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NON-TRIVIAL SOLUTIONS FOR p-HARMONIC TYPE EQUATIONS VIA A LOCAL MINIMUM THEOREM FOR FUNCTIONALS

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Abstract. In this paper, we establish existence results and energy estimates of weak solutions for an equation involving a p-harmonic operator, subject to Dirichlet boundary conditions in a bounded smooth open domain of $\mathbb{R}^N$. A critical point result for differentiable functionals is exploited, in order to prove that the problem admits at least one non-trivial weak solution.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded smooth open domain and let $p > 1$. The aim of this paper is to study the following Dirichlet problem

$$\begin{cases} \Delta(a(x,\ \Delta u)) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, n denotes the outward unit normal to $\partial\Omega$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$(\mathrm{f_1}) \qquad |\mathrm{f(x,t)}| \leq \mathrm{a_1} + \mathrm{a_2}|\mathrm{t}|^{\mathrm{q}-1}, \qquad \forall(\mathrm{x,t}) \in \Omega \times \mathbb{R},$$

for some non-negative constants a_1, a_2, where $q \in (1, p^*)$ and

$$p^* := \begin{cases} \dfrac{pN}{N-2p} & \text{if } p < \frac{N}{2}, \\ +\infty & \text{if } p \geq \frac{N}{2}. \end{cases}$$

Regarding the function $a : \Omega \times \mathbb{R} \to \mathbb{R}$, we assume that $A : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, $A(x,\xi)$ is continuous in $\bar{\Omega} \times \mathbb{R}$, with continuous derivative with respect to ξ, $a = D_\xi A = A'$, having the following properties:

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(a) $A(x, 0) = 0, \quad \forall x \in \Omega.$

(b) a satisfies the growth condition: There exists a constant $c_1 > 0$ such that

$$|a(x, \xi)| \le c_1(1 + |\xi|^{p-1}), \quad \forall x \in \Omega,\ \xi \in \mathbb{R}.$$

(c) A is strictly convex, that is $\forall x \in \Omega,\ t \in [0, 1],\ \xi, \eta \in \mathbb{R}$,

$$A(x, t\xi + (1-t)\eta) \le tA(x, \xi) + (1-t)A(x, \eta).$$

The above strictly inequality holds if and only if $\xi \neq \eta$ and $t \in (0, 1)$.

(d) A satisfies the ellipticity condition: there exists a constant $c_2 > 0$ such that

$$A(x, \xi) \ge c_2|\xi|^p, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}.$$

The simplest case occurs when $a(x, s) = |s|^{p-2}s$, thus (1.1) reduces to a p-harmonic equation with Dirichlet boundary conditions.

More precisely, employing a critical point result for differentiable functionals, the main goal here is to obtain some sufficient conditions to guarantee that, problem (1.1) has at least one weak solution (see Theorem 3.1).

A special case of our main result reads as follows.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying a $(q-1)$-sublinear growth at infinity for some $q \in (1, p^*)$, i.e.,*

$$\lim_{|t|\to\infty} \frac{f(t)}{|t|^{q-1}} = 0.$$

In addition, if $f(0) = 0$, assume also that

$$\lim_{t\to 0^+} \frac{\displaystyle\int_0^t f(\xi)\, d\xi}{t} = +\infty.$$

Then, there exists $\lambda^\star > 0$, such that, for any $\lambda \in (0, \lambda^\star)$ the following p-harmonic problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(u), & \text{in } \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in W_0^{2,p}(\Omega)$. Also, $\lambda^\star = +\infty$, provided $q \in (1, p)$.

Moreover,

$$\lim_{\lambda\to 0^+} \int_\Omega |\Delta u_\lambda(x)|^p\, dx = 0$$

and the function

$$\lambda \mapsto \frac{1}{p}\int_\Omega |\Delta u_\lambda(x)|^p\,dx - \lambda\int_\Omega\left(\int_0^{u_\lambda(x)} f(\xi)\,d\xi\right)dx$$

is negative and strictly decreasing in $(0,\lambda^\star)$.

Finally, we cite the manuscripts [2, 3, 4, 5, 8], where the existence of multiple solutions for this type of nonlinear differential equations was studied.

