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REMARKS ON THE QUALITATIVE QUESTIONS FOR BIHARMONIC OPERATORS

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Abstract. In this article, we obtain several remarks on the qualitative questions such as Picone's identity, Morse index and Hardy-Rellich type inequality for biharmonic operators.

1. INTRODUCTION

In the recent years there has been a good amount of interest on the qualitative questions such as stability criteria, Picone's identity, Morse index, Sturm comparison theorem for Laplace as well as p-Laplace operators but very little is known for biharmonic operators.

It is a well-known fact that in the qualitative theory of elliptic PDEs, Picone's identity plays an important role. The classical Picone's identity says that if u and v are differentiable functions such that $v > 0$ and $u \geq 0$, then

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0, \tag{1.1}$$

see [17]. (1.1) has an enormous applications to second-order elliptic equations and systems, see for instance [3, 4, 5, 16] and the references therein. Let us write briefly the recent developments on Picone's identity. In order to apply (1.1) to p-Laplace equations, (1.1) is extended by W. Allegretto and Y. X. Huang [6]. The extension to (1.1) is as follows:

Theorem 1.1. [6] *Let $v > 0$ and $u \geq 0$ be differentiable functions. Denote*

$$L(u, v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla v|^{p-2}\nabla v$$

and

$$R(u, v) = |\nabla u|^p - \nabla(\frac{u^p}{v^{p-1}})|\nabla v|^{p-2}\nabla v.$$

Then $L(u, v) = R(u, v)$. Moreover, $L(u, v) \geq 0$ and $L(u, v) = 0$ a.e. in Ω if and only if $\nabla(\frac{u}{v}) = 0$ a.e. in Ω.

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Recently, the second author obtain a nonlinear analogue of (1.1) in [19]. The nonlinear analogue of (1.1) reads as follows:

Theorem 1.2. [19] *Let v be a differentiable function in Ω such that $v \neq 0$ in Ω and u be a non-constant differentiable function in Ω. Let $f(y) \neq 0, \forall\, 0 \neq y \in \mathbb{R}$ and suppose that there exists $\alpha > 0$ such that $f'(y) \geq \frac{1}{\alpha}, \forall\, 0 \neq y \in \mathbb{R}$. Denote*

$$L(u, v) = \alpha |\nabla u|^2 - \frac{|\nabla u|^2}{f'(v)} + \left(\frac{u\sqrt{f'(v)}\nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}} \right)^2 \tag{1.2}$$

and

$$R(u, v) = \alpha |\nabla u|^2 - \nabla \left(\frac{u^2}{f(v)} \right) \nabla v. \tag{1.3}$$

Then $L(u, v) = R(u, v)$. Moreover, $L(u, v) \geq 0$ and $L(u, v) = 0$ in Ω if and only if $u = c_1 v + c_2$ for some arbitrary constants c_1, c_2.

K. Bal [7] extended the nonlinear Picone's identity of [19] to deal with p-Laplace equations. The extension reads as follows:

Theorem 1.3. [7] *Let $v > 0$ and $u \geq 0$ be two non-constant differentiable functions in Ω. Also assume that $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$ for all y. Define*

$$L(u,v) = |\nabla u|^p - \frac{pu^{p-1}\nabla u |\nabla v|^{p-2}\nabla v}{f(v)} + \frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2}$$

and

$$R(u,v) = |\nabla u|^p - \nabla(\frac{u^p}{f(v)})|\nabla v|^{p-2}\nabla v.$$

Then $L(u,v) = R(u,v) \geq 0$. Moreover $L(u,v) = 0$ a.e. in Ω if and only if $\nabla(\frac{u}{v}) = 0$ a.e. in Ω.

There are also several interesting articles dealing with Picone's identity in different contexts. We just name a few articles, for instance, for a Picone type identity to higher order half linear differentiable operators, we refer to [15] and the references therein, for Picone identities to half-linear elliptic operators with $p(x)$-Laplacians, we refer to [21] and for Picone-type identity to pseudo p-Laplacian with variable power, we refer to [8]. In [10], D. R. Dunninger established a Picone identity for a class of fourth order elliptic differential inequalities. This identity says that if $u, v, a\Delta u, A\Delta v$ are twice continuously differentiable functions with $v(x) \neq 0$ and a and A are positive weights, then

$$\begin{aligned} &\operatorname{div}\left[u\nabla(a\Delta u) - a\Delta u \nabla u - \frac{u^2}{v}\nabla(A\Delta v) + A\Delta v \cdot \nabla\left(\frac{u^2}{v}\right)\right] \\ &= -\frac{u^2}{v}\Delta(A\Delta v) + u\Delta(a\Delta u) + (A-a)(\Delta u)^2 \\ &\quad -A\left(\Delta u - \frac{u}{v}\Delta v\right)^2 + A\frac{2\Delta v}{v}\left(\nabla u - \frac{u}{v}\nabla v\right)^2. \end{aligned} \tag{1.4}$$

