TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 4, pp. 1305-1316, August 2015
DOI: 10.11650/tjm.19.2015.5481
This paper is available online at http://journal.taiwanmathsoc.org.tw

AN INVERSE NODAL PROBLEM AND AMBARZUMYAN PROBLEM FOR THE PERIODIC p-LAPLACIAN OPERATOR WITH INTEGRABLE POTENTIALS

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Abstract. In this note, we solve the inverse nodal problem and Ambarzumyan problem for the p-Laplacian coupled with periodic or anti-periodic boundary conditions. We also extend some results in a previous paper to p-Laplacian with L^1 potentials, and for arbitrary linear separated boundary conditions. There we prove a generalized Riemann-Lebesgue Lemma which is of independent interest.

1. INTRODUCTION

An inverse nodal problem is a problem of understanding the potential function through the nodal points of eigenfunctions, without any other spectral information. An Ambarzumyan problem is the unique determination of potential q, when its associated spectrum $\sigma(q) = \sigma(0)$. Both problems have been well studied for the classical Sturm-Liouville operator (see [8, 9, 11, 14]). In a previous paper, we studied the p-Laplacian operator with C^1-potentials and solved the inverse nodal problem and Ambarzumyan problem for Dirichlet boundary conditions [10]. Now we want to extend the results to periodic/anti-periodic boundary conditions, and to L^1 potentials, which is the most general class of potentials.

Consider the equation

$$-\left(y'^{(p-1)}\right)' = (p-1)(\lambda - q(x))y^{(p-1)} , \tag{1.1}$$

where $f^{(p-1)} = |f|^{p-1}\mathrm{sgn} f$. Assume that $q(1+x) = q(x)$ for $x \in \mathbb{R}$, then (1.1) can be coupled with periodic or anti-periodic boundary conditions respectively:

$$y(0) = y(1) , \quad y'(0) = y'(1) \tag{1.2}$$

Received October 23, 2014, accepted January 26, 2015.
Communicated by Tai-Chia Lin.
2010 *Mathematics Subject Classification*: 34A55, 34B24, 47A75.
Key words and phrases: p-Laplacian, Inverse nodal problem, Ambarzumyan problem, Periodic eigenvalues.

or

$$y(0) = -y(1)\ , \quad y'(0) = -y'(1). \tag{1.3}$$

When $p = 2$, the above is the classical Hill's equation. It follows from Floquet theory that there are countably many interlacing pairs of periodic and anti-periodic eigenvalues of Hill's operator. However, Floquet theory does not apply when $p \neq 2$. Let σ_{2k} (resp. σ_{2k-1}) denote the set of periodic (resp. anti-periodic) eigenvalues of (1.1) which admit eigenfunctions with exactly $2k$ (resp. $2k-1$) zeros in $[0,1)$. In 2001, Zhang [15] used a rotation number function to show the existence of the minimal eigenvalue $\underline{\lambda}_n = \min \sigma_n$ and the maximal eigenvalue $\overline{\lambda}_n = \max \sigma_n$ respectively. Binding and Rynne studied in more detail in a series of papers [3, 4, 5] and showed that

(i) σ_{2k} and σ_{2k-1} are nonempty and compact. Also for all $\lambda \in \sigma_{2k}$,

$$\overline{\lambda}_{2k-1} < \underline{\lambda}_{2k} \leq \lambda \leq \overline{\lambda}_{2k} < \underline{\lambda}_{2k+1},$$

while $\sigma_0 = \{\lambda_0\}$ contains only one simple eigenvalue.

(ii) There exists a sequence of variational periodic eigenvalues $\{\gamma_n\}$ and variational anti-periodic eigenvalues $\{\delta_n\}$, such that $\gamma_0 = \lambda_0$ and for all $k \geq 1$,

$$\overline{\lambda}_{2k} = \gamma_{2k} \geq \underline{\lambda}_{2k} = \gamma_{2k-1} > \overline{\lambda}_{2k-1} = \delta_{2k} \geq \underline{\lambda}_{2k-1} = \delta_{2k-1}.$$

