(B) \leqslant m$. Then, (2.9) is equivalent to $i_-(E_ACE_A) = m - r(A)$ and $i_-(F_BCF_B) = m - r(B)$. Hence, the equivalence of statements (1)-(4) is proved.

Since the matrix inequality $(1.2)$, which can be written as

$$AXB + (AXB)^* = C + VV^*,$$

where $VV^* > 0$, i.e., $r(V) = m$. ■

Similarly, we can prove the following results.

**Theorem 2.4.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$, $C \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}^{n\times p}$ be variable matrix, and denote $G = \begin{pmatrix} A & B^* \end{pmatrix}$. If $r(G) = m$, then the following statements are equivalent:*

(1) *Matrix inequality $AXB + (AXB)^* \leqslant C$ is solvable;*

(2) *$E_ACE_A \geqslant 0$, $F_BCF_B \geqslant 0$;*

(3) $i_-\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} = r(A)$, $i_-\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} = r(B)$.

*In this case, a general solution can be expressed as*

$$X = \frac{1}{2}(X_1 + X_2^*), \tag{2.10}$$

*where*

$$\begin{aligned} X_1 &= A^\dagger(C - VV^*)B^\dagger - A^\dagger B^* M^\dagger (C - VV^*)B^\dagger \\ &\quad - A^\dagger S(B^*)^\dagger (C - VV^*)N^\dagger A^* B^\dagger - A^\dagger S Y_1 E_N A^* B^\dagger + F_A Y_2 + Y_3 E_B, \\ X_2 &= M^\dagger (C - VV^*)(A^*)^\dagger + S^\dagger S(B^*)^\dagger (C - VV^*)N^\dagger \\ &\quad + F_M(Y_1 - S^\dagger S Y_1 N N^\dagger) + Y_4 F_A, \\ V &= (E_A C E_A)^{\frac{1}{2}} + (F_B A A^\dagger)^\dagger [(F_B C F_B)^{\frac{1}{2}} T - (E_A C E_A)^{\frac{1}{2}}] + AA^\dagger F_{F_B AA^\dagger} Z, \end{aligned}$$

*where* $M = E_A B^*$, $N = A^* F_B$, $S = B^* F_M$, *and* $Z \in \mathbb{C}^{m\times m}$, $T \in \mathbb{U}_m$, $Y_i$ $(i = 1, 2, 3, 4)$ *are arbitrary matrices over complex field with appropriate sizes.*

**Theorem 2.5.** *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$, $C \in \mathbb{C}_H^m$ *be given,* $X \in \mathbb{C}^{n\times p}$ *be variable matrix, and denote* $G = \begin{pmatrix} A & B^* \end{pmatrix}$. *If* $r(G) = m$, *then the following statements are equivalent:*

(1) *Matrix inequality* $AXB + (AXB)^* < C$ *is solvable;*

(2) $i_+(E_A C E_A) = m - r(A)$ *and* $i_+(F_B C F_B) = m - r(B)$*;*

(3) $E_A C E_A > 0$, $F_B C F_B > 0$, $r(E_A C E_A) = m - r(A)$ *and* $r(F_B C F_B) = m - r(B)$*;*

(4) $i_+\begin{pmatrix} C & A \\ A^* & 0 \end{pmatrix} = m$, $i_+\begin{pmatrix} C & B^* \\ B & 0 \end{pmatrix} = m$.

*In this case, the general solution of* $X$ *can be expressed by* $(2.10)$ *with* $Z$ *in* $V$ *given by Theorem* $2.4$ *such that* $r(V) = m$.

## 3. Applications

In the following contents, we consider some applications of Theorem 2.2 in the Re-nnd generalized inverses and Re-nnd solution. Next, we first prove an auxiliary result.

**Lemma 3.1.** *Let* $A \in \mathbb{C}^{m\times m}$ *satisfy* $r(A^2) = r(A)$. *Then*

(1) $r(F_A E_A) = r(F_A) = r(E_A)$*;*

(2) $F_A E_A (F_A E_A)^\dagger = F_A$, $(F_A E_A)^\dagger F_A E_A = E_A$, $E_{F_A E_A} = A^\dagger A$, $F_{F_A E_A} = AA^\dagger$.