In this context, there is a natural question to ask. Can we establish a nonlinear analogue of (1.4)? More precisely, the aim of this article is to establish a nonlinear analogue of Picone's identity which could deal with biharmonic operators and using Picone's identity, we establish several qualitative results. In the best of our knowledge, we are not aware on these results proved by Picone's identity or by other techniques.

The plan of this paper is as follows. Section 2 deals with nonlinear analogue of Picone's identity which could deal with biharmonic operators. In Section 3, we give several applications of Picone's identity to biharmonic operators.

2. Nonlinear Analogue of Picone's Identity

In this section, we establish a nonlinear analogue of Picone's identity. The next lemma can be obtained from (1.4) with some assumptions. Since the proof is short and interesting so we write it independently here with more useful insights.

Lemma 2.1. (Picone's identity). *Let u and v be twice continuously differentiable functions in Ω such that $v > 0,\ -\Delta v > 0$ in Ω. Denote*

$$L(u,\, v) = \left(\Delta u - \frac{u}{v}\Delta v\right)^2 - \frac{2\Delta v}{v}\left(\nabla u - \frac{u}{v}\nabla v\right)^2$$

and

$$R(u,\, v) = |\Delta u|^2 - \Delta\left(\frac{u^2}{v}\right)\Delta v.$$

Then (i) $L(u,\, v) = R(u,\, v)$, (ii) $L(u,\, v) \geq 0$ and (iii) $L(u,\, v) = 0$ in Ω if and only if $u = \alpha v$ for some $\alpha \in \mathbb{R}$.

Proof. Let us expand $R(u,\, v)$:

$$\begin{aligned} R(u,\, v) &= |\Delta u|^2 - \Delta\left(\frac{u^2}{v}\right)\Delta v \\ &= |\Delta u|^2 + \frac{u^2}{v^2}|\Delta v|^2 - \frac{2u}{v}\Delta u\Delta v - \frac{2}{v}|\nabla u|^2\Delta v + \frac{4u}{v^2}\nabla u\nabla v\Delta v - \frac{2u^2}{v^3}|\nabla v|^2\Delta v \\ &= \left(\Delta u - \frac{u}{v}\Delta v\right)^2 - \frac{2\Delta v}{v}\left(\nabla u - \frac{u}{v}\nabla v\right)^2 \\ &= L(u,\, v), \end{aligned}$$

which proves the first part. Now using the fact that $v > 0,\ -\Delta v > 0$ in Ω, one can see that $L(u,\, v) \geq 0$ and therefore (ii) is proved. Now $L(u,\, v) = 0$ in Ω implies that

$$0 = \left(\Delta u - \frac{u}{v}\Delta v\right)^2 - \frac{2\Delta v}{v}\left(\nabla u - \frac{u}{v}\nabla v\right)^2,$$

that is,

$$0 \le -\frac{2\Delta v}{v}\left(\nabla u - \frac{u}{v}\nabla v\right)^2 = -\left(\Delta u - \frac{u}{v}\Delta v\right)^2 \le 0,$$

which implies that there exists some $\alpha \in \mathbb{R}$ such that $u = \alpha v$. Conversely, when $u = \alpha v$, one can see easily that $L(u,\, v) = 0$, and therefore (iii) is proved. ■

Remark 2.2. We note that the above lemma also holds if we replace $v > 0$ and $-\Delta v > 0$ in Ω by $v < 0$ and $-\Delta v < 0$ in Ω, respectively.

In the next proposition, we establish a nonlinear analogue of Picone's identity for biharmonic operators.

