TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 5, pp. 1401-1414, October 2015
DOI: 10.11650/tjm.19.2015.5547
This paper is available online at http://journal.taiwanmathsoc.org.tw

GROWTH AND DIFFERENCE PROPERTIES OF MEROMORPHIC SOLUTIONS ON DIFFERENCE EQUATIONS

Zong-Xuan Chen and Kwang Ho Shon*

Abstract. Consider the difference Riccati equation $f(z+1) = \frac{a(z)f(z)+b(z)}{c(z)f(z)+d(z)}$, where $a,\ b,\ c,\ d$ are polynomials, we precisely estimate growth of meromorphic solutions.

To the difference Riccati equation $f(z+1) = \frac{A(z)+f(z)}{1-f(z)}$, where $A(z) = \frac{m(z)}{n(z)}$, $m(z),\ n(z)$ are irreducible nonconstant polynomials, we precisely estimate exponents of convergence of zeros and poles of meromorphic solutions $f(z)$, their differences $\Delta f(z) = f(z+1) - f(z)$ and divided differences $\frac{\Delta f(z)}{f(z)}$.

1. Introduction and Results

Yanagihara [13] studied meromorphic solutions of nonlinear difference equations, and obtained the following difference analogue of Malmquist's theorem.

Theorem A. (see [13]). *If the first order difference equation*

$$w(z+1) = R(z,\ w), \tag{1.1}$$

where $R(z,\ w)$ is rational in both arguments, admits a nonrational meromorphic solution of finite order, then $\deg_w(R) = 1$.

Equation (1.1) with $\deg_w(R) = 1$ is called the difference Riccati equation

$$w(z+1) = \frac{\alpha(z)w(z) + \beta(z)}{\gamma(z)w(z) + \delta(z)}. \tag{1.2}$$

Received November 10, 2014, accepted March 17, 2015.
Communicated by Yingfei Yi.
2010 *Mathematics Subject Classification*: 30D35, 39A10.
Key words and phrases: Difference Riccati equation, Growth order, Zero, Pole.
The first author was supported by the National Natural Science Foundation of China (No. 11171119). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2013R1A1A2008978).
*Corresponding author.

Recently, a number of papers (including [1-6, 8, 9, 11, 12, 15, 16]) focus on complex difference equations and differences analogues of Nevanlinna's theory.

Halburd and Korhonen [6] use value distribution theory to single out the difference Painlev$\acute{e}$ II equation from a large class of difference equations of the form

$$y(z+1)+y(z-1)=\frac{c_2y^2+c_1y+c_0}{y^2-p^2},$$

where $c_j's,\ p\ (\not\equiv 0)$ are rational functions. In their proof, Halburd and Korhonen are concerned with the difference Riccati equation of the form

$$w(z+1)=\frac{A+\delta w(z)}{\delta-w(z)}, \tag{1.3}$$

where A is a polynomial, $\delta=\pm 1$ (see [6, p.197]).

From this, we see that the difference Riccati equation is an important class of difference equations, it will play an important role for research of difference Painlev$\acute{e}$ equations.

Considering the growth of meromorphic solutions of complex difference Riccati equations is an important problem. In [8], Ishizaki considered growth of transcendental meromorphic solutions of a difference Riccati equation (1.3) and obtained the following theorem.

Theorem B. (see [8]). *Suppose that $A(z)$ is a rational function, and suppose that difference Riccati equation*

$$f(z+1)=\frac{A+f(z)}{1-f(z)}, \tag{1.4}$$

possesses a rational solution $a(z)$. Then (1.4) *has no transcendental meromorphic solutions of order less than $1/2$.*

Theorem B is an important result on difference equations, and shows that every transcendental meromorphic solution of (1.4) satisfies its order of growth $\geq 1/2$ if (1.4) has a rational solution.

In this paper, we assume the reader is familiar with basic notions of Nevanlinna's value distribution theory (see [10, 14]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function f; and $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote, respectively, the exponents of convergence of zeros and poles of f.

Chen [2] considered the growth of transcendental meromorphic solutions to the particular difference Riccati equation, the Pielou logistic equation, and obtained the following theorem.

Theorem C. (see [2]). *Let $P(z),\ Q(z),\ R(z)$ be polynomials with $P(z)Q(z)R(z)$ $\not\equiv 0$, and $y(z)$ be a transcendental meromorphic solution with finite order of the Pielou* *logistic equation*

$$y(z+1) = \frac{R(z)y(z)}{Q(z)+P(z)y(z)}. \tag{1.5}$$

Then

$$\lambda\left(\frac{1}{y}\right) = \sigma(y) \geq 1. \tag{1.6}$$

The following example shows that result of Theorem C is sharp.