*Proof.* On account of Lemma 1.6, we have

$$r(F_AE_A) = r(E_AF_A) = r\begin{pmatrix} A & I_m \\ 0 & A \end{pmatrix} - 2r(A) = m - r(A) = r(F_A) = r(E_A),$$

which means that (1) holds, and $R(F_AE_A) = R(F_A)$, $R[(F_AE_A)^*] = R(E_A)$. Hence, (2) can be proved by Lemma 1.7. ∎

In the following theorem, we present a general expression of $A_{re}^{(1,2,3)}$.

**Theorem 3.1.** *Let $A \in \mathbb{C}^{m\times m}$. Then the following statements are equivalent:*
$(1)$ $A_{re}^{(1,2,3)}$ *exists;*
$(2)$ $(A^\dagger)^2A$ *is Re-nnd and* $r(A) = r(A^2)$*;*
$(3)$ $A^2A^\dagger$ *is Re-nnd and* $r(A) = r(A^2)$*;*
$(4)$ $A^*A^2$ *is Re-nnd and* $r(A) = r(A^2)$*;*
$(5)$ $A^{\#}AA^\dagger$ *is Re-nnd and* $r(A) = r(A^2)$.

*In this case, a general expression of* $A_{re}^{(1,2,3)}$ *can be expressed as*

$$A_{re}^{(1,2,3)} = A^\dagger + \frac{1}{2}(\tilde{X}_1 + \tilde{X}_2^*), \tag{3.1}$$

*where*

$$\begin{aligned} \tilde{X}_1 = {} & F_A[VV^* - A^\dagger - (A^\dagger)^*]AA^\dagger - F_AM^\dagger[VV^* - A^\dagger - (A^\dagger)^*]AA^\dagger \\ & -F_ASAA^\dagger[VV^* - A^\dagger - (A^\dagger)^*]N^\dagger AA^\dagger - F_ASY_1E_NF_AAA^\dagger, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \tilde{X}_2 = {} & M^\dagger[VV^* - A^\dagger - (A^\dagger)^*]F_A + AA^\dagger S^\dagger SAA^\dagger[VV^* - A^\dagger - (A^\dagger)^*]N^\dagger \\ & +AA^\dagger F_M(Y_1 - S^\dagger SY_1NN^\dagger)F_A, \end{aligned} \tag{3.3}$$

$$V = J - (E_AF_A)^\dagger J,$$

*with* $J = [(A^\dagger)^2A + ((A^\dagger)^2A)^*]^{\frac{1}{2}}$, $M = A^\dagger A^2A^\dagger$, $N = F_AE_A$, $S = AA^\dagger F_M$, *and* $Y_1 \in \mathbb{C}^{m\times m}$ *is arbitrary.*

*Proof.* The equivalence of the statements (1)-(5) is given by [Theorem 2.1, 2].

Since $A^{(1,2,3)} = A^\dagger + F_AXAA^\dagger$, then $A_{re}^{(1,2,3)}$ exists if and only if there exists some $X$ such that $A^{(1,2,3)}$ is Re-nnd, i.e.,

$$F_AXAA^\dagger + (F_AXAA^\dagger)^* \geqslant -[A^\dagger + (A^\dagger)^*]. \tag{3.4}$$

According to $r(A) = r(A^2)$, it follows that $r(G) = r\begin{pmatrix} F_A & AA^\dagger \end{pmatrix} = m$. By Theorem 2.2, the solution of $X$ to (3.4) can be written as

$$X = \frac{1}{2}(X_1 + X_2^*),$$

where

$$\begin{aligned} X_1 &= F_A[VV^* - A^\dagger - (A^\dagger)^*]AA^\dagger - F_AAA^\dagger M^\dagger[VV^* - A^\dagger - (A^\dagger)^*]AA^\dagger \\ &\quad -F_ASAA^\dagger[VV^* - A^\dagger - (A^\dagger)^*]N^\dagger F_AAA^\dagger \\ &\quad -F_ASY_1E_NF_AAA^\dagger + A^\dagger AY_2 + Y_3E_A, \\ X_2 &= M^\dagger[VV^* - A^\dagger - (A^\dagger)^*]F_A + S^\dagger SAA^\dagger[VV^* - A^\dagger - (A^\dagger)^*]N^\dagger \\ &\quad +F_M(Y_1 - S^\dagger SY_1NN^\dagger) + Y_4A^\dagger A, \\ V &= J - F_A(E_AF_A)^\dagger E_AA^\dagger AJ + F_AF_{E_AF_A}Z = J - (E_AF_A)^\dagger J. \end{aligned}$$