Proposition 2.3. (Nonlinear analogue of Picone's identity). *Let u and v be twice continuously differentiable functions in Ω such that $v > 0$, $-\Delta v > 0$ in Ω. Let $f\colon (0,\,\infty) \to (0,\infty)$ be a C^2 function such that $f''(y) \le 0,\ f'(y) \ge 1, \forall\, 0 \ne y \in \mathbb{R}$. Denote*

$$\begin{aligned} L(u,\, v) \,=\, & |\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)} + \left(\frac{\Delta u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta v\right)^2 \\ & - \frac{2\Delta v}{f(v)}\left(\nabla u - \frac{uf'(v)}{f(v)}\nabla v\right)^2 + \frac{u^2 f''(v)}{f(v)}|\nabla v|^2 \Delta v \end{aligned}$$

and

$$R(u,\, v) \,=\, |\Delta u|^2 - \Delta\left(\frac{u^2}{f(v)}\right)\Delta v.$$

Then (i) $L(u,\, v) = R(u,\, v)$, (ii) $L(u,\, v) \ge 0$ and (iii) $L(u,\, v) = 0$ in Ω if and only if $u = cv + d$ for some $c, d \in \mathbb{R}$.

Proof. Let us expand $R(u,v)$:

$$\begin{aligned} R(u,\, v) \,=\, & |\Delta u|^2 - \Delta\left(\frac{u^2}{f(v)}\right)\Delta v \\ =\, & |\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)} + \left(\frac{|\Delta u|^2}{f'(v)} + \frac{u^2 f'(v)}{f^2(v)}|\Delta v|^2 - \frac{2u\Delta u \Delta v}{f(v)}\right) \\ & - \frac{2\Delta v}{f(v)}\left(|\nabla u|^2 + \frac{u^2 f'^2(v)}{f^2(v)}|\nabla v|^2 - \frac{2uf'(v)}{f(v)}\nabla u \cdot \nabla v\right) + \frac{u^2 f''(v)}{f^2(v)}|\nabla v|^2 \Delta v \\ =\, & |\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)} + \left(\frac{\Delta u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta v\right)^2 \\ & - \frac{2\Delta v}{f(v)}\left(\nabla u - \frac{uf'(v)}{f(v)}\nabla v\right)^2 + \frac{u^2 f''(v)}{f^2(v)}|\nabla v|^2 \Delta v \\ =\, & L(u,v), \end{aligned}$$

which proves the first part. Now using the fact that $-\Delta v > 0$, $f'(y) \geq 1$, and $f''(y) \leq 0$, $\forall\, 0 \neq y \in \mathbb{R}$, we get $L(u,v) \geq 0$ and therefore (ii) is proved. Next we prove (iii). We have

$$L(u,\,v) \;=\; \underbrace{|\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)}}_{\text{(I)}} + \underbrace{\left(\frac{\Delta u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta v\right)^2}_{\text{(II)}}$$

$$\underbrace{-\frac{2\Delta v}{f(v)}\left(\nabla u - \frac{uf'(v)}{f(v)}\nabla v\right)^2}_{\text{(III)}} + \underbrace{\frac{u^2 f''(v)}{f(v)}|\nabla v|^2 \Delta v.}_{\text{(IV)}}$$

From our assumptions on v and f, we conclude that each of the terms (I), (II), (III) and (IV) in the expression for $L(u,v)$ is nonnegative. Hence $L(u,v) = 0$ in Ω implies that each of (I), (II), (III) and (IV) is zero. In particular

$$|\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)} = 0 \tag{2.1}$$

and

$$\nabla u - \frac{uf'(v)}{f(v)}\nabla v = 0. \tag{2.2}$$

On solving (2.1), we get

$$f'(v) = 1 \Rightarrow f(v) = v + c_1, \tag{2.3}$$

where c_1 is a constant. On using (2.3) in (2.2), we get

$$(\nabla u)(v + c_1) - u\nabla(v + c_1) = 0 \;\Rightarrow \nabla\left(\frac{u}{v + c_1}\right) = 0 \;\text{ i.e., }\; u = cv + d$$

for some constants c and d. Conversely, let us assume (2.1) holds. We need to show that $L(u,v) = 0$. From (2.1), we get that $f'(v) = 1$ and therefore $f''(v) = 0$. Now it remains to show that

$$\left(\frac{\Delta u}{\sqrt{f'(v)}} - \frac{u}{f(v)}\sqrt{f'(v)}\Delta v\right) = 0 \;\text{ i.e., } f(v)\Delta u = uf'(v)\Delta v.$$