In conclusion, we cite a recent monograph by Kristály, Radulescu and Varga [6] as a general reference on variational methods adopted here.

2. Preliminaries

In order to prove our main result, stated in Theorem 3.1, in the following we will perform the variational principle of Ricceri established in [7]. For the sake of clarity, we recall it here below in the form given in [1].

Theorem 2.1. *Let* X *be a reflexive real Banach space, let* $\Phi,\Psi : X \to \mathbb{R}$ *be two Gâteaux differentiable functionals such that* Φ *is strongly continuous, sequentially weakly lower semicontinuous and coercive in* X *and* Ψ *is sequentially weakly upper semicontinuous in* X. *Let* I_λ *be the functional defined as* $I_\lambda := \Phi - \lambda\Psi$, $\lambda\in\mathbb{R}$, *and for any* $r > \inf_X \Phi$ *let* φ *be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}\big((-\infty,r)\big)} \frac{\displaystyle\sup_{v\in\Phi^{-1}\big((-\infty,r)\big)} \Psi(v)-\Psi(u)}{r-\Phi(u)}. \tag{2.1}$$

Then, for any $r > \inf_X \Phi$ *and any* $\lambda \in (0, 1/\varphi(r))$, *the restriction of the functional* I_λ *to* $\Phi^{-1}\big((-\infty,r)\big)$ *admits a global minimum, which is a critical point (precisely a local minimum) of* I_λ *in* X.

Now, let us denote by X the Sobolev space $W_0^{2,p}(\Omega)$, endowed with the norm

$$\|u\| := \left(\int_\Omega |\Delta u(x)|^p\,dx\right)^{1/p}.$$

We recall that (see [9, page 1026]) if $p > N/2$, the embedding $X \hookrightarrow C^0(\bar\Omega)$ is compact, and if $p \le N/2$, the embedding $X \hookrightarrow L^q(\Omega)$ for all $q\in[1,p^*)$ is compact.

Hence, for the case where $p > N/2$, there exists $k>0$ such that

$$\|u\|_\infty \le k\|u\|, \qquad \forall u\in X,$$

and for the case where $p \leq N/2$, there exists $S_q > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq S_q \|u\|, \qquad \forall u \in X.$$

We say that a function $u \in X$ is a *weak solution* of problem (1.1), if u satisfies

$$\int_\Omega a(x,\ \Delta u(x))\Delta v(x)\,dx - \lambda \int_\Omega f(x,u(x))v(x)\,dx = 0,$$

for every $v \in X$.

3. Main Results

In this section we establish the main abstract result of this paper.

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that condition* (f_1) *holds. In addition, if $f(x,0) = 0$ for a.e. $x \in \Omega$, assume also that*

(f_2) *there are a non-empty open set $D \subseteq \Omega$ and a set $B \subseteq D$ of positive Lebesgue measure such that*

$$\limsup_{t\to 0^+} \frac{\operatorname*{ess\,inf}_{x\in B} F(x,t)}{t} = +\infty,$$

and

$$\liminf_{t\to 0^+} \frac{\operatorname*{ess\,inf}_{x\in D} F(x,t)}{t} > -\infty,$$

where F is the primitive of the nonlinearity f with respect to the second variable, i.e., $F(x,t) := \displaystyle\int_0^t f(x,\xi)\,d\xi$.

Further, assume that a and A are continuous functions and satisfy conditions (a)-(d). *Then, there exists $\lambda^\star > 0$, such that, for any $\lambda \in (0,\lambda^\star)$ problem* (1.1) *admits at least one non-trivial weak solution $u_\lambda \in X$. Also, $\lambda^\star = +\infty$, provided $q \in (1,p)$.*

Moreover,

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$$

and the function

$$\lambda \mapsto \int_\Omega A(x,\ \Delta u_\lambda(x))\,dx - \lambda \int_\Omega F(x,u_\lambda(x))\,dx$$

is negative and strictly decreasing in $(0,\lambda^\star)$.