Substituting $X$ into $A^{(1,2,3)} = A^\dagger + F_AXAA^\dagger$ and denote

$$\begin{aligned} \tilde{X}_1 &= F_AX_1AA^\dagger = F_A[VV^* - A^\dagger - (A^\dagger)^*]AA^\dagger - F_AM^\dagger[VV^* - A^\dagger - (A^\dagger)^*]AA^\dagger \\ &\quad -F_ASAA^\dagger[VV^* - A^\dagger - (A^\dagger)^*]N^\dagger AA^\dagger - F_ASY_1E_NF_AAA^\dagger, \\ \tilde{X}_2 &= AA^\dagger X_2F_A = M^\dagger[VV^* - A^\dagger - (A^\dagger)^*]F_A + AA^\dagger S^\dagger SAA^\dagger[VV^* - A^\dagger \\ &\quad -(A^\dagger)^*]N^\dagger + AA^\dagger F_M(Y_1 - S^\dagger SY_1NN^\dagger)F_A, \end{aligned}$$

which yields (3.1), (3.2) and (3.3) respectively. The proof is complete. ∎

Together with $A^{(1,2,4)} = \left((A^*)^{(1,2,3)}\right)^*$ and Theorem 3.1, the following corollary is evident.

**Corollary 3.1.** *Let $A \in \mathbb{C}^{m\times m}$. Then the following statements are equivalent:*

(1) $A_{re}^{(1,2,4)}$ *exists;*

(2) $A(A^\dagger)^2$ *is Re-nnd and* $r(A) = r(A^2)$*;*

(3) $A^\dagger A^2$ *is Re-nnd and* $r(A) = r(A^2)$*;*

(4) $A^2A^*$ *is Re-nnd and* $r(A) = r(A^2)$*;*

(5) $A^\dagger AA^\#$ *is Re-nnd and* $r(A) = r(A^2)$.

*In this case, a general expression of $A_{re}^{(1,2,4)}$ can be expressed as*

$$A_{re}^{(1,2,4)} = A^\dagger + \frac{1}{2}(\tilde{X}_1^* + \tilde{X}_2),$$

*where*

$$\begin{aligned}
\tilde{X}_1 &= E_A[VV^* - A^\dagger - (A^\dagger)^*]A^\dagger A - E_A M^\dagger[VV^* - A^\dagger - (A^\dagger)^*]A^\dagger A \\
&\quad -E_A S A^\dagger A[VV^* - A^\dagger - (A^\dagger)^*]N^\dagger A^\dagger A - E_A S Y_1 E_N E_A A^\dagger A, \\
\tilde{X}_2 &= M^\dagger[VV^* - A^\dagger - (A^\dagger)^*]E_A + A^\dagger A S^\dagger S A^\dagger A[VV^* - A^\dagger - (A^\dagger)^*]N^\dagger \\
&\quad +A^\dagger A F_M(Y_1 - S^\dagger S Y_1 N N^\dagger)E_A, \\
V &= J - (F_A E_A)^\dagger J,
\end{aligned}$$

*with* $J = [A(A^\dagger)^2 + (A(A^\dagger)^2)^*]^{\frac{1}{2}}$, $M = A(A^\dagger)^2 A$, $N = E_A F_A$, $S = A^\dagger A F_M$, *and* $Y_1 \in \mathbb{C}^{m\times m}$ *is arbitrary.*

**Remark.** Recently, Liu and Fang [4] gave some representations for $A_{re}^{(1,2,3)}$ and $A_{re}^{(1,2,4)}$, which are limited. The general representations for $A_{re}^{(1,2,3)}$ and $A_{re}^{(1,2,4)}$ are provided by Theorem 3.1 and Corollary 3.1 respectively.