From (2.1), we get

$$0 = f(v)\nabla u - uf'(v)\nabla v. \tag{2.4}$$

A simple differentiation in (2.4) yields

$$0 = f(v)\Delta u + f'(v)\nabla u \cdot \nabla v - f'(v)\nabla u \cdot \nabla v - uf''(v)|\nabla v|^2 - uf'(v)\Delta v.$$

Now using the fact that $f''(v) = 0$, one can see easily that

$$f(v)\Delta u = uf'(v)\Delta v,$$

which completes the proof. ∎

3. Applications

This section deals with the applications of Lemma 2.1 and Proposition 2.3. For the existence of positive solution to fourth order elliptic equations, we refer to the paper of Goncalves et al. [14] and for the existence and uniqueness of a solution to the variational inequality to biharmonic operators, we refer to the work of H. Brézis and G. Stampacchia [9]. In the next theorem, we obtain a Hardy-Rellich type inequality. For the details on the Hardy-Rellich inequality and its generalizations and applications, we refer the reader to [1, 13].

Theorem 3.1. *Assume that there is a C^2 function v satisfying*

$$\Delta^2 v \geq \lambda g f(v),\ v > 0,\ -\Delta v > 0 \text{ in } \Omega, \tag{3.1}$$

for some $\lambda > 0$ and a nonnegative continuous function g on Ω and f satisfies the conditions of Proposition 2.3. Then for any $u \in C_0^\infty(\Omega)$, we have

$$\int_\Omega |\Delta u|^2 dx \geq \lambda \int_\Omega g|u|^2 dx. \tag{3.2}$$

Proof. Take $\phi \in C_0^\infty(\Omega)$, by Proposition 2.3, we have

$$\begin{aligned} 0 &\leq \int_\Omega L(\phi, v)dx = \int_\Omega R(\phi, v)dx \\ &= \int_\Omega |\Delta\phi|^2 dx - \int_\Omega \Delta\left(\frac{\phi^2}{f(v)}\right)\Delta v dx \\ &= \int_\Omega |\Delta\phi|^2 dx - \int_\Omega (\Delta^2 v)\cdot\frac{\phi^2}{f(v)}dx, \quad \text{(on integration)}, \\ &\leq \int_\Omega |\Delta\phi|^2 dx - \lambda\int_\Omega \phi^2 g dx \quad \text{(by (3.1))}. \end{aligned}$$

Letting $\phi = u$ yields

$$\int_\Omega |\Delta u|^2 dx \geq \lambda \int_\Omega g|u|^2 dx.$$ ∎

The next lemma deals with a necessary condition for the nonnegative solutions of biharmonic operators.

Lemma 3.2. *Let* $u \in H^2(\Omega) \cap H_0^1(\Omega)$ *be a nonnegative weak solution (not identically zero) of*

$$\Delta^2 u = a(x)u \ \textit{in}\ \Omega, \quad u = \Delta u = 0 \ \textit{on}\ \partial\Omega, \tag{3.3}$$

where $0 \leq a \in L^\infty(\Omega)$, *then* $-\Delta u > 0$ *in* Ω.

Proof. Let $-\Delta u = v$. Then writing (3.3) into system form, we get

$$\begin{cases} -\Delta u = v \ \text{ in } \ \Omega, \\ -\Delta v = a(x)u \ \text{ in } \ \Omega, \\ u = 0 = v \ \ \text{ on } \ \partial\Omega. \end{cases} \tag{3.4}$$

Since $a(x) \geq 0$ in Ω, so by maximum principle, we get $v \geq 0$. By strong maximum principle, either $v > 0$ or $v \equiv 0$ in Ω. If $v \equiv 0$, then we have

$$-\Delta u = 0 \ \text{ in } \ \Omega; \ \ v = 0 \ \text{ on } \partial\Omega.$$

Again by maximum principle, we get $u \equiv 0$, which is a contradiction and therefore $v > 0$ in Ω and hence

$$-\Delta u > 0 \ \text{ in } \Omega. \qquad \blacksquare$$

Next, we consider the following singular system of fourth order elliptic equations:

$$\begin{aligned} &\Delta^2 u = f(v) \text{ in}\, \Omega, \\ &\Delta^2 v = \tfrac{(f(v))^2}{u} \text{ in}\, \Omega, \\ &u > 0,\, v > 0 \quad \text{in}\, \Omega, \\ &u = \Delta u = \ 0 = v = \Delta v \ \text{ on } \partial\Omega, \end{aligned} \tag{3.5}$$

where f is defined as in Proposition 2.3. In the next theorem, we show a linear relationship between the components u and v, where $(u,\, v)$ is a solution of (3.5).