Furthermore, letting μ_n $(n \geq 1)$ and ν_n $(n \geq 0)$ be the Dirichlet and Neumann eigenvalues which admit eigenfunctions with exactly n zeros in $[0,1)$, we have

$$\begin{array}{rcl} \overline{\lambda}_{2k} \geq & \mu_{2k}, \nu_{2k} & \geq \underline{\lambda}_{2k} \\ \overline{\lambda}_{2k-1} \geq & \mu_{2k-1}, \nu_{2k-1} & \geq \underline{\lambda}_{2k-1}. \end{array}$$

The variational periodic eigenvalues $\{\gamma_n\}$ are defined by the Ljusternik-Schnirelmann construction. Define

$$W_P^{1,p}(0,1) = \{w \in W^{1,p}(0,1) : \ w(0) = w(1),\ w'(0) = w'(1)\}.$$

Let $M = \{u \in W_P^{1,p}(0,1) : \ \int_0^1 |u|^p = 1\}$, and

$$\mathcal{A} = \{A \subset M : \ A \text{ is non-empy, compact and symmetric } (A = -A)\}\,.$$

Hence we define the Krasnoselskij genus of $A \in \mathcal{A}$ by

$$\varphi(A) = \min\{m \in \mathbb{N} : \ \text{ there exists a continuous, odd } f : A \to \mathbb{R}^m \backslash \{0\}\}\,.$$

Thus for any integer $n \geq 0$, let $\mathcal{F}_n = \{A \in \mathcal{A} : \ \varphi(A) \geq n\}$. Then

$$\gamma_n := \min_{A \in \mathcal{F}_{n+1}} \max_{u \in A} \int_0^1 \left(\frac{|u'|^p}{p-1} + q|u|^p \right).$$

The set of variational anti-periodic eigenvalues $\{\delta_n\}$ is defined in a similar manner.

(iii) In general, non-variational eigenvalues may exist in σ_{2k} and σ_{2k-1} for all $k \geq 1$.

Some of the above properties are similar to the linear case, but others are not. This makes the study of p-Laplacian operators more interesting.

From now onward, by a periodic eigenvalue λ_{2k}, we mean an element of σ_{2k}, whether it is variational or non-variational or not. By an anti-periodic eigenvalue λ_{2k-1}, we mean an element of σ_{2k-1}, variational or non-variational.

In 2008, Brown and Eastham [6] derived a sharp asymptotic expansion of periodic eigenvalues of the p-Laplacian with locally integrable and absolutely continuous $(r-1)$ derivative potentials respectively. Below is a version of their theorem for periodic eigenvalues of the p-Laplacian (1.1), (1.2).

Theorem 1.1. ([6, Theorem 3.1]). *Let q be 1-periodic and locally integrable in $(-\infty,\infty)$. Then the periodic eigenvalue λ_{2k} satisfies*

$$\lambda_{2k}^{1/p} = 2k\widehat{\pi} + \frac{1}{p(2k\widehat{\pi})^{p-1}}\int_0^1 q(t)dt + o(\frac{1}{k^{p-1}}), \tag{1.4}$$

where $\widehat{\pi} = \frac{2\pi}{p\sin(\frac{\pi}{p})}$.

By a similar argument, the asymptotic expansion of the anti-periodic eigenvalue λ_{2n-1} satisfies

$$\lambda_{2k-1}^{1/p} = (2k-1)\widehat{\pi} + \frac{1}{p((2k-1)\widehat{\pi})^{p-1}}\int_0^1 q(t)dt + o(\frac{1}{k^{p-1}}). \tag{1.5}$$

We denote by $\{x_i^{(n)}\}_{i=0}^{n-1}$ the zeros of the eigenfunction corresponding to a periodic/ anti-periodic eigenvalue λ_n, and define the nodal length $\ell_i^{(n)} = x_{i+1}^{(n)} - x_i^{(n)}$ and $j = j_n(x) = \max\{i : \ x_i^{(n)} \leq x\}$. Our main theorem is as follows.