**Theorem 3.2.** *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$ *and* $C \in \mathbb{C}^{m\times p}$ *be given matrices. Then, the following statements are equivalent:*

(1) *There exists a Re-nnd solution to consistent matrix equation* $AXB = C$*;*

(2) $E_H[A^\dagger C B^\dagger + (A^\dagger C B^\dagger)^*]E_H \geqslant 0$*;*

(3) $i_-\begin{pmatrix} 0 & -C & A \\ -C^* & 0 & B^* \\ A^* & B & 0 \end{pmatrix} = r\begin{pmatrix} A^* & B \end{pmatrix}$*;*

(4) $F_G \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} F_G \leqslant 0$, *where* $G = \begin{pmatrix} A^* & B \end{pmatrix}$.

*In this case, a general expression for this Re-nnd solution can be written as*

$$X = A^\dagger C B^\dagger + \begin{pmatrix} F_A & 0 \end{pmatrix}\begin{pmatrix} W_1 \\ W_2^* \end{pmatrix} + \begin{pmatrix} W_1 \\ W_2^* \end{pmatrix}^* \begin{pmatrix} 0 \\ E_B \end{pmatrix}, \tag{3.5}$$

*where*

$$\begin{aligned}
\begin{pmatrix} W_1 \\ W_2^* \end{pmatrix} &= \frac{1}{2}(H^\dagger(VV^* - [A^\dagger C B^\dagger + (A^\dagger C B^\dagger)^*])(I_n + E_H) \\
&\quad +F_H Y_1 + [(H^\dagger Y_2)^* - H^\dagger Y_2]H^*),
\end{aligned} \tag{3.6}$$

*with* $H = \begin{pmatrix} F_A & E_B \end{pmatrix}$, $V = \left(E_H[A^\dagger C B^\dagger + (A^\dagger C B^\dagger)^*]E_H\right)^{\frac{1}{2}} + HH^\dagger Z$, $Z \in \mathbb{C}^{n\times n}$, $Y_1 \in \mathbb{C}^{2n\times n}$ *and* $Y_2 \in \mathbb{C}^{n\times 2n}$ *are arbitrary.*

*Proof.* According to the assumption and Lemma 1.8, we have that $AA^\dagger C = C$ and $CB^\dagger B = C$, the solution $X$ to $AXB = C$ can be expressed as

$$X = A^\dagger C B^\dagger + F_A W_1 + W_2 E_B, \tag{3.7}$$

where $W_1, W_2 \in \mathbb{C}^{n\times n}$ are arbitrary.

Hence, $X$ is Re-nnd if and only if

$$\begin{pmatrix} F_A & E_B \end{pmatrix} \begin{pmatrix} W_1 \\ W_2^* \end{pmatrix} + \begin{pmatrix} W_1^* & W_2 \end{pmatrix} \begin{pmatrix} F_A \\ E_B \end{pmatrix} \geqslant -[A^\dagger CB^\dagger + (A^\dagger CB^\dagger)^*]$$

is solvable. On account of Corollary 2.1, statement (2) and (3.6) are obvious. Combining (3.7) and (3.6) produces (3.5).

Next, we show that statements (2), (3) and (4) are equivalent. By Lemma 1.2, statement (2) is equivalent to $i_-\left(E_H[A^\dagger CB^\dagger + (A^\dagger CB^\dagger)^*]E_H\right) = 0$, i.e.,

$$r\begin{pmatrix} F_A & E_B \end{pmatrix} = i_- \begin{pmatrix} A^\dagger CB^\dagger + (A^\dagger CB^\dagger)^* & E_{A^*} & E_B \\ E_{A^*} & 0 & 0 \\ E_B & 0 & 0 \end{pmatrix}. \tag{3.8}$$

In view of Lemma 1.6, we can compute that

$$\begin{aligned} & r\begin{pmatrix} F_A & E_B \end{pmatrix} \\ &= r\begin{pmatrix} I_n & A^* & 0 \\ I_n & 0 & B \end{pmatrix} - r(A) - r(B) = n + r\begin{pmatrix} A^* & B \end{pmatrix} - r(A) - r(B). \end{aligned} \tag{3.9}$$