Example 1.1. The function $y(z) = \frac{z2^z}{2^z-1}$ satisfies the Pielou logistic equation

$$y(z+1) = \frac{2(z+1)y(z)}{z+y(z)},$$

where $y(z)$ satisfies

$$\lambda(y) = 0 \quad \text{and} \quad \lambda\left(\frac{1}{y}\right) = \sigma(y) = 1.$$

Theorem C reminds us to improve result of Theorem B. In this paper, we consider a more general difference Riccati equation than (1.4), and obtain a more precise result than one of Theorem B, that is, prove the following Theorem 1.1.

Theorem 1.1. *Let $a,\ b,\ c,\ d$ be rational functions, $ac \not\equiv 0$ and $ad-bc \not\equiv 0$. If a difference Riccati equation*

$$f(z+1) = \frac{a(z)f(z)+b(z)}{c(z)f(z)+d(z)} \tag{1.7}$$

has a rational solution $B(z)$, then every transcendental meromorphic solution $f(z)$ with finite order of (1.7) *satisfies*

$$\lambda\left(\frac{1}{f}\right) = \sigma(f) \geq 1. \tag{1.8}$$

Remark 1.1. By Theorems C and 1.1, it seems reasonable to conjecture that in Theorem 1.1, the condition "(1.7) has a rational solution $B(z)$" can be omitted.

The other main goal of this paper is to investigate value distribution of a meromorphic solution $f(z)$, and its difference $\Delta f(z) = f(z+1) - f(z)$, and divided difference $\frac{\Delta f(z)}{f(z)}$ of (1.4).

For the meromorphic function $f(z)$ of small growth, zeros of $\Delta f(z)$ and $\frac{\Delta f(z)}{f(z)}$ are investigated in many papers. Bergweiler and Langley [1] obtained the following theorem.

Theorem D. (see [1]). *There exists $\delta_0 \in (0, 1/2)$ with the following property. Let f be a transcendental entire function with order*

$$\sigma(f) \le \sigma < \frac{1}{2} + \delta_0 < 1,$$

where σ is a nonnegative real number satisfying $\sigma < \frac{1}{2} + \delta_0$. Then

$$G(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z+1) - f(z)}{f(z)}$$

has infinitely many zeros.

In [1], Bergweiler and Langley raised that it seems reasonable to conjecture that the conclusion of Theorem D holds for $\sigma(f) < 1$. Now this conjecture is still open. But for an entire function of $\sigma(f) \ge 1$, the conclusion of Theorem D does not hold. For example, $f(z) = e^z$ satisfies $\frac{\Delta f(z)}{f(z)} = e - 1$ which has only finitely many zeros.

When f is meromorphic, Bergweiler and Langley [1] consider the existence of zeros of the difference $\Delta f(z) = f(z+1) - f(z)$, also gave a construction theorem to show that even if for a transcendental meromorphic function $f(z)$ of lower order 0, $\Delta f(z)$ may have only finitely many zeros.

Langley [11] considered existence of zeros of difference and divided difference of meromorphic functions, and proved the following theorem.

Theorem E. (see [11]). *Let f be a transcendental meromorphic function of order less than $1/6$, then at least one of $\Delta f(z)$ and $\frac{\Delta f(z)}{f(z)}$ has infinitely many zeros.*

Theorem E shows that the condition "order less than $1/6$" can only guarantee that one of $\Delta f(z)$ and $\frac{\Delta f(z)}{f(z)}$ has infinitely many zeros.

From Theorem C and Example 1.1, we see that although every transcendental meromorphic solution $y(z)$ of (1.5) satisfies $\lambda\left(\frac{1}{y}\right) = \sigma(y) \ge 1$, y may have only finitely many zeros. But we discover that for transcendental meromorphic solutions $y(z)$ of some difference Riccati equations, $\Delta y(z)$ and $\frac{\Delta y(z)}{y(z)}$ have infinitely many zeros, and prove the following theorem.

Theorem 1.2. *Let $A(z)$ be a non-constant rational function. Suppose that a difference Riccati equation*

$$f(z+1) = \frac{A(z) + f(z)}{1 - f(z)} \tag{1.9}$$

has a rational solution $B(z)$. Suppose that $f(z)$ is a transcendental meromorphic solution with finite order of (1.9). *Then*

(i) $\lambda(f) = \lambda\left(\frac{1}{f}\right) = \sigma(f) \geq 1$;

(ii) *if $A(z) = a(z)^2$, where $a(z)$ is a nonconstant rational function, then*

$$\lambda(\Delta f(z)) = \lambda\left(\frac{1}{\Delta f(z)}\right) = \sigma(f) \geq 1$$

and

$$\lambda\left(\frac{\Delta f(z)}{f(z)}\right) = \lambda\left(\frac{1}{\Delta f(z)/f(z)}\right) = \sigma(f) \geq 1.$$

2. Proof of Theorem 1.1

We need the following lemmas and remark to prove Theorem 1.1.