Theorem 1.2. *Let $q \in L^1(0,1)$ be 1-periodic. Define $F_n(x)$ as the following:*

(a) *For the periodic case, let*

$$F_{2k}(x) = p(2k\widehat{\pi})^p[(2k)\ell_j^{(2k)} - 1] + \int_0^1 q(t)dt,$$

(b) *For the anti-periodic case, let*

$$F_{2k-1}(x) = p((2k-1)\widehat{\pi})^p[(2k-1)\ell_j^{(2k-1)} - 1] + \int_0^1 q(t)dt.$$

Then both $\{F_{2k}\}$ and $\{F_{2k-1}\}$ converges to q pointwise a.e. and in $L^1(0,1)$.

Thus either one of the sequences $\{F_{2k}\}/\{F_{2k-1}\}$ will be sufficient to reconstruct q. Note that here $q \in L^1(0,1)$. Furthermore, the map between the nodal space and the set of admissible potentials are homeomorphic after a partition (cf. [10]). The same idea also works for linear separated boundary value problems with integrable potentials.

Using the eigenvalue asymptotics above, the Ambarzumyan problems for the periodic and anti-periodic boundary conditions can also be solved.

Theorem 1.3. *Let $q \in L^1(0,1)$ be periodic of period 1.*

(a) *If a sequence of periodic eigenvalues $\{\lambda_{2k}\}_{k=0}^{\infty}$ for (1.1) such that $\lambda_{2k} \in \sigma_{2k}$, is given by $\lambda_{2k} = (2k\widehat{\pi})^p$ for all $k \in \mathbb{N} \cup \{0\}$, then $q = 0$ on $[0,1]$.*

(b) *If a set of anti-periodic eigenvalue $\{\lambda_{2k-1}\}_{k=1}^{\infty}$ for (1.1) such that $\lambda_{2k-1} \in \sigma_{2k-1}$, is given by $\lambda_{2k-1} = ((2k-1)\widehat{\pi})^p$ for all $k \in \mathbb{N}$, with $\lambda_1 = \min \sigma_1$, and $\int_0^1 q(t) S_p(\widehat{\pi} t)^p dt = 0$, then $q = 0$ on $[0,1]$.*

Note that this sequence might not exploit all the periodic eigenvalues, as we know that the set σ_{2k} $(k \geq 1)$ contains at least two variational periodic eigenvalues ($\underline{\lambda}_{2k}$ and $\overline{\lambda}_{2k}$), as well as some non-variational periodic eigenvalues, as explained above. In fact, it has been shown that when $p \neq 2$, the set σ_{2k} can have arbitrarily many elements for C^1 potentials (cf. [3, Theorem 1.3]). The situation for anti-periodic eigenvalues is similar.

In Section 2, we shall apply Theorem 1.1 to study the problems involving periodic and anti-periodic boundary conditions. There Theorem 1.1 and Theorem 1.2 will be proved. In section 3, we shall deal with the case of linear separated boundary conditions.

Recently, we worked on a Tikhonov regularization approach of the inverse nodal problem for p-Laplacian [7]. The approach helps to obtain a more practical approximation of the potential function for Dirichlet p-Laplacian eigenvalue problem. The present work will be useful in making a similar approach for the periodic p-Laplacian eigenvalue problem.

2. Proof of Main Results

Fix $p > 1$ and assume that $q = 0$ and $\lambda = 1$. Then (1.1) becomes

$$-(y'^{(p-1)})' = (p-1) y^{(p-1)}.$$

Let S_p be the solution satisfying the initial conditions $S_p(0) = 0$, $S_p'(0) = 1$. It is well known that S_p and its derivative S_p' are periodic functions on $\mathbb{R}$ with period $2\widehat{\pi}$. The two functions also satisfy the following identities (cf. [6, 10]).

Lemma 2.1. *(a)* $|S_p(x)|^p + |S_p'(x)|^p = 1$ *for any* $x \in \mathbb{R}$*;*

(b) $(S_p S_p'^{(p-1)})' = |S_p'|^p - (p-1)|S_p|^p = 1 - p|S_p|^p = (1-p) + p|S_p'|^p$.