Theorem 3.3. *Let* $(u, v) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ *be a weak solution of* (3.5) *and* f *satisfy the conditions of Proposition 2.3. Then* $u = c_1 v + c_2$, *where* c_1, c_2 *are constants.*

Proof. Since $(u, v) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ is a weak solution of (3.5), for any $\phi_1, \phi_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\int_\Omega \Delta u \Delta \phi_1 dx = \int_\Omega f(v)\phi_1 dx \tag{3.6}$$

and

$$\int_\Omega \Delta v \Delta \phi_2 dx = \int_\Omega \frac{f^2(v)}{u}\phi_2 dx. \tag{3.7}$$

Now choosing $\phi_1 = u$ and $\phi_2 = \dfrac{u^2}{f(v)}$ in (3.6) and (3.7), respectively, we obtain

$$\int_\Omega |\Delta u|^2 dx = \int_\Omega f(v)u dx = \int_\Omega \Delta v \Delta \left(\frac{u^2}{f(v)}\right) dx$$

and therefore, we have

$$\int_\Omega R(u,v)dx = \int_\Omega \left[|\Delta u|^2 - \Delta v \Delta \left(\frac{u^2}{f(v)}\right)\right] dx = 0.$$

By the positivity of $R(u,v)$, we get $R(u,v) = 0$ and by Lemma 3.2, we have

$$-\Delta u > 0, \quad -\Delta v > 0 \quad \text{in } \Omega.$$

Now an application of Proposition 2.3 yields that $u = c_1 v + c_2$ for some constants c_1 and c_2. ∎

Let us consider the following weighted eigenvalue problem

$$\Delta^2 u = \lambda a(x) u \ \text{ in } \Omega, \quad u = \Delta u = 0 \ \text{ on } \partial\Omega, \tag{3.8}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded subset with smooth boundary, $N > 4$, and $0 \leq a \in L^\infty(\Omega)$. We recall that a value $\lambda \in \mathbb{R}$ is an eigenvalue of (3.8) if and only if there exists $u \in H^2(\Omega) \cap H_0^1(\Omega) \backslash \{0\}$ such that

$$\int_\Omega \Delta u \cdot \Delta \phi dx = \lambda \int_\Omega a(x) u \phi dx, \ \ \forall\ \phi \in H^2(\Omega) \cap H_0^1(\Omega) \tag{3.9}$$

and u is called an eigenfunction associated with λ. The least positive eigenvalue of (3.8) is defined as

$$\lambda_1 = \inf\left\{\int_\Omega |\Delta u|^2 dx : u \in H^2(\Omega) \cap H_0^1(\Omega) \ \text{ and } \ \int_\Omega a(x)|u|^2 dx = 1\right\}.$$

Lemma 3.4. λ_1 *is attained.*

Proof. For showing the above infimum is attained, let us introduce the functionals

$$J,\ G : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow \mathbb{R}$$

defined by

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx, \quad G(u) = \frac{1}{2}\int_\Omega a(x)|u|^2 dx, \quad u \in H^2(\Omega) \cap H_0^1(\Omega).$$

It is easy to see that J and G are C^1 functionals. By definition, $\lambda \in \mathbb{R}$ is an eigenvalue of (3.8) if and only if there exists $u \in H^2(\Omega) \cap H_0^1(\Omega) \backslash \{0\}$ such that

$$J'(u) = \lambda G'(u).$$

Let us define

$$M = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) : \frac{1}{2}\int_\Omega a(x)|u|^2 dx = 1 \right\}.$$

Since $a \geq 0$ so $M \neq \emptyset$ and M is a C^1 manifold in $H^2(\Omega) \cap H_0^1(\Omega)$. It is also easy to see that J is coercive and (sequentially) weakly lower semicontinuous on M and M is a weakly closed subset of $H^2(\Omega) \cap H_0^1(\Omega)$. Now by an application of Theorem 1.2 [18], J is bounded from below on M and attains its infimum in M. Also by Lagrange's multiplier rule

$$J'(u) = \lambda_1 G'(u)$$

and therefore λ_1 is attained. ■

In the next lemma, we show that the first eigenfunction u corresponding to the first eigenvalue λ_1 of (3.8) is of one sign. We use the following theorem.