Proof. Our aim is to apply Theorem 2.1 to problem (1.1). To this end, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \int_\Omega A(x,\ \Delta u(x))\,dx, \qquad \Psi(u) := \int_\Omega F(x, u(x))\,dx,$$

for every $u \in X$, and set $I_\lambda := \Phi - \lambda\Psi$.

Clearly, Φ and Ψ are well defined and continuously Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \int_\Omega a(x,\ \Delta u(x))\Delta v(x)\,dx,$$
$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)\,dx,$$

for every $v \in X$ (see [3, Lemma 2.2]).

By [3, Lemma 2.4], Φ is sequentially weakly lower semicontinuous and Ψ is sequentially weakly (upper) continuous. By condition (d), for all $u \in X$, we have

$$\Phi(u) = \int_\Omega A(x,\ \Delta u(x))\,dx \geq c_2 \int_\Omega |\Delta u(x)|^p\,dx = c_2\|u\|^p. \tag{3.1}$$

Hence, Φ is coercive in X and $\inf_{u\in X} \Phi(u) = 0$.

Now, let $r > 0$. It is easy to see that $\varphi(r) \geq 0$ for any $r > 0$, where φ is defined by (2.1).

Then, by Theorem 2.1,

(3.2) for any $r > 0$ and any $\lambda \in \Big(0, 1/\varphi(r)\Big)$ the restriction of I_λ to $\Phi^{-1}\big((-\infty, r)\big)$ admits a global minimum $u_{\lambda,r}$,

which is a critical point (namely a local minimum) of I_λ in X.

Let $\lambda^\star$ be defined as follows

$$\lambda^\star := \sup_{r>0} \frac{1}{\varphi(r)}.$$

Note that $\lambda^\star > 0$, since $\varphi(r) \geq 0$ for any $r > 0$.

Now, fix $\bar\lambda \in (0, \lambda^\star)$. It is easy to see that

(3.3) there exists $\bar r_{\bar\lambda} > 0$ such that $\bar\lambda \leq 1/\varphi(\bar r_{\bar\lambda})$.

Then, by (3.2) applied with $r = \bar r_{\bar\lambda}$, we have that for any λ such that

$$0 < \lambda < \bar\lambda \leq 1/\varphi(\bar r_{\bar\lambda}),$$

the function $u_\lambda := u_{\lambda,\bar{r}_{\bar{\lambda}}}$ is a global minimum of the functional I_λ restricted to $\Phi^{-1}\big((-\infty, \bar{r}_{\bar{\lambda}})\big)$, i.e.,

$$I_\lambda(u_\lambda) \le I_\lambda(u) \text{ for any } u \in X \text{ such that } \Phi(u) < \bar{r}_{\bar{\lambda}} \tag{3.4}$$

and

$$\Phi(u_\lambda) < \bar{r}_{\bar{\lambda}}, \tag{3.5}$$

and also u_λ is a critical point of I_λ in X and so it is a weak solution of problem (1.1).

Now, we show that $\lambda^\star = +\infty$, provided $q \in (1, p)$. To this end, by (f_1), one has

$$|F(x,t)| \le a_1|t| + \frac{a_2}{q}|t|^q, \tag{3.6}$$

for any $(x,t) \in \Omega \times \mathbb{R}$.

Also, by (3.1), for any $u \in X$ such that $\Phi(u) < r$, with $r > 0$, we have

$$\|u\|^p < \frac{r}{c_2}.$$

Now, we discuss two cases.

Case 1. If $p < N/2$, from (3.6), for any $u \in X$ such that $\Phi(u) < r$, we obtain

$$\begin{aligned}
\Psi(u) &= \int_\Omega F(x, u(x))\, dx \\
&\le a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q}\|u\|^q_{L^q(\Omega)} \\
&\le a_1 S_1\|u\| + \frac{a_2 S_q}{q}\|u\|^q \\
&< a_1 S_1\Big(\frac{r}{c_2}\Big)^{1/p} + \frac{a_2 S_q^q}{q}\Big(\frac{r}{c_2}\Big)^{q/p},
\end{aligned}$$

so that

$$\sup_{u\in\Phi^{-1}\left((-\infty,r)\right)} \Psi(u) \le \frac{a_1 S_1}{c_2^{1/p}} r^{1/p} + \frac{a_2 S_q^q}{q c_2^{q/p}} r^{q/p}$$

for any $r > 0$. Now, by definition of φ, for any $r > 0$ we have

$$\varphi(r) \le \frac{\sup_{u\in\Phi^{-1}\left((-\infty,r)\right)} \Psi(u)}{r} \le \frac{a_1 S_1}{c_2^{1/p}} r^{1/p-1} + \frac{a_2 S_q^q}{q c_2^{q/p}} r^{q/p-1}.$$