Next we define a generalized Prüfer substitution using S_p and S_p':

$$y(x) = r(x) S_p(\lambda^{1/p}\theta(x)), \quad y'(x) = \lambda^{1/p} r(x) S_p'(\lambda^{1/p}\theta(x)) \ . \tag{2.1}$$

By Lemma 2.1, one obtains ([10])

$$\theta'(x) = 1 - \frac{q(x)}{\lambda} |S_p(\lambda^{1/p}\theta(x))|^p \ . \tag{2.2}$$

Theorem 2.2. *In the periodic/anti-periodic eigenvalue problem, if* $q \in L^1(0,1)$ *is periodic of period* 1*, then*

$$q(x) = \lim_{n\to\infty} p\lambda_n \left(\frac{\lambda_n^{1/p}\ell_j^{(n)}}{\widehat{\pi}} - 1 \right) ,$$

pointwise a.e. and in $L^1(0,1)$*, where* $j = j_n(x) = \max\{k : x_k^{(n)} \le x\}$.

The proof below works for both even and odd n's, i.e. for both periodic and anti-periodic problems. Some of the arguments above are motivated by [9]. See also [11].

Proof. First, integrating (2.2) from $x_k^{(n)}$ to $x_{k+1}^{(n)}$ with $\lambda = \lambda_n$, we have

$$\begin{aligned} \frac{\widehat{\pi}}{\lambda_n^{1/p}} &= \ell_k^{(n)} - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \frac{q(t)}{\lambda_n} |S_p(\lambda_n^{1/p}\theta(t))|^p dt , \\ &= \ell_k^{(n)} - \frac{1}{p\lambda_n}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)dt - \frac{1}{\lambda_n}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p})dt . \end{aligned}$$

Hence,

$$\ell_k^{(n)} = \frac{\widehat{\pi}}{\lambda_n^{1/p}} + \frac{1}{p\lambda_n}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)dt + \frac{1}{\lambda_n}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p})dt . \tag{2.3}$$

and

$$\begin{aligned} & p\lambda_n \left(\frac{\lambda_n^{1/p}\ell_k^{(n)}}{\widehat{\pi}} - 1 \right) \\ &= \frac{\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)dt + \frac{p\lambda_n^{1/p}}{\widehat{\pi}}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p})dt \end{aligned} . \tag{2.4}$$

Now, for $x \in (0,1)$, let $j = j_n(x) = \max\{k : x_k^{(n)} \le x\}$. Then $x \in I_j^{(n)} := [x_j^{(n)}, x_{j+1}^{(n)})$ and, for large n,

$$I_j^{(n)} \subset B(x, \frac{2\widehat{\pi}}{\lambda_n^{1/p}}) ,$$

where $B(t,\varepsilon)$ is the open ball with centre t and radius ε. That is, the sequence of intervals $\{I_j^{(n)} : \ n \text{ is sufficiently large}\}$ shrinks to x nicely (cf. Rudin [13, p.140]). Since $q \in L^1(0,1)$ and $\frac{\lambda_n^{1/p}\ell_k^{(n)}}{\widehat{\pi}} = 1 + o(1)$, we define the sequence of functions

$$h_n := \frac{\lambda_n^{1/p}}{\widehat{\pi}} \sum_{k=0}^{n-1} \left(\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q \right) \chi_{I_k^{(n)}},$$

which is convergent to q pointwise a.e. $x \in (0,1)$. Furthermore,

$$|h_n| \le g_n := \frac{\lambda_n^{1/p}}{\widehat{\pi}} \sum_{k=0}^{n-1} \left(\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} |q| \right) \chi_{I_k^{(n)}},$$

and as n tends to infinity,

$$\int_0^1 g_n(t)dt = \sum_{k=0}^{n-1} \frac{\lambda_n^{1/p} \ell_k^{(n)}}{\widehat{\pi}} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} |q(t)|dt \to \|q\|_1 \ .$$

Thus when n is large, $|h_n - q| \le (2g_n + |q|)$ and the integral of the latter converges to $3\|q\|_1$. By the general Lebesgue dominated convergence theorem [12, p.89], h_n converges to q in $L^1(0,1)$.