Theorem 3.5. (Dual cone decomposition theorem). [12] *Let H be a Hilbert space with scalar product $(\cdot, \cdot)_H$. Let $K \subset H$ be a closed, convex nonempty cone. Let K^* be its dual cone, namely*

$$K^* = \{w \in H \mid (w, v)_H \leq 0, \ \forall \, v \in K\}.$$

Then for any $u \in H$, there exists a unique $(u_1, u_2) \in K \times K^*$ such that

$$u = u_1 + u_2, \quad (u_1, u_2)_H = 0. \tag{3.10}$$

In particular,

$$||u||_H^2 = ||u_1||_H^2 + ||u_2||_H^2.$$

Moreover, if we decompose arbitrary $u, v \in H$ according to (3.10), *then this implies*

$$||u - v||_H^2 \geq ||u_1 - v_1||_H^2 + ||u_2 - v_2||_H^2.$$

In particular, the projection onto K is Lipschitz continuous.

For a proof of the above theorem, we refer to Theorem 3.4 [12].

Lemma 3.6. *The eigenfunction u corresponding to the first eigenvalue λ_1 of* (3.8) *is of one sign.*

Proof. Using Theorem 3.5, and classical maximum principle for $-\Delta$, Ferrero et al. [11] obtain the positivity of the minimizers of the problem

$$S_q = \min_{w \in X/\{0\}} \frac{||\Delta w||_2^2}{||w||_q^2}, \ \ 1 \leq q < \frac{2n}{n-4},$$

where $X = H^2(B) \cap H_0^1(B)$, B denotes the unit ball in $\mathbb{R}^n$. The same proof works for eigenfunction u corresponding to the first eigenvalue λ_1 of (3.8) in Ω. For this, we refer to [11] and omit the details. ■

Remark 3.7. Using Lemma B1, p. 271 [20], we see that the $u \in L^p(\Omega), \ \forall\ 1 \leq p < \infty$, where u is the eigenfunction corresponding to the first eigenvalue λ_1. Furthermore, with the additional L^p-estimates due to Agmon, Douglis and Nirenberg [2], it can be shown that $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$, see on p. 274 [20] for the complete details.

Next, we show the strict monotonicity of the principle eigenvalue λ_1.

Theorem 3.8. *Suppose $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. Then $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$, if both exist.*

Proof. Let u_i be a positive eigenfunction associated with $\lambda_1(\Omega_i)$, $i = 1, 2$, then by Remark 3.7, $u_i \in C^4(\Omega_i) \cap C^3(\bar{\Omega}_i)$ for $i = 1, 2$ and we have the following

$$\begin{cases} \Delta^2 u_1 = \lambda_1(\Omega_1) a(x) u_1 \text{ in } \Omega_1, \\ u_1 > 0 \text{ in } \Omega_1, \\ u_1 = 0 = \Delta u_1 \text{ on } \partial\Omega_1 \end{cases} \tag{3.11}$$

and

$$\begin{cases} \Delta^2 u_2 = \lambda_1(\Omega_2) a(x) u_2 \text{ in } \Omega_2, \\ u_2 > 0 \text{ in } \Omega_2, \\ u_2 = 0 = \Delta u_2 \text{ on } \partial\Omega_2. \end{cases} \tag{3.12}$$

For $\phi \in C_c^\infty(\Omega_1)$,

$$\begin{aligned} 0 \leq \int_{\Omega_1} L(\phi, u_2)\, dx &= \int_{\Omega_1} R(\phi, u_2)\, dx \\ &= \int_{\Omega_1} \left(|\Delta\phi|^2 - \Delta(\frac{\phi^2}{u_2}) \Delta u_2 \right) dx \\ &= \int_{\Omega_1} |\Delta\phi|^2 dx - \int_{\Omega_1} \frac{\phi^2}{u_2} \Delta^2 u_2 dx. \end{aligned} \tag{3.13}$$

On using (3.12) in (3.13), it is easy to see that

$$0 \le \int_{\Omega_1} |\Delta\phi|^2\,dx - \lambda_1(\Omega_2)\int_{\Omega_1} a(x)\phi^2\,dx. \tag{3.14}$$