Lemma 2.1. (see [3]). *Let $F(z)$, $P_n(z), \ldots, P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$. Suppose that $f(z)$ is a meromorphic solution with infinitely many poles of*

$$P_n(z)f(z+n) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = F(z)$$

or

$$P_n(z)f(z+n) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = 0.$$

Then $\sigma(f) \geq 1$.

Remark 2.1. Following Hayman [7, pp. 75-76], we define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r \geq 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 2.2. [1] *Let g be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an ε-set E such that as $z \to \infty$ in $\mathbb{C} \setminus E$,*

$$\frac{g'(z+c)}{g(z+c)} \to 0, \quad \frac{g(z+c)}{g(z)} \to 1 \quad g(z+c) - g(z) = cg'(z)(1+o(1))$$

uniformly in c for $|c| \leq h$. Further, E may be chosen so that for large z not in E the function g has no zeros or poles in $|\zeta - z| \leq h$.

Lemma 2.3. [5, 9] *Let $w(z)$ be a nonconstant finite order meromorphic solution of*

$$P(z, w) = 0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a)\not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r,a)=S(r,w)$, then

$$m\left(r,\frac{1}{w-a}\right)=S(r,w).$$

Proof of Theorem 1.1. Suppose that f is a transcendental meromorphic solution with finite order of (1.7). Without less of generality, we may suppose that $a,\ b,\ c,\ d$ are polynomials. Set

$$y(z)=\frac{1}{f(z)-B(z)}, \tag{2.1}$$

where $B(z)$ is the rational solution of (1.7). By the condition of the theorem, we clearly see that $B(z)\not\equiv 0$. By (2.1) we have $T(r,\ y)=T(r,\ f)+S(r,\ f)$ and $S(r,\ y)=S(r,\ f)$. Substituting (2.1) into (1.7), and considering $B(z+1)=\frac{a(z)B(z)+b(z)}{c(z)B(z)+d(z)}$, we obtain

$$(c(z)B(z+1)-a(z))y(z+1)+(c(z)B(z)+d(z))y(z)+c(z)=0. \tag{2.2}$$

Set $B(z)=\frac{h(z)}{H(z)}$, where $h(z)$ and $H(z)$ are nonzero polynomials. Substituting $B(z)=\frac{h(z)}{H(z)}$ into (2.2), we obtain

$$\begin{aligned}&[c(z)h(z+1)-a(z)H(z+1)]H(z)y(z+1)\\ &+[c(z)h(z)+d(z)H(z)]H(z+1)y(z)\\ =&-c(z)H(z)H(z+1).\end{aligned} \tag{2.3}$$

Now we prove $c(z)h(z+1)-a(z)H(z+1)\not\equiv 0$. In fact, if $c(z)h(z+1)-a(z)H(z+1)\equiv 0$, then $B(z+1)=\frac{h(z+1)}{H(z+1)}=\frac{a(z)}{c(z)}$, so that, since $B(z)$ is the solution of (1.7), by (1.7), we obtain

$$\frac{a(z)}{c(z)}=\frac{a(z)a(z-1)+b(z)c(z-1)}{c(z)a(z-1)+d(z)c(z-1)},$$

that is,

$$a(z)d(z)c(z-1)-c(z)b(z)c(z-1)\equiv 0.$$

Since $c(z-1)\not\equiv 0$, we have $a(z)d(z)-c(z)b(z)\equiv 0$. This contradicts our condition $a(z)d(z)-c(z)b(z)\not\equiv 0$.

Now we prove $c(z)h(z)+d(z)H(z)\not\equiv 0$. Suppose that $c(z)h(z)+d(z)H(z)\equiv 0$. Then $B(z)=\frac{h(z)}{H(z)}=-\frac{d(z)}{c(z)}$. Substituting $B(z)=-\frac{d(z)}{c(z)}$ into (1.7), and noting $a(z)d(z)-c(z)b(z)\not\equiv 0$ and $c(z+1)\not\equiv 0$, we obtain

$$-\frac{d(z+1)}{c(z+1)}=\frac{a(z)\left(-\frac{d(z)}{c(z)}\right)+b(z)}{c(z)\left(-\frac{d(z)}{c(z)}\right)+d(z)}=\frac{-a(z)d(z)+b(z)c(z)}{0}=\infty.$$

It is a contradiction. Hence $c(z)h(z) + d(z)H(z) \not\equiv 0$.