On the other hand, let $q_{k,n} := \frac{1}{\ell_k^{(n)}} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)dt$. Then $\sum_{k=0}^{n-1} q_{k,n}\chi_{I_k^{(n)}}$ converges to q pointwise a.e. Let $\phi_n(t) = |S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p}$. Then for a.e. $x \in (0,1)$,

$$\begin{aligned} T_n(x) &:= \frac{p\lambda_n^{1/p}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t)\phi_n(t)dt \ , \\ &= \frac{p\lambda_n^{1/p}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (q(t) - q_{j,n})\phi_n(t)dt + \frac{p\lambda_n^{1/p}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_{j,n}\phi_n(t)dt \ , \\ &:= A_n(x) + B_n(x) \ . \end{aligned}$$

By Lemma 2.1(b) and (2.2),

$$\begin{aligned} B_n(x) &= \frac{p\lambda_n^{1/p} q_{j,n}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left(|S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p} \right) \left(\theta'(t) + \frac{q(t)}{\lambda_n} |S_p(\lambda_n^{1/p}\theta(t))|^p \right) dt, \\ &= -\frac{pq_{j,n}}{\widehat{\pi}} S_p(\lambda_n^{1/p}\theta(t)) S_p'(\lambda_n^{1/p}\theta(t))^{(p-1)} \Big|_{x_j^{(n)}}^{x_{j+1}^{(n)}} + O(\lambda_n^{-1+1/p}) \ , \\ &= O(\lambda_n^{-1+1/p}) \ . \end{aligned}$$

Also,

$$\begin{aligned} |A_n(x)| &\le \frac{p\lambda_n^{1/p}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} |q(t) - q_{j,n}| \, |S_p(\lambda_n^{1/p}\theta(t))|^p - \frac{1}{p}| dt \ , \\ &\le \frac{(p-1)\lambda_n^{1/p}}{\widehat{\pi}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} |q(t) - q_{j,n}| \, dt \ , \end{aligned}$$

which converges to 0 pointwise a.e. because the sequence of intervals $\{I_j^{(n)} : n$ is sufficiently large$\}$ shrinks to x nicely. We conclude that $T_n \to 0$ a.e. $x \in (0,1)$. Finally, applying the general Lebesgue dominated convergence theorem as above, $T_n \to 0$ in $L^1(0,1)$. Therefore, the left hand side of (2.4) converges to q pointwise a.e. and in $L^1(0,1)$. ∎

Proof of Theorem 1.2. By the eigenvalue estimates (1.4) and (1.5), we have

$$p\lambda_{2k}\Big(\frac{\lambda_{2k}^{1/p}\ell_{j_{2k}(x)}^{(2k)}}{\widehat{\pi}} - 1\Big) = p(2k\widehat{\pi})^p(2k\ell_j^{(2k)} - 1) + 2k\ell_{j_{2k}(x)}^{(2k)}\int_0^1 q(t)dt + o(1)\ . \tag{2.5}$$

Hence by Theorem 2.2 and the fact that $2k\ell_j^{(2k)} = 1 + o(1)$,

$$F_{2k}(x) \equiv p(2k\widehat{\pi})^p(2k\ell_j^{(2k)} - 1) + \int_0^1 q(t)dt$$

also converges to q pointwise a.e. and in $L^1(0,1)$. The proof for (b) is the same. ∎

Proof of Theorem 1.3. By (1.4), we have $\displaystyle\int_0^1 q(t)dt = 0$. Also as the least periodic eigenvalue $\lambda_0 = 0$ is variational, we take the constant function 1 as a test function. Then

$$0 = \lambda_0 \le \int_0^1 q = 0.$$

Therefore 1 is the first periodic eigenfunction, and $q = 0$. This proves (a).

For part (b), since $\lambda_{2k-1} = ((2k-1)\widehat{\pi})^p$ for $k \in \mathbb{N}$, we have, by (1.5), $\displaystyle\int_0^1 q(t)dt = 0$. Moreover, $v(x) = p^{1/p}S_p(\widehat{\pi}x)$ satisfies anti-periodic boundary conditions and $\|v\|_{L^p} = 1$. Note that by Lemma 2.1(b),

$$\int_0^1 |S_p'(\widehat{\pi}t)|^p dt - \frac{p-1}{p} = \int_0^1 |S_p(\widehat{\pi}t)|^p dt - \frac{1}{p} = 0\ .$$