Letting $\phi = u_1$ in (3.14), we obtain

$$0 \le \int_{\Omega_1} L(u_1,u_2)dx = (\lambda_1(\Omega_1) - \lambda_1(\Omega_2))\int_{\Omega_1} a(x)u_1^2\,dx.$$

This gives $\lambda_1(\Omega_1) - \lambda_1(\Omega_2) \ge 0$. Now if $\lambda_1(\Omega_1) - \lambda_1(\Omega_2) = 0$ then $L(u_1,u_2) = 0$ and an application of Lemma 2.1 implies that $u_1 = cu_2$, which is not possible as $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. This completes the proof. ■

In the next theorem, using Picone's identity (Lemma 2.1), we show that λ_1 is simple, i.e., the eigenfunctions associated to it are a constant multiple of each other.

Theorem 3.9. λ_1 *is simple.*

Proof. Let u and v be two eigenfunctions associated with λ_1. In view of Remark 3.7, we may assume that $u,\, v \in C^4(\Omega) \cap C^3(\bar{\Omega})$. From Lemma 3.6, without any loss of generality, we may also assume that u and v are positive in Ω. Now by Lemma 3.2, we have

$$-\Delta u > 0, \quad -\Delta v > 0 \ \text{ in } \Omega.$$

Let $\epsilon > 0$. From Lemma 2.1, we have

$$\begin{aligned} 0 &\le \int_\Omega L(u,\, v+\epsilon)dx \\ &= \int_\Omega R(u,\, v+\epsilon)dx \\ &= \int_\Omega \left[|\Delta u|^2 - \Delta\left(\frac{u^2}{v+\epsilon}\right)\Delta v\right]dx \\ &= \lambda_1\int_\Omega a(x)u^2dx - \int_\Omega \Delta\left(\frac{u^2}{v+\epsilon}\right)\Delta v dx. \end{aligned} \tag{3.15}$$

In view of Remark 3.7, $\dfrac{u^2}{v+\epsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ and is admissible in the weak formulation of $\Delta^2 v = \lambda_1 a(x)v$, i.e.,

$$\int_\Omega \Delta v\Delta\left(\frac{u^2}{v+\epsilon}\right)dx = \lambda_1\int_\Omega a(x)v\left(\frac{u^2}{v+\epsilon}\right)dx. \tag{3.16}$$

From (3.15) and (3.16), we get

$$0 \le \int_\Omega L(u,\, v+\epsilon)dx = \lambda_1 \int_\Omega a(x)\left[u^2 - v\left(\frac{u^2}{v+\epsilon}\right)\right]dx.$$

Letting $\epsilon \to 0$, in the above inequality, we get

$$L(u,\, v) = 0$$

and again by an application of Lemma 2.1, there exists $\alpha \in \mathbb{R}$ such that

$$u = \alpha\, v,$$

which proves the simplicity of λ_1. ■

Next, we show the sign changing nature of any eigenfunction v associated to a positive eigenvalue $0 < \lambda \neq \lambda_1$.

Proposition 3.10. *Any eigenfunction v associated to a positive eigenvalue $0 < \lambda \neq \lambda_1$ changes sign.*

Proof. Assume by contradiction that $v \ge 0$, the case $v \le 0$ can be dealt similarly. By Lemma 3.2, $v > 0$ in Ω. Let $\phi > 0$ be an eigenfunction associated with $\lambda_1 > 0$. For any $\epsilon > 0$, we apply Lemma 2.1 to the pair $\phi,\, v+\epsilon$ and get

$$\begin{aligned} 0 &\le \int_\Omega L(\phi,\, v+\epsilon)dx \\ &= \int_\Omega R(\phi,\, v+\epsilon)dx \\ &= \int_\Omega \left[|\Delta\phi|^2 - \Delta\left(\frac{\phi^2}{v+\epsilon}\right)\Delta v\right]dx \\ &= \int_\Omega \left[\lambda_1 a(x)\phi^2 - \Delta\left(\frac{\phi^2}{v+\epsilon}\right)\Delta v\right]dx. \end{aligned} \tag{3.17}$$

Again, we note that $\frac{\phi^2}{v+\epsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ and is admissible in the weak formulation of

$$\Delta^2 v = \lambda a(x) v \text{ in } \Omega;\;\; v = \Delta v = 0 \;\text{ on } \partial\Omega.$$