Now we divide this into three cases to prove $\sigma(y) \geq 1$.

Case 1. Suppose that $y(z)$ has infinitely many poles. Thus, the equation (2.3) satisfies the conditions of Lemma 2.1. By Lemma 2.1, we obtain $\sigma(y) \geq 1$.

Case 2. Suppose that $y(z)$ is an entire function. Thus, by (2.3) and results above, $y(z)$ satisfies the equation

$$A_1(z)y(z+1) + A_0(z)y(z) = F(z), \tag{2.4}$$

where

$$A_1(z) = [c(z)h(z+1) - a(z)H(z+1)]H(z) \not\equiv 0,$$

$$A_0(z) = [c(z)h(z) + d(z)H(z)]H(z+1) \not\equiv 0,$$

$$F(z) = -c(z)H(z)H(z+1) \not\equiv 0,$$

and A_j $(j = 0,\ 1)$ and $F(z)$ are all nonzero polynomials. In what follows, without loss of generality, we suppose that $\deg\ A_1 \leq \deg\ A_0$ (if $\deg\ A_1 \geq \deg\ A_0$, then we can use the same method to prove it).

Suppose that $\sigma(y) < 1$. We will deduce a contradiction.

First, suppose that $\deg\ A_1 < \deg\ A_0$. By Lemma 2.2 and $\sigma(y) < 1$, we see that there exists an ε-set E_1 such that as $z \to \infty$ in $\mathbb{C} \setminus E_1$,

$$y(z+1) = y(z)(1 + o_1(1)), \tag{2.5}$$

where $o_1(1)$ satisfy $o_1(1) \to 0$ as $z \to \infty$ in $\mathbb{C} \setminus E_1$. Set $H_1 = \{|z| = r :\ z \in E_1\}$. Then by Remark 2.1, H_1 is of finite logarithmic measure. We take z such that $|z| = r \notin H_1,\ |y(z)| = M(r,\ y)$. For r sufficiently large, $\left|\frac{A_1(z)}{A_0(z)}\right| < \frac{1}{3}$. Thus, by (2.4), (2.5) and $\left|\frac{A_1(z)}{A_0(z)}\right| < \frac{1}{3}$, it follows that when $|y(z)| = M(r,\ y)$,

$$\begin{aligned} |F(z)| &= |A_0(z)|M(r,\ y)\left|1 + \frac{A_1(z)}{A_0(z)}(1 + o_1(1))\right| \\ &\geq \frac{1}{2}|A_0(z)|M(r,\ y), \qquad |z| = r \notin H_1. \end{aligned} \tag{2.6}$$

Since y is transcendental and $F,\ A_0$ are polynomials, we see (2.6) is a contradiction.

Secondly, we suppose that $\deg\ A_1 = \deg\ A_0$. Set

$$A_0(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0, \quad A_1(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0,$$

where $a_n,\ a_{n-1}, \ldots, a_0;\ b_n,\ b_{n-1},\ \ldots,\ b_0$ are constants, $a_n b_n \neq 0$. By (2.4) and (2.5), we have

$$F(z) = y(z)(A_0(z) + A_1(z)(1 + o_1(1))), \quad |z| = r \notin H_1. \tag{2.7}$$

Clearly, $A_0(z)+A_1(z)(1+o_1(1))\not\equiv 0$. We take z_r such that $|z_r|=r\notin H_1,\ |y(z_r)|=M(r,\ y)$. Now we divide this proof into two subcases.

Subcase 2(1). Suppose that there exists a subsequence $\{z_n\}\subset\{z_r\}$ satisfying

$$\lim_{n\to\infty}(A_0(z_n)+A_1(z_n)(1+o_1(1)))=A,\quad (0<|A|<\infty \text{ or } A=\infty). \tag{2.8}$$

Thus, by (2.7) and (2.8), we obtain when $0<|A|<\infty$, $|z_n|=r_n$

$$|F(z_n)|\ge \frac{1}{2}|A|M(r_n,\ y),$$

or when $|A|=\infty$, $|z_n|=r_n$

$$|F(z_n)|\ge M(r_n,\ y),$$

all are contrary.

Subcase 2(2). Now suppose that there do not exist any subsequence $\{z_n\}$ of $\{z_r\}$ satisfying (2.8). Thus,

$$\lim_{r\to\infty}(A_0(z_r)+A_1(z_r)(1+o_1(1)))=0,\quad |z_r|=r\notin H_1,\ |y(z_r)|=M(r,\ y).$$

So that, we have $\frac{A_0(z_r)}{A_1(z_r)}\to -1$, $a_n=-b_n$ and $A_0(z_r)+A_1(z_r)=-A_1(z_r)o_1(1)$. We again divide Subcase 2(2) into two subcases.