Now $\lambda_1 = \widehat{\pi}^p$ is the first minimal anti-periodic eigenvalue, so it is a variational one. We let v be a test function, and obtain by variational principle and the hypothesis, that

$$\widehat{\pi}^p \le \int_0^1 \frac{p\widehat{\pi}^p}{p-1}|S_p'(\widehat{\pi}t)|^p dt + p\int_0^1 q(t)S_p(\widehat{\pi}t)^p dt = \widehat{\pi}^p\ .$$

This implies v is the first eigenfunction. Thus $q = 0$ a.e. in $(0,1)$. ∎

3. Linear Separated Boundary Conditions

Consider the one-dimensional p-Laplacian with linear separated boundary conditions

$$\begin{cases} y(0)S_p'(\alpha) + y'(0)S_p(\alpha) = 0 \\ y(1)S_p'(\beta) + y'(1)S_p(\beta) = 0 \end{cases}, \tag{3.1}$$

where $\alpha, \beta \in [0, \widehat{\pi})$. Letting μ_n be the nth eigenvalue whose associated eigenfunction has exactly $n-1$ zeros in $(0,1)$, the generalized phase θ_n as given in (2.2) satisfies

$$\begin{aligned} \theta_n(0) &= \frac{-1}{\mu_n^{1/p}} \widetilde{CT}_p^{-1}\Big(-\frac{\widetilde{CT}_p(\alpha)}{\mu_n^{1/p}}\Big); \\ \theta_n(1) &= \frac{1}{\mu_n^{1/p}} \left(n\widehat{\pi} - \widetilde{CT}_p^{-1}\Big(-\frac{\widetilde{CT}_p(\beta)}{\mu_n^{1/p}}\Big) \right), \end{aligned} \tag{3.2}$$

where the function $CT_p(\gamma) := \frac{S_p(\gamma)}{S_p'(\gamma)}$ is an analogue of cotangent function, while $\widetilde{CT}_p(\gamma) := CT_p(\gamma)$ if $\gamma \neq 0$; and $\widetilde{CT}_p(0) := 0$. Also $\widetilde{CT}_p^{-1}$ stands for the inverse of $\widetilde{CT}_p$, taking values only in $[0, \widehat{\pi})$.

Let $\phi_n(x) = |S_p(\mu_n^{1/p}\theta_n(x)|^p - \frac{1}{p}$. Below we shall state a general Riemann-Lebesgue lemma, which shows that $\int_0^1 g\phi_n \to 0$ for any $g \in L^1(0,1)$, when μ_n's are associated with certain linear separated boundary conditions. In the case of periodic boundary conditions, Brown and Eastham [6] used a Fourier series expansion of ϕ_n where $\phi_n(\mu_n^{1/p}\theta_n(x)) \approx \phi_n(\alpha + 2n\widehat{\pi}x)$ and apply Plancherel Theorem to show convergence.

Lemma 3.1. *Let f_n be uniformly bounded and integrable on $(0,1)$. Suppose that*

(i) *there exists a partition $\{x_0^n = 0 < x_1^n < \cdots < x_n^n = 1\}$ such that $\Delta x_k^n := x_{k+1}^n - x_k^n = o(1)$ as $n \to \infty$;*

(ii) *$F_k^n(x) := \int_{x_k^n}^x f_n(t)\,dt$ satisfies $F_k^n(x) = O(\frac{1}{n})$ for $x \in (x_k^n, x_{k+1}^n)$ and $F_k^n(x_{k+1}^n) = o(\frac{1}{n})$ for all $0 \le k \le n-1$, as $n \to \infty$.*

Then for any $g \in L^1(0,1)$, $\int_0^1 gf_n \to 0$ as $n \to \infty$.

Proof. Let $|f_n| \le M$. We divide the proof into two parts. First, suppose that $g \in C^1[0,1]$. We can find a constant $M_1 > 0$ such that $|g|, |g'| \le M_1$. Given any $\epsilon > 0$, then for sufficiently large n, we have $\triangle x_k^n \le \epsilon$, and $|F_k^n(x_{k+1}^n)| \le \frac{\epsilon}{2nM_1}$, $|F_k^n(x)| \le \frac{1}{2M_1 n}$ for $x \in (x_k^n, x_{k+1}^n)$ for all $0 \le k \le n-1$. Using integration by parts,

$$\begin{aligned} \left|\int_0^1 gf_n\right| &= \sum_{k=0}^{n-1}\left|\int_{x_k^n}^{x_{k+1}^n} gf_n\right| = \sum_{k=0}^{n-1}\left|\left(g(x_{k+1}^n)F_k^n(x_{k+1}^n) - \int_{x_k^n}^{x_{k+1}^n} g'F_k^n\right)\right| \\ &\le \epsilon. \end{aligned}$$