Together with Lemma 1.2 and the fact $i_\pm(M) = i_\pm(PMP^*)$, where $M$ is Hermitian, and $P$ is nonsingular, then we have

$$\begin{aligned} & i_- \begin{pmatrix} A^\dagger CB^\dagger + (A^\dagger CB^\dagger)^* & E_{A^*} & E_B \\ E_{A^*} & 0 & 0 \\ E_B & 0 & 0 \end{pmatrix} \\ &= i_- \begin{pmatrix} A^\dagger CB^\dagger + (A^\dagger CB^\dagger)^* & I_n & I_n & 0 & 0 \\ I_n & 0 & 0 & A^* & 0 \\ I_n & 0 & 0 & 0 & B \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & B^* & 0 & 0 \end{pmatrix} - r(A) - r(B) \\ &= i_- \begin{pmatrix} 0 & I_n & I_n & -(CB^\dagger)^* & 0 \\ I_n & 0 & 0 & A^* & 0 \\ I_n & 0 & 0 & 0 & B \\ -CB^\dagger & A & 0 & 0 & 0 \\ 0 & 0 & B^* & 0 & 0 \end{pmatrix} - r(A) - r(B) \end{aligned} \tag{3.10}$$

$$= i_-\begin{pmatrix} 0 & I_n & I_n & 0 & 0 \\ I_n & 0 & 0 & A^* & 0 \\ I_n & 0 & 0 & 0 & B \\ 0 & A & 0 & 0 & C \\ 0 & 0 & B^* & C^* & 0 \end{pmatrix} - r(A) - r(B)$$

$$= i_-\begin{pmatrix} 0 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & A^* & 0 \\ 0 & 0 & 0 & -A^* & B \\ 0 & A & -A & 0 & C \\ 0 & 0 & B^* & C^* & 0 \end{pmatrix} - r(A) - r(B)$$

$$= i_-\begin{pmatrix} 0 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A^* & B \\ 0 & 0 & -A & 0 & C \\ 0 & 0 & B^* & C^* & 0 \end{pmatrix} - r(A) - r(B)$$

$$= i_-\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + i_-\begin{pmatrix} 0 & -A^* & B \\ -A & 0 & C \\ B^* & C^* & 0 \end{pmatrix} - r(A) - r(B)$$

$$= n + i_-\begin{pmatrix} 0 & A^* & B \\ A & 0 & -C \\ B^* & -C^* & 0 \end{pmatrix} - r(A) - r(B)$$

$$= n + i_-\begin{pmatrix} 0 & -C & A \\ -C^* & 0 & B^* \\ A^* & B & 0 \end{pmatrix} - r(A) - r(B).$$

Substituting (3.9) and (3.10) into (3.8) yields statements (3) and (4). ■

Three simple consequences of Theorem 3.2 are given below.

**Corollary 3.2.** *Let $A, C \in \mathbb{C}^{m\times n}$ be given. Then there exists a Re-nnd solution to consistent matrix equation $AX = C$ if and only if $A^\dagger CA^\dagger A$ or $CA^*$ is Re-nnd. In this case, a general expression for this Re-nnd solution can be written as*

$$X = A^\dagger C + \frac{1}{2}F_A\left(VV^* - [A^\dagger C + (A^\dagger C)^*]\right)(I_n + A^\dagger A) + F_A Y F_A,$$

*where $V = [A^\dagger CA^\dagger A + (A^\dagger CA^\dagger A)^*]^{\frac{1}{2}} + F_A Z$, $Y, Z \in \mathbb{C}^{n\times n}$ are arbitrary with $Y^* = -Y$.*

**Corollary 3.3.** *Let $B, C \in \mathbb{C}^{n\times m}$ be given. Then there exists a Re-nnd solution to consistent matrix equation $XB = C$ if and only if $BB^\dagger CB^\dagger$ or $B^*C$ is Re-nnd. In* *this case, a general expression for this Re-nnd solution can be written as*

$$X = CB^\dagger + \frac{1}{2}(I_n + BB^\dagger)\left(VV^* - [CB^\dagger + (CB^\dagger)^*]\right)E_B + E_B Y E_B,$$

*where* $V = [BB^\dagger CB^\dagger + (BB^\dagger CB^\dagger)^*]^{\frac{1}{2}} + E_B Z$, $Y, Z \in \mathbb{C}^{n\times n}$ *are arbitrary with* $Y^* = -Y$.