Subcase 2(2(i)). Suppose that $A_0(z_r)+A_1(z_r)=-A_1(z_r)o_1(1)\to 0$. Then $A_0(z)\equiv -A_1(z)$ since A_0 and A_1 are polynomials. By Lemma 2.2 and $\sigma(y)<1$, we see that there exists an ε-set E_2 such that as $z\to\infty$ in $\mathbb{C}\setminus E_2$,

$$y(z+1)-y(z)=y'(z)(1+o_2(1)),\quad (o_2(1)\to 0). \tag{2.9}$$

Set $H_2=\{|z|=r:\ z\in E_2\}$. Then by Remark 2.1, H_2 is of finite logarithmic measure. Thus, by (2.4), (2.9) and $A_0(z)\equiv -A_1(z)$, we obtain

$$F(z)=-A_1(z)y(z)+A_1(z)y(z+1)=A_1(z)y'(z)(1+o_2(1)). \tag{2.10}$$

We take z such that $|z|=r\notin H_2,\ |y'(z)|=M(r,\ y')$, by (2.10), we have

$$|F(z)|=|A_1(z)y'(z)(1+o_2(1))|\ge \frac{1}{2}|A_1(z)|M(r,\ y').$$

It is a contradiction.

Subcase 2(2(ii)). Suppose that $A_0(z_r)+A_1(z_r)=-A_1(z_r)o_1(1)\not\to 0$. Then $A_0(z)\not\equiv -A_1(z)$. Since $a_n=-b_n$, we may suppose that

$$A_1(z)=\alpha(z)+\beta_1(z),\quad A_0(z)=-\alpha(z)+\beta_0(z), \tag{2.11}$$

where $\alpha(z)$ and $\beta_j(z)$ $(j = 0, 1)$ are polynomials, and $\deg \beta_j < \deg \alpha$ $(j = 0, 1)$. By (2.4), (2.5), (2.9) and (2.11), we have that

$$\begin{aligned} F(z) &= -\alpha(z)y(z) + \beta_0(z)y(z) + \alpha(z)y(z+1) + \beta_1(z)y(z+1) \\ &= \alpha(z)y'(z)(1 + o_2(1)) + y(z)(\beta_0(z) + \beta_1(z)(1 + o_1(1))), \\ & |z| = r \notin H_1 \bigcup H_2. \end{aligned} \tag{2.12}$$

By Wiman-Valiron theory (see [10]), we see that there exists a set $H_3 \subset (1,\ \infty)$ of finite logarithmic measure, such that

$$\frac{y'(z)}{y(z)} = \frac{\nu(r,\ y)}{z}(1 + o_3(1)), \quad |z| = r \notin H_3, \quad o_3(1) \to 0,$$

where z satisfy $|z| = r$ and $|y(z)| = M(r,\ y)$, $\nu(r,\ y)$ is the central index of $y(z)$. So that

$$\left|\frac{y'(z)}{y(z)}\right| = \frac{\nu(r,\ y)}{|z|}|(1 + o_3(1))| \geq \frac{1}{2|z|}\nu(r,\ y), \quad |z| = r \notin H_3. \tag{2.13}$$

By $\deg \beta_j < \deg \alpha$ $(j = 1, 2)$ and $\nu(r,\ y) \to \infty$, we have that

$$\frac{\beta_0(z) + \beta_1(z)(1 + o_1(1))}{\alpha(z)\frac{1}{2|z|}\nu(r,\ y)} \to 0. \tag{2.14}$$

Thus, by (2.12)–(2.14), we deduce that as z satisfy $|y(z)| = M(r,\ y)$, $|z| = r \notin H_1 \bigcup H_2 \bigcup H_3$, $r \to \infty$,

$$\begin{aligned} |F(z)| &= |y(z)| \left|\alpha(z)\frac{y'(z)}{y(z)}(1 + o_2(1)) + \beta_0(z) + \beta_1(z)(1 + o_1(1))\right| \\ &\geq |y(z)| \left||\alpha(z)|\frac{1}{2|z|}\nu(r,\ y) - |\beta_0(z) + \beta_1(z)(1 + o_1(1))|\right| \\ &\geq |y(z)||\alpha(z)|\frac{1}{4|z|}\nu(r,\ y) \\ &= M(r,\ y)|\alpha(z)|\frac{1}{4|z|}\nu(r,\ y). \end{aligned} \tag{2.15}$$

Since y is a transcendental entire function, $\nu(r,\ y) \to \infty$ and $F,\ \alpha$ are polynomials, we see (2.15) is a contradiction.