Take any $g \in L^1(0,1)$. Then there is a C^1 function $\tilde{g}$ on $[0,1]$ such that $\int_0^1 |\tilde{g} - g| < \epsilon$. Hence

$$\int_0^1 g f_n = \int_0^1 (g - \tilde{g}) f_n + \int_0^1 \tilde{g} f_n.$$

Here $|\int_0^1 (g - \tilde{g}) f_n| \leq M\epsilon$, and by above, the term $\int_0^1 \tilde{g} f_n$ can be arbitrarily small when n is large enough. Hence the theorem is valid. ■

Corollary 3.2. *Consider the p-Laplacian (1.1) with boundary conditions (3.1). Define $\phi_n(x) = |S_p(\mu_n^{1/p}\theta_n(x))|^p - \frac{1}{p}$, then for any $g \in L^1(0,1)$, $\int_0^1 g\phi_n \to 0$.*

Proof. Since $\theta_n(0)$ and $\theta_n(1)$ are as given in (3.2), ϕ_n is uniformly bounded on $[0,1]$. Take x_k^n be such that $\theta(x_k^n) = \frac{k\widehat{\pi}}{\mu_n^{1/p}}$. Also by integrating the phase equation (2.2), $\mu_n^{1/p} = O(n)$, and

$$\Delta x_n = O(\frac{1}{\mu_n^{1/p}}) = O(\frac{1}{n}).$$

Hence by Lemma 2.1(b) and (3.1), we have for $k = 1, \ldots, n-2$,

$$\begin{aligned}\int_{x_k^n}^{x_{k+1}^n} \phi_n(x)\,dx &= \frac{-1}{p\mu_n^{1/p}} \int_{x_k^n}^{x_{k+1}^n} \frac{1}{\theta_n'(x)} \frac{d}{dx}\left[S_p(\mu_n^{1/p}\theta_n(x)) S_p'(\mu_n^{1/p}\theta_n(x))^{(p-1)}\right] dx \\ &= \frac{-1}{p\mu_n^{1/p}} \left[S_p(\mu_n^{1/p}\theta_n(x)) S_p'(\mu_n^{1/p}\theta_n(x))^{(p-1)}\right]_{x_k^n}^{x_{k+1}^n} + O(\frac{1}{\mu_n}) \\ &= O(\frac{1}{\mu_n}) = o(\frac{1}{n}) \ ,\end{aligned}$$

since $S_p(k\widehat{\pi}) = 0$. It is also clear that $\int_{x_k^n}^{x} \phi_n(x)\,dx = O(\frac{1}{n})$. Thus we may apply Lemma 3.1 to complete the proof. ■

Theorem 3.3. *When $q \in L^1(0,1)$, the eigenvalues μ_n of the Dirichlet p-Laplacian (1.1) satisfies, as $n \to \infty$,*

$$\mu_n^{1/p} = n\widehat{\pi} + \frac{1}{p(n\widehat{\pi})^{p-1}} \int_0^1 q(t)dt + o(\frac{1}{n^{p-1}}) \ . \tag{3.3}$$

Furthermore, F_n converges to q pointwise and in $L^1(0,1)$, where

$$F_n(x) := p(n\widehat{\pi})^p (n\ell_j^{(n)} - 1) + \int_0^1 q(t)\,dt.$$

Proof. Integrating (2.2) from 0 to 1, we have

$$\begin{aligned}\mu_n^{1/p} &= n\widehat{\pi} + \frac{1}{p\mu_n^{1-1/p}}\int_0^1 q(t)|S_p(\mu_n^{1/p}\theta(t))|^p dt \ , \\ &= n\widehat{\pi} + \frac{1}{p\mu_n^{1-1/p}}\int_0^1 q(t)dt + \frac{1}{p\mu_n^{1-1/p}}\int_0^1 q(t)(|S_p(\mu_n^{1/p}\theta(t))|^p - \frac{1}{p})dt \ .\end{aligned}$$