**Corollary 3.4.** *Let* $A \in \mathbb{C}^{m\times n}$ *and* $C \in \mathbb{C}^{m\times m}$ *be given matrices. Then there exists a Re-nnd solution to consistent matrix equation* $AXA^* = C$ *if and only if* $A^\dagger C(A^\dagger)^*$ *or* $C$ *is Re-nnd. In this case, a general expression for this Re-nnd solution can be written as*

$$\begin{aligned}
X \;=\; & A^\dagger C(A^\dagger)^* + \frac{1}{2}F_A(I_n+F_A)^{-1}\left(VV^* - A^\dagger(C+C^*)(A^\dagger)^*\right)(I_n + A^\dagger A)\\
& + \frac{1}{2}(I_n + A^\dagger A)\left(VV^* - A^\dagger(C+C^*)(A^\dagger)^*\right)F_A(I_n+F_A)^{-1}F_A\\
& + \frac{1}{2}[F_A\tilde{Y}_1 - F_A(I_n+F_A)^{-1}F_A(\tilde{Y}_1+\tilde{Y}_2)] + \frac{1}{2}[F_A\tilde{Y}_3^*F_A\\
& - F_A(I_n+F_A)^{-1}(\tilde{Y}_3+\tilde{Y}_4)F_A]\\
& + \frac{1}{2}[\tilde{Y}_2^*F_A - (\tilde{Y}_1+\tilde{Y}_2)^*F_A(I_n+F_A)^{-1}F_A] + \frac{1}{2}[F_A\tilde{Y}_4F_A\\
& - F_A(\tilde{Y}_3+\tilde{Y}_4)^*F_A(I_n+F_A)^{-1}F_A],
\end{aligned}$$

*where* $V = \left(A^\dagger(C+C^*)(A^\dagger)^*\right)^{\frac{1}{2}} + F_A Z$, $Z \in \mathbb{C}^{n\times n}$ *and* $\tilde{Y}_i \in \mathbb{C}^{n\times n}$ $(i = 1, 2, 3, 4)$ *are arbitrary.*

*Proof.* It is easy to verify that

$$\begin{aligned}
H^\dagger \;&=\; \begin{pmatrix} F_A & F_A\end{pmatrix}^\dagger = \begin{pmatrix} F_A - F_A(I_n+F_A)^{-1}F_A \\ (I_n+F_A)^{-1}F_A\end{pmatrix}\\
&=\; \begin{pmatrix} F_A(I_n+F_A)^{-1} \\ F_A(I_n+F_A)^{-1}\end{pmatrix} = \begin{pmatrix} F_A(I_n+F_A)^{-1}F_A \\ F_A(I_n+F_A)^{-1}F_A\end{pmatrix},\\
H^\dagger H \;&=\; \begin{pmatrix} F_A(I_n+F_A)^{-1}F_A & F_A(I_n+F_A)^{-1}F_A \\ (I_n+F_A)^{-1}F_A & (I_n+F_A)^{-1}F_A\end{pmatrix}, \quad HH^\dagger = F_A.
\end{aligned}$$