Hence $\sigma(y) \geq 1$.

Case 3. Suppose that $y(z)$ has only finitely many poles. We see that (2.4) holds. Then set $y(z) = \frac{y^*(z)}{G(z)}$, where $y^*(z)$ is an entire function, and $G(z)$ is a polynomial. Substituting $y(z) = \frac{y^*(z)}{G(z)}$ into (2.4), we have

$$A_1(z)G(z)y^*(z+1) + A_0(z)G(z+1)y^*(z) = F(z)G(z)G(z+1). \tag{2.16}$$

Thus, by the result of Case 2, we obtain $\sigma(y^*) \geq 1$.

Hence, $\sigma(f) = \sigma(y) = \sigma(y^*) \geq 1$.

In what follows, we prove $\lambda(1/f) = \sigma(f)$. Set $y_1(z) = \frac{1}{f(z)}$. Then $T(r,\ y_1) = T(r,\ f) + O(1)$. Substituting $y_1(z) = \frac{1}{f(z)}$ into (1.7), we obtain

$$\frac{1}{y_1(z+1)} = \frac{a(z) + b(z)y_1(z)}{c(z) + d(z)y_1(z)}.$$

Thus, we have that

$$D(z,\ y_1) := y_1(z+1)(a(z) + b(z)y_1(z)) - (c(z) + d(z)y_1(z)) = 0$$

and

$$D(z,\ 0) = -c(z) \not\equiv 0.$$

By Lemma 2.3, we obtain

$$m\left(r,\ \frac{1}{y_1}\right) = S(r,\ y_1).$$

Hence,

$$N\left(r,\ \frac{1}{y_1}\right) = T(r,\ y_1) + S(r,\ y_1),$$

that is, $N(r,\ f) = T(r,\ f) + O(1) + S(r,\ y_1)$. By $S(r,\ y_1) = o\{T(r,\ y_1)\}$, we see $S(r,\ y_1) = o\{T(r,\ f)\}$. Thus, $N(r,\ f) = T(r,\ f)(1+o(1))$. Hence, $\lambda(1/f) = \sigma(f)$.

Thus, Theorem 1.1 is proved.

3. Proof of Theorem 1.2

We need the following lemmas for proof of Theorem 1.2.

Lemma 3.1. (see [4]). *Let $\delta = \pm 1$ be a constant and $A(z) = \frac{m(z)}{n(z)}$ be an irreducible non-constant rational function, where $m(z)$ and $n(z)$ are polynomials with $\deg m(z) = m$ and $\deg n(z) = n$. If $f(z)$ is a finite order transcendental meromorphic solution of* (1.9), *then*

(i) *if $\sigma(f) > 0$, then f has at most one Borel exceptional value;*

(ii) $\lambda(f) = \lambda(\frac{1}{f}) = \sigma(f)$;

(iii) *if $A(z) \not\equiv -z^2 - z + 1$, then the exponent of convergence of fixed points of f satisfies $\tau(f) = \sigma(f)$.*

Lemma 3.2. *Suppose that $a(z)$ is a nonconstant rational function and $f(z)$ is a transcendental meromorphic function. Then, $a(z)^2 + f(z)^2$ and $1 - f(z)$ (or $f(z)$) have at most finitely many common zeros.*

Proof. Suppose that z_0 is a common zero of $a(z)^2 + f(z)^2$ and $1 - f(z)$. Then, $a(z_0)^2 + f(z_0)^2 = 0$. Thus, $f(z_0) = \pm ia(z_0)$. Substituting $f(z_0) = \pm ia(z_0)$ into $1 - f(z)$, we obtain $1 \mp ia(z_0) = 0$. Since $1 \mp ia(z)$ has only finitely many zeros, we see that $a(z)^2 + f(z)^2$ and $1 - f(z)$ have at most finitely many common zeros. Similarly, we can prove $a(z)^2 + f(z)^2$ and $f(z)$ have at most finitely many common zeros.

Proof of Theorem 1.2.

Suppose that f is a transcendental meromorphic solution with finite order of (1.9).