Then by Corollary 3.2, we have

$$\int_0^1 q(t)(|S_p(\mu_n^{1/p}\theta(t))|^p - \frac{1}{p})dt = o(1) \ ,$$

for any $q \in L^1(0,1)$. Hence (3.3) holds. Furthermore, by Theorem 2.2, we can obtain the reconstruction formula with pointwise and L^1 convergence. ■

Remark. In the same way, the Ambarzumyan Theorems for Neumann as well as Dirichlet boundary conditions as given in [10, Theorems 1.3 and 5.1] can also be proved for L^1 potentials. Furthermore, the above method can also be used to show Theorem 1.1 by reducing the periodic problem to a Dirichlet problem by a translation of the first nodal length, as in [8].

In fact, for general linear separated boundary problems (3.1),

$$\begin{aligned}\mu_n^{1/p} = n_{\alpha\beta}\widehat{\pi} + \frac{(\widetilde{CT}_p(\beta))^{(p-1)} - (\widetilde{CT}_p(\alpha))^{(p-1)}}{(n_{\alpha\beta}\widehat{\pi})^{p-1}} \\ + \frac{1}{p(n_{\alpha\beta}\widehat{\pi})^{p-1}}\int_0^1 q(x)\,dx + o(\frac{1}{n^{p-1}}),\end{aligned} \tag{3.4}$$

where

$$n_{\alpha\beta} = \begin{cases} n & \text{if } \alpha = \beta = 0 \\ n - 1/2 & \text{if } \alpha > 0 = \beta \text{ or } \beta > 0 = \alpha \\ n - 1 & \alpha, \beta > 0 \end{cases} .$$

This is because, after an integration of (2.2),

$$\theta_n(1) - \theta_n(0) = 1 - \frac{1}{\mu_n}\int_0^1 q(x)|S_p(\mu_n^{1/p}\theta(x))|^p\,dx + o(\frac{1}{\mu_n}). \tag{3.5}$$

By (3.2), if $\alpha = 0$, then $\theta_n(0) = 0$. Similarly $\theta_n(1) = 0$ if $\beta = 0$. Now, let $y = CT_p^{-1}(x)$. Then $x = CT_p(y)$ and hence

$$y' = \frac{-|x|^{p-2}}{1+|x|^p} = -|x|^{p-2}(1 + O(|x|^p)),$$

when $|x|$ is sufficiently small. Since $y(0) = \frac{\widehat{\pi}}{2}$, we have

$$y(x) = \frac{\widehat{\pi}}{2} - \frac{x^{(p-1)}}{p-1} + O(x^{2p-1}) \ .$$

Therefore, when n is sufficiently large,

$$\theta_n(0) = \frac{\widehat{\pi}}{2\mu_n^{1/p}} + \frac{(CT_p(\alpha))^{(p-1)}}{(p-1)\mu_n^{(p-1)/p}} + O(\mu_n^{\frac{1-2p}{p}}).$$

Similarly, when $\beta \neq 0$,

$$\theta_n(1) = \frac{(n-\frac{1}{2})\widehat{\pi}}{\mu_n^{1/p}} + \frac{(CT_p(\beta))^{(p-1)}}{(p-1)\mu_n^{(p-1)/p}} + O(\mu_n^{\frac{1-2p}{p}}).$$

Hence (3.4) is valid. Furthermore, F_n converges to q pointwise and in $L^1(0,1)$, where

$$F_n(x) := p(n_{\alpha\beta}\widehat{\pi})^p \left[(n_{\alpha\beta} + \frac{(\widetilde{CT}_p(\beta))^{(p-1)} - (\widetilde{CT}_p(\alpha))^{(p-1)}}{(n_{\alpha\beta}\widehat{\pi})^{p-1}})\ell_j^{(n)} - 1 \right] + \int_0^1 q(t)\, dt.$$

Acknowledgments

The authors are supported in part by Ministry of Science and Technology (formerly National Science Council) under contract numbers NSC 98-2115-M-110-006, NSC 97-2115-M-005-MY2 and NSC 97-2115-M-022-001.

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