Since $\Phi(0) = \Psi(0) = 0$, namely,

$$\frac{1}{\varphi(r)} \ge \frac{q c_2^{q/p}}{a_1 S_1 q c_2^{(q-1)/p} r^{(1-p)/p} + a_2 S_q^q r^{(q-p)/p}},$$

so that

$$\lambda^\star = \sup_{r>0} \frac{1}{\varphi(r)} \geq \sup_{r>0} \frac{qc_2^{q/p}}{a_1 S_1 q c_2^{(q-1)/p} r^{(1-p)/p} + a_2 S_q^q r^{(q-p)/p}} = +\infty,$$

provided $q \in (1, p)$. Hence, $\lambda^\star = +\infty$ if $q \in (1, p)$.

Case 2. If $p \geq N/2$, from (3.6), for any $u \in X$ such that $\Phi(u) < r$, we obtain

$$\begin{aligned}\Psi(u) &= \int_\Omega F(x, u(x))\, dx \\ &\leq \operatorname{meas}(\Omega) \left(a_1 \|u\|_\infty + \frac{a_2}{q} \|u\|_\infty^q \right) \\ &\leq \operatorname{meas}(\Omega) \left(a_1 k \|u\| + \frac{a_2 k^q}{q} \|u\|^q \right) \\ &< \operatorname{meas}(\Omega) \left(a_1 k \Big(\frac{r}{c_2}\Big)^{1/p} + \frac{a_2 k^q}{q} \Big(\frac{r}{c_2}\Big)^{q/p} \right),\end{aligned}$$

so that

$$\sup_{u \in \Phi^{-1}\left((-\infty, r)\right)} \Psi(u) \leq \operatorname{meas}(\Omega) \left(\frac{a_1 k}{c_2^{1/p}} r^{1/p} + \frac{a_2 k^q}{q c_2^{q/p}} r^{q/p} \right)$$

for any $r > 0$. Now, by definition of φ, for any $r > 0$ we have

$$\varphi(r) \leq \frac{\sup_{u \in \Phi^{-1}\left((-\infty, r)\right)} \Psi(u)}{r} \leq \operatorname{meas}(\Omega) \left(\frac{a_1 k}{c_2^{1/p}} r^{1/p-1} + \frac{a_2 k^q}{q c_2^{q/p}} r^{q/p-1} \right).$$

Namely,

$$\lambda^\star = \sup_{r>0} \frac{1}{\varphi(r)} \geq \sup_{r>0} \frac{qc_2^{q/p}}{\operatorname{meas}(\Omega) \left(a_1 k q c_2^{(q-1)/p} r^{(1-p)/p} + a_2 k^q r^{(q-p)/p} \right)} = +\infty,$$

provided $q \in (1, p)$. Hence, we obtain again $\lambda^\star = +\infty$ if $q \in (1, p)$.

Now, we have to show that for any $\lambda \in (0, \lambda^\star)$ the solution u_λ is not trivial. If $f(\cdot, 0) \neq 0$, we have $u_\lambda \not\equiv 0$ in X, since the trivial function does not solve problem (1.1).