(i) By Theorem 1.1 and Lemma 3.1, we have that

$$\lambda(f) = \lambda\left(\frac{1}{f}\right) = \sigma(f) \geq 1.$$

(ii) By (1.9), we obtain

$$\Delta f(z) = \frac{a(z)^2 + f(z)^2}{1 - f(z)} = \frac{(f(z) - ia(z))(f(z) + ia(z))}{1 - f(z)}. \tag{3.1}$$

By (i), we see that $\lambda\left(\frac{1}{f}\right) = \sigma(f)$. If z_0 is a pole of $f(z)$ of order $k_0 \geq 1$ (is not a pole of $a(z)$), then z_0 must be a pole of $\frac{a(z)^2+f(z)^2}{1-f(z)}$ of order k_0. Thus, by (3.1), we see that z_0 is a pole of $\Delta f(z)$ of order k_0. Hence, we obtain $\lambda\left(\frac{1}{\Delta f(z)}\right) \geq \lambda\left(\frac{1}{f(z)}\right)$. Combining this and the result of (i), we obtain

$$\lambda\left(\frac{1}{\Delta f(z)}\right) = \lambda\left(\frac{1}{f(z)}\right) = \sigma(f(z)).$$

By Lemma 3.2, we see that $a(z)^2 + f(z)^2$ and $1 - f(z)$ have at most finitely many common zeros. Since $a(z)$ is the rational function and $f(z)$ is transcendental, we see that zeros of $f(z) - ia(z)$ must not be poles of $f(z) + ia(z)$ except finitely many exceptional. Thus, to prove $\lambda(\Delta f(z)) = \sigma(f(z))$, by (3.1), we only need to prove that

$$\lambda(f(z) - ia(z)) = \sigma(f(z)) \tag{3.2}$$

or

$$\lambda(f(z) + ia(z)) = \sigma(f(z)). \tag{3.3}$$

In what follows, we prove that (3.2) holds. Suppose that

$$\lambda(f(z) - ia(z)) = \lambda_1 < \sigma(f(z)).$$

Thus, $f(z) - ia(z)$ can be rewritten as the form

$$f(z) - ia(z) = z^s \frac{p_1(z)}{q_1(z)} e^{h_1(z)} = \frac{p(z)}{q(z)} e^{h_1(z)} = \frac{p(z)}{H(z)}, \tag{3.4}$$

where $p_1(z)$ and $q_1(z)$ are canonical products (or polynomials) formed by nonzero zeros and poles of $f(z) - ia(z)$, respectively, $h_1(z)$ is a nonzero polynomial such that $\deg h_1(z) \le \sigma(f(z))$, s is an integer, if $s \ge 0$, then $p(z) = z^s p_1(z),\ q(z) = q_1(z)$; if $s < 0$, then $p(z) = p_1(z),\ q(z) = z^{-s} q_1(z)$, so that

$$\lambda(p(z)) = \sigma(p(z)) = \lambda\left(f(z) - ia(z)\right) = \lambda_1 < \sigma(f(z)), \tag{3.5}$$

and

$$\lambda(q(z)) = \sigma(q(z)) = \lambda\left(\frac{1}{f(z) - ia(z)}\right) \le \sigma(f(z)),$$

and $H(z) = q(z)e^{-h_1(z)}$ is an entire function. By (3.4) and (3.5), we have $\sigma(H(z)) = \sigma(f(z))$. By (3.4), we have $f(z) = p(z)y_1(z) + ia(z)$, where $y_1(z) = \frac{1}{H(z)}$. So $\sigma(y_1(z)) = \sigma(H(z)) = \sigma(f(z))$.

Substituting $f(z) = p(z)y_1(z) + ia(z)$ into (1.9), we obtain

$$D(z, y_1(z)) := [ia(z+1) + p(z+1)y_1(z+1)][1 - ia(z) - p(z)y_1(z)]$$

and

$$-a(z)^2 - [ia(z) + p(z)y_1(z)] = 0. \tag{3.6}$$

By (3.6), we have that

$$D(z,0) = ia(z+1)(1 - ia(z)) - a(z)^2 - ia(z) = (i + a(z))(a(z+1) - a(z)). \tag{3.7}$$

If $i + a(z) \equiv 0$, then $a(z) \equiv -i$. This contradicts our condition that $a(z)$ is a nonconstant rational function. If $a(z+1) - a(z) \equiv 0$, then $a(z)$ is either a constant or a periodic function. This also contradicts our condition. Both cases show $D(z,\ 0) \not\equiv 0$ in (3.7).