This implies that

$$\int_\Omega \Delta v \Delta\left(\frac{\phi^2}{v+\epsilon}\right)dx = \lambda \int_\Omega a(x) v \frac{\phi^2}{v+\epsilon}dx. \tag{3.18}$$

From (3.17) and (3.18), we get

$$0 \le \int_\Omega \left[\lambda_1 a(x)\phi^2 - \lambda a(x) v\frac{\phi^2}{v+\epsilon}\right]dx.$$

Letting $\epsilon \to 0$ in the above inequality, we get

$$0 \leq (\lambda_1 - \lambda) \int_\Omega a(x)\phi^2 dx,$$

which is a contradiction, because $\int_\Omega a(x)\phi^2 dx > 0$ and hence v must change sign. ∎

For the application of Lemma 2.1 on Morse index, let us consider the following boundary value problem

$$\Delta^2 u = a(x)G(u) \text{ in } \Omega; \quad u = \Delta u = 0 \text{ on } \partial\Omega, \tag{3.19}$$

where $a \in C^\alpha(\Omega),\ 0 < \alpha < 1$ and $G \in C^1(\mathbb{R}, \mathbb{R})$. For the existence of positive solution to the equations similar to (3.19), we refer the reader to [14]. By the standard elliptic regularity theory, $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$. We shall assume that there exists a positive C^4 solution u of the boundary value problem (3.19). For the solution $u \in C^4(\Omega)$, the Morse index is defined via the eigenvalue problem for the linearization at u.

Definition 3.11 (Morse index). The Morse index of a solution u of (3.19) is the number of negative eigenvalues of the linearized operator

$$\Delta^2 - a(x)G'(u) \tag{3.20}$$

acting on $H^2(\Omega) \cap H_0^1(\Omega)$, i.e., the number of eigenvalues λ such that $\lambda < 0$, and the boundary value problem

$$\Delta^2 w - a(x)G'(u)w = \lambda w \text{ in } \Omega; \ w = 0 = \Delta w \text{ on } \partial\Omega \tag{3.21}$$

has a nontrivial solution w in $H^2(\Omega) \cap H_0^1(\Omega)$.

The next theorem gives an application of Lemma 2.1.

Theorem 3.12. *Let us consider* (3.19). *Let* $a \in C^\alpha(\Omega),\ 0 < \alpha < 1$ *and* $G \in C^1(\mathbb{R}, \mathbb{R})$ *be such that*

$$\frac{G(v)}{v} \geq G'(0) \geq 0, \ \forall\, 0 < v \in \mathbb{R}.$$

Then the trivial solution of (3.19) *has Morse index* 0.

Proof. Let $v \in C^2(\bar{\Omega})$ be a positive weak solution of (3.19). Then

$$\int_\Omega \Delta v \Delta\psi dx = \int_\Omega a(x)G(v)\psi dx, \ \forall\, \psi \in C_c^\infty(\Omega). \tag{3.22}$$

For any $w \in C_c^\infty(\Omega)$, let us take $\frac{w^2}{v}$ as a test function in (3.22) and obtain

$$\int_\Omega \Delta v \Delta \left(\frac{w^2}{v}\right) dx = \int_\Omega a(x)\frac{G(v)}{v} w^2 dx. \tag{3.23}$$

Since v is a positive solution of (3.19), using the fact that $G(v) \geq 0$ and in view of Lemma 3.2, one can see that

$$-\Delta v > 0.$$

Now an application of Lemma 2.1 for $u = w$ yields that

$$\begin{aligned} &\int_\Omega |\Delta w|^2 dx \\ &\geq \int_\Omega \Delta v \Delta \left(\frac{w^2}{v}\right) dx \\ &= \int_\Omega a(x)\frac{G(v)}{v} w^2 dx \\ &\geq \int_\Omega a(x) G'(0) w^2 dx. \end{aligned} \tag{3.24}$$

Consider the eigenvalue problem associated with the linearization for (3.19) at 0, which is

$$\Delta^2 w - a(x)G'(0)w = \lambda w \text{ in } \Omega; \; w = 0 = \Delta w \text{ on } \partial\Omega. \tag{3.25}$$

By the variational characterization of the eigenvalue in (3.25), from (3.24), one can see that $\lambda \geq 0$, which proves the claim. ∎

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