Let us consider the case when $f(\cdot, 0) = 0$ and let us fix $\bar{\lambda} \in (0, \lambda^\star)$ and $\lambda \in (0, \bar{\lambda})$. Finally, let u_λ be as in (3.4) and (3.5). We will prove that $u_\lambda \not\equiv 0$ in X. To this end, let us show that

$$\limsup_{\|u\| \to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.7}$$

For this, first note that, by (a) and (c), we have

$$A(x,t\xi) \leq tA(x,\xi),$$

for all $x \in \Omega$, $t \in [0,1]$ and $\xi \in \mathbb{R}$. Thus, for all $t \in [0,1]$ and $u \in X$, we have

$$\begin{aligned}\Phi(tu) &= \int_\Omega A(x,\Delta(tu(x)))\,dx \\ &\leq t\int_\Omega A(x,\Delta(u(x)))\,dx \\ &= t\Phi(u).\end{aligned}$$

Due to (f_2), we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and a constant $\kappa > 0$ such that

$$\lim_{n\to\infty} \frac{\operatorname*{ess\,inf}_{x\in B} F(x,\xi_n)}{\xi_n} = +\infty,$$

and

$$\operatorname*{ess\,inf}_{x\in D} F(x,\xi_n) \geq \kappa\xi_n,$$

for n sufficiently large.

Now, fix a set $C \subset B$ of positive measure and a function $v \in X$ such that:

(i) $v(x) \in [0,1]$, for every $x \in \bar{\Omega}$;

(ii) $v(x) = 1$, for every $x \in C$;

(iii) $v(x) = 0$, for every $x \in \Omega \setminus D$.

Hence, fix $M > 0$ and consider a real positive number η with

$$M < \frac{\eta\,\mathrm{meas}(C) + \kappa\displaystyle\int_{D\setminus C} v(x)\,dx}{\Phi(v)}.$$

Then, there is $\nu \in \mathbb{N}$ such that $\xi_n < 1$ and

$$\operatorname*{ess\,inf}_{x\in B} F(x,\xi_n) \geq \eta\xi_n,$$

for every $n > \nu$.

Finally, let $w_n := \xi_n v$ for every $n \in \mathbb{N}$. It is easy to see that $w_n \in X$ for any $n \in \mathbb{N}$. Now, for every $n > \nu$, bearing in mind the properties of the function v ($0 \leq w_n(x) < \sigma$ for n sufficiently large), one has

$$\begin{aligned}\frac{\Psi(w_n)}{\Phi(w_n)} &= \frac{\displaystyle\int_C F(x,\xi_n)\,dx + \int_{D\setminus C} F(x,\xi_n v(x))\,dx}{\Phi(w_n)} \\ &\geq \frac{\eta\,\mathrm{meas}(C) + \kappa\displaystyle\int_{D\setminus C} v(x)\,dx}{\Phi(v)} > M.\end{aligned}$$

Since M could be arbitrarily large, it follows that

$$\lim_{n\to\infty} \frac{\Psi(w_n)}{\Phi(w_n)} = +\infty,$$

from which (3.7) clearly follows.

Hence, there exists a sequence $\{w_n\} \subset X$ strongly converging to zero, such that, for every n sufficiently large, $w_n \in \Phi^{-1}\big((-\infty, \bar{r}_{\bar{\lambda}})\big)$, and

$$I_\lambda(w_n) := \Phi(w_n) - \lambda\Psi(w_n) < 0. \tag{3.8}$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}\big((-\infty, \bar{r}_{\bar{\lambda}})\big)$ (see (3.4)), by (3.8) we conclude that

$$I_\lambda(u_\lambda) \leq I_\lambda(w_n) < 0 = I_\lambda(0), \tag{3.9}$$

so that $u_\lambda \not\equiv 0$ in X. Thus, u_λ is a nontrivial weak solution of problem (1.1).

Moreover, from (3.9) we easily see that the map

$$(0, \lambda^\star) \ni \lambda \mapsto I_\lambda(u_\lambda) \text{ is negative.} \tag{3.10}$$

Now, we claim that

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Indeed, let again $\bar{\lambda} \in (0, \lambda^\star)$ and $\lambda \in (0, \bar{\lambda})$. Bearing in mind (3.1) and the fact that $\Phi(u_\lambda) < \bar{r}_{\bar{\lambda}}$ for any $\lambda \in (0, \bar{\lambda})$ (see (3.5)), one has that

$$c_2\|u_\lambda\|^p \leq \Phi(u_\lambda) < \bar{r}_{\bar{\lambda}},$$

that is,

$$\|u_\lambda\|^p < \frac{\bar{r}_{\bar{\lambda}}}{c_2}.$$

Again, we consider two cases.