Thus, by Lemma 2.3 and $D(z,\ 0) \not\equiv 0$, we obtain

$$m\left(r,\ \frac{1}{y_1(z)}\right) = S(r,\ y_1(z)).$$

So that,

$$\begin{aligned} N(r,\ H(z)) = N\left(r,\ \frac{1}{y_1(z)}\right) &= T(r,\ y_1(z)) + S(r,\ y_1(z)) \\ &= T(r,\ H(z)) + S(r,\ H(z)). \end{aligned} \tag{3.8}$$

But, since $H(z)$ is the entire function, we have $N(r,\ H(z)) \equiv 0$. This contradicts (3.8). Hence, (3.2) holds, that is,

$$\lambda(\Delta f(z)) = \sigma(f(z)).$$

Finally, we prove that

$$\lambda\left(\frac{\Delta f(z)}{f(z)}\right) = \lambda\left(\frac{1}{\Delta f(z)/f(z)}\right) = \sigma(f) \geq 1.$$

By (3.1), we have that

$$\frac{\Delta f(z)}{f(z)} = \frac{a(z)^2 + f(z)^2}{(1 - f(z))f(z)} = \frac{(f(z) - ia(z))(f(z) + ia(z))}{(1 - f(z))f(z)}. \tag{3.9}$$

By Lemma 3.2, we see that $a(z)^2+f(z)^2$ and $(1-f(z))f(z)$ have at most finitely many common zeros. So that, zeros of $f(z)$ must be poles of $\frac{\Delta f(z)}{f(z)}$, at most except finitely many exceptional points. Thus, by the result of (i), we have $\lambda(f(z)) = \sigma(f(z))$, hence,

$$\lambda\left(\frac{1}{\Delta f(z)/f(z)}\right) = \lambda(f(z)) = \sigma(f(z)). \tag{3.10}$$

By Lemma 3.2 and (3.9), we see that to prove $\lambda\left(\frac{\Delta f(z)}{f(z)}\right) = \sigma(f(z))$, we only need to prove (3.2) holds. Above, we have proved that (3.2) holds. Hence, $\lambda\left(\frac{\Delta f(z)}{f(z)}\right) = \sigma(f(z))$.

Thus, Theorem 1.2 is proved.

Acknowledgments

The authors are grateful to the referee for a number of helpful suggestions to improve the paper.

References

1. W. Bergweiler and J. K. Langley, Zeros of differences of meromorphic functions, *Math. Proc. Camb. Phil. Soc.*, **142** (2007), 133-147.
2. Z. X. Chen, On growth, zeros and poles of meromorphic solutions of linear and nonlinear difference equations, *Science China Math.*, **54(10)** (2011), 2123-2133.
3. Z. X. Chen, Growth and zeros of meromorphic solution of some linear difference equations, *J. Math. Anal. Appl.*, **373** (2011), 235-241.
4. Z. X. Chen and K. H. Shon, Some results on difference Riccati equations, *Acta Math. Sinica, English Series*, **27(6)** (2011), 1091-1100.
5. R. G. Halburd and R. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, *J. Math. Anal. Appl.*, **314** (2006), 477-487.
6. R. G. Halburd and R. Korhonen, Existence of finite-order meromorphic solutions as a detector of integrability in difference equations, *Physica D.*, **218** (2006), 191-203.

7. W. K. Hayman. Slowly growing integral and subharmonic functions, *Comment Math. Helv.*, **34** (1960), 75-84.

8. K. Ishizaki, On difference Riccati equations and second order linear difference equations, *Aequationes Math.*, **81** (2011), 185-198

9. I. Laine and C. C. Yang, Clunie theorems for difference and q-difference polynomials, *J. Lond. Math. Soc.*, **76(3)** (2007), 556-566.

10. I. Laine, *Nevanlinna Theory and Complex Differential Equations*, W. de Gruyter, Berlin, 1993.

11. J. K. Langley, Value distribution of differences of meromorphic functions, *Rocky Mountain J. Math.*, **41** (2011), 275-291.

12. S. Li and Z. S. Gao, Finite order meromorphic solutions of linear difference equations, *Proc. Japan Acad., Ser. A*, **87(5)** (2011), 73-76.

13. N. Yanagihara, Meromorphic solutions of some difference equations, *Funkcial. Ekvac.*, **23** (1980), 309-326.

14. L. Yang, *Value Distribution Theory*, Science Press, Beijing, 1993.

15. R. R. Zhang and Z. X. Chen, On meromorphic solutions of Riccati and linear difference equations, *Acta Math. Sci.*, **33B(5)** (2013), 1243-1254.

16. X. M. Zheng and J. Tu, Growth of meromorphic solutions of linear difference equations, *J. Math. Anal. Appl.*, **384** (2011), 349-356.

Zong-Xuan Chen
School of Mathematical Sciences
South China Normal University
Guangzhou 510631
P. R. China
E-mail: chzx@vip.sina.com

Kwang Ho Shon
Department of Mathematics
College of Natural Sciences
Pusan National University
Busan 609-735
Republic of Korea
E-mail: khshon@pusan.ac.kr