So, $V = \left(A^\dagger(C+C^*)(A^\dagger)^*\right)^{\frac{1}{2}} + F_A Z$. Denote $Y_1 = \begin{pmatrix}\tilde{Y}_1\\ \tilde{Y}_2\end{pmatrix}$, $Y_2 = \begin{pmatrix}\tilde{Y}_3 & \tilde{Y}_4\end{pmatrix}$. By Theorem 3.2, we get

$$\begin{aligned}
\begin{pmatrix}F_A & 0\end{pmatrix} F_H Y_1 &= F_A\tilde{Y}_1 - F_A(I_n+F_A)^{-1}F_A(\tilde{Y}_1+\tilde{Y}_2),\\
\begin{pmatrix}F_A & 0\end{pmatrix}[(H^\dagger Y_2)^* - H^\dagger Y_2]H^* &= F_A\tilde{Y}_3^*F_A - F_A(I_n+F_A)^{-1}(\tilde{Y}_3+\tilde{Y}_4)F_A,\\
Y_1^* F_H \begin{pmatrix}0\\ F_A\end{pmatrix} &= \tilde{Y}_2^*F_A - (\tilde{Y}_1+\tilde{Y}_2)^*F_A(I_n+F_A)^{-1}F_A,\\
H[H^\dagger Y_2 - (H^\dagger Y_2)^*]\begin{pmatrix}0\\ F_A\end{pmatrix} &= F_A\tilde{Y}_4F_A - F_A(\tilde{Y}_3+\tilde{Y}_4)^*F_A(I_n+F_A)^{-1}F_A.
\end{aligned}$$

According to the above analyses, this corollary is obvious. ∎

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## References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Second ed., Springer, New York, 2003.

[2] D. S. Cvetković-Ilić, *Re-nnd solutions of the matrix equation $AXB = C$*, J. Aust. Math. Soc. **84** (2008), 63–72. `http://dx.doi.org/10.1017/s1446788708000207`

[3] X. Liu, *Comments on "The common Re-nnd and Re-pd solutions to the matrix equations $AX = C$ and $XB = D$"*, Appl. Math. Comput. **236** (2014), 663–668. `http://dx.doi.org/10.1016/j.amc.2014.03.074`

[4] X. Liu and R. Fang, *Notes on Re-nnd generalized inverses*, Filomat, 2014, (in press). `http://dx.doi.org/10.2298/fil1505121l`

[5] Y. Liu and Y. Tian, *Max-min problems on the ranks and inertias of the matrix expressions $A - BXC \pm (BXC)^*$ with applications*, J. Optim. Theory. Appl. **148** (2011), 593–622. `http://dx.doi.org/10.1007/s10957-010-9760-8`

[6] J. Nikolov and D. S. Cvetković-Ilić, *Re-nnd generalized inverses*, Linear Algebra Appl. 439 (2013), 2999–3007. `http://dx.doi.org/10.1016/j.laa.2013.08.002`

[7] R. Penrose and J. A. Todd, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406–413. `http://dx.doi.org/10.1017/s0305004100030401`

[8] X. Sheng and G. Chen, *Some generalized inverses of partition matrix and quotient identity of generalized Schur complement*, Appl. Math. Comput. **196** (2008), 174–184. http://dx.doi.org/10.1016/j.amc.2007.05.050

[9] Y. Tian, *Maximization and minimization of the rank and inertia of the Hermitian matrix expression* $A - BX - (BX)^*$ *with applications*, Linear Algebra Appl. **434** (2011), 2109–2139. http://dx.doi.org/10.1016/j.laa.2010.12.010

[10] Y. Tian and D. Rosen, *Solving the matrix inequality* $AXB + (AXB)^* \geqslant C$, Math. Inequal. Appl. **12** (2012), 537–548. http://dx.doi.org/10.7153/mia-15-47

[11] Q. Wang, J. van der Woude and H. Chang, *A system of real quaternion matrix equations with applications*, Linear Algebra Appl. **431** (2009), 2291–2303. http://dx.doi.org/10.1016/j.laa.2009.02.010

[12] Z. Xiong and Y. Qin, *The common Re-nnd and Re-pd solutions to the matrix equations* $AX = C$ *and* $XB = D$, Appl. Math. Comput. **218** (2011), 3330–3337. http://dx.doi.org/10.1016/j.amc.2011.08.074

[13] X. Zhang, *The general common Hermitian nonnegative-definite solution to the matrix equations* $AXA^* = BB^*$ *and* $CXC^* = DD^*$ *with applications instatistics*, Journal of Multivariate Analysis **93** (2005), 257–266. http://dx.doi.org/10.1016/j.jmva.2004.04.009

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