Case 1. If $p < N/2$, we have

$$\begin{aligned} \left|\int_\Omega f(x, u_\lambda(x))u_\lambda(x)\,dx\right| &\leq a_1\|u_\lambda\|_{L^1(\Omega)} + a_2\|u_\lambda\|^q_{L^q(\Omega)} \\ &\leq a_1 S_1\|u_\lambda\| + a_2 S_q^q\|u_\lambda\|^q \\ &< a_1 S_1\Big(\frac{\bar{r}_{\bar{\lambda}}}{c_2}\Big)^{1/p} + a_2 S_q^q\Big(\frac{\bar{r}_{\bar{\lambda}}}{c_2}\Big)^{q/p} =: M_{\bar{r}_{\bar{\lambda}}}, \end{aligned} \tag{3.11}$$

for every $\lambda \in (0, \bar{\lambda})$.

Case 2. If $p \geq N/2$, we have

$$\begin{aligned} &\left| \int_\Omega f(x, u_\lambda(x)) u_\lambda(x)\, dx \right| \\ &\leq \text{meas}(\Omega)\, (a_1 \|u_\lambda\|_\infty + a_2 \|u_\lambda\|_\infty^q) \\ &\leq \text{meas}(\Omega)\, (a_1 k \|u_\lambda\| + a_2 k^q \|u_\lambda\|^q) \\ &< \text{meas}(\Omega) \left(a_1 k \Big(\frac{\bar r_{\bar\lambda}}{c_2}\Big)^{1/p} + a_2 k^q \Big(\frac{\bar r_{\bar\lambda}}{c_2}\Big)^{q/p} \right) =: N_{\bar r_{\bar\lambda}}, \end{aligned} \tag{3.12}$$

for every $\lambda \in (0, \bar\lambda)$.

Since u_λ is a critical point of I_λ, then $I'_\lambda(u_\lambda)(v) = 0$, for any $v \in X$ and every $\lambda \in (0, \bar\lambda)$. In particular, $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_\Omega f(x, u_\lambda(x)) u_\lambda(x)\, dx, \tag{3.13}$$

for every $\lambda \in (0, \bar\lambda)$. On the other hand, since A is convex with $A(x, 0) = 0$ for all $x \in \Omega$, we have

$$a(x, \xi) \cdot \xi \geq A(x, \xi) \geq c_2 |\xi|^p, \tag{3.14}$$

for all $\xi \in \mathbb{R}$. Then, from (3.13) and (3.14), it follows that

$$0 \leq c_2 \|u_\lambda\|^p \leq \Phi'(u_\lambda)(u_\lambda) = \lambda \int_\Omega f(x, u_\lambda(x)) u_\lambda(x)\, dx,$$

for any $\lambda \in (0, \bar\lambda)$. Taking into account (3.11) or (3.12) and letting $\lambda \to 0^+$, we get $\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$, as claimed.

Finally, we show that the map

$$\lambda \mapsto I_\lambda(u_\lambda) \text{ is strictly decreasing in } (0, \lambda^\star).$$

Indeed, we observe that for any $u \in X$, one has

$$I_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right). \tag{3.15}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \bar\lambda < \lambda^\star$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi^{-1}\big((-\infty, \bar r_{\bar\lambda})\big)$ for $i = 1, 2$. Also, let

$$m_{\lambda_i} := \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}\left((-\infty, \bar r_{\bar\lambda})\right)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v) \right),$$

for every $i = 1, 2$.

Clearly, (3.10) together (3.15) and the positivity of λ imply that

$$m_{\lambda_i} < 0, \quad \text{for } i = 1, 2. \tag{3.16}$$

Moreover,

$$m_{\lambda_2} \leq m_{\lambda_1}, \tag{3.17}$$

thanks to $0 < \lambda_1 < \lambda_2$. Then, by (3.15)-(3.17) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \bar{\lambda})$. The arbitrariness of $\bar{\lambda} < \lambda^\star$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^\star)$. Thus, the proof is complete. ■

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