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EXISTENCE OF PERIODIC SOLUTIONS OF SEASONALLY FORCED SIR MODELS WITH IMPULSE VACCINATION

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Abstract. In this paper, we study periodic oscillation of seasonally forced epidemiological models with impulse vaccination where periodicity occurs in contact rate. Using the famous Mawhin's coincidence degree method, we get the existence of positive periodic solutions of seasonally forced SIR models with impulse vaccination at fixed time. Some numerical simulations are presented to illustrate the effectiveness of such pulse vaccination strategy.

1. Introduction

In the past decades, tens of millions of human being suffered or died from various infectious diseases. Many infectious diseases, such as measles, chickenpox, mumps, rubella, pertussis and influenza, show seasonal patterns of incidence [2, 10, 12]. The cause of seasonal patterns may vary from the periodic contact rates [11, 12], periodic fluctuation in birth and death rates [23, 24, 25], and periodic vaccination program [8]. Thus, it is natural to model these diseases by following seasonally forced epidemiological models:

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \beta(t)SI - \mu S, \\ \dfrac{dI(t)}{dt} = \beta(t)SI - (\mu+\gamma)I, \\ \dfrac{dR(t)}{dt} = \gamma I - \mu R, \end{cases}$$

in which

- S, I, R are the fractions of the susceptible, infective and recovered population,
- μ and γ denote the birth (death) rate and recovery rate respectively, which are positive constant,

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- $\beta(t)$ is the seasonally-dependent transmission rate, which is a positive continuous T-periodic function.

Recently, G. Katriel [17] got the existence of periodic positive solutions for the periodically forced SIR model by Leray-Schauder degree theory provided $\frac{1}{T}\int_0^T \beta(t)dt > \gamma + \mu$. Jódar, Villanueva and Arenas [16] obtained that a T-periodic solution exists for a more general system by Gaines-Mawhin's continuation theorem, whenever the condition $\min_{t\in\mathbb{R}} \beta(t) > \gamma + \mu$ holds. Coincidence degree theory is a very powerful technique especially in proof of existence of solutions in nonlinear equations. It has many applications in the existence of periodic solutions for periodically forced SIR model with saturated incidence rates [3, 4, 22].

Pulse vaccination has been testified to be an effective strategy in preventing viral infections. The pulse vaccination scheme proposes to vaccinate a fraction p of the entire susceptible population in a single pulse, which can be formulated as

$$S(t_i^+) := \lim_{h\to 0^+} S(t_i + h) = S(t_i) - pS(t_i),$$

where $S(t)$ is left continuous satisfying $S(t_i) = S(t_i^-) := \lim_{h\to 0^-} S(t_i + h)$. This part of susceptible population has been converted into recovered population

$$R(t_i^+) = R(t_i) + pS(t_i).$$

The theoretical study on pulse vaccination strategy was firstly presented by Agur et al. [1], then studied by many authors [6, 15, 19, 26, 27, 28, 29]. A comprehensive introduction on vaccination strategies can be found in [18].

There are some research activities about the existence of periodic solutions of impulsive differential equation [5, 7, 13, 14, 20]. In this paper, we study the existence of periodic solutions of SIR model with both periodic transmission rate and pulse vaccination:

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \beta(t)SI - \mu S, \\ \dfrac{dI(t)}{dt} = \beta(t)SI - (\mu+\gamma)I, \\ \dfrac{dR(t)}{dt} = \gamma I - \mu R, \\ \triangle S\big|_{t=nT+t_i} = -J_i(S(t), I(t))\big|_{t=nT+t_i}, \\ \triangle I\big|_{t=nT+t_i} = 0, \\ \triangle R\big|_{t=nT+t_i} = J_i(S(t), I(t))\big|_{t=nT+t_i}, \end{cases} \tag{1.1}$$

where $\triangle S\big|_{t=nT+t_i} = S(nT + t_i^+) - S(nT + t_i)$, $0 \le t_1 < \cdots < t_k < T$, $n \in \mathbb{N}$. The susceptible population will be vaccinated for four times largely because the susceptible population can be divided into many groups and all groups can not be vaccinated at the same time. Our vaccination strategies concern the impact of infected population, which can be formulated as

$$J_i(S(t), I(t))\big|_{t=nT+t_i} = p_i(1 - e^{-\alpha I(nT+t_i)})S(nT + t_i), \ i = 1, \dots, k,$$

where $0 \le p_i < 1$, $\alpha > 0$ large enough.

Denote the basic reproduction number

$$\mathcal{R}_0 = \frac{\bar{\beta}}{\gamma + \mu},$$

with $\bar{\beta} = \frac{1}{T}\int_0^T \beta(t)dt$. The following theorem gives the main results of this paper.

Theorem 1. *Let $\mathcal{R}_0 > 1$, there exists at least one T-periodic solution $(S(t), I(t), R(t))$ of* (1.1)*, all of whose components are positive.*

Remark. When $p_i \equiv 0$, $i = 1, \dots, k$, system (1.1) is the usual seasonally forced SIR model without pulse vaccination. This problem has been considered in [17] by Leary-Schauder degree theory. But the proof of Theorem 1 gives another version's proof by Mawhin's coincidence degree method.

The rest of this paper is organized as follows. In Section 2, we give the outline of Gaines and Mawhin's continuation theorem and some notations. In Section 3, we choose a suitable region to our problem. In Section 4, we establish the results on existence of periodic solutions of our impulsive systems. In Section 5, we present some numerical experiments to illustrate the effectiveness of our pulse vaccination strategy.

2. Preliminaries

It is not difficult to observe that $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} \equiv 0$ in system (1.1). Since $S(t)$, $I(t)$, $R(t)$ are fractions of the population, we have $S(t) + I(t) + R(t) = 1$ for all t. Because R does not appear in the first two equations in (1.1), it is sufficient to consider the existence of periodic solutions of following systems

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \beta(t)SI - \mu S, \\ \dfrac{dI(t)}{dt} = \beta(t)SI - (\mu + \gamma)I, \\ \triangle S\big|_{t=nT+t_i} = -J_i(S(t), I(t))\big|_{t=nT+t_i}, \\ \triangle I\big|_{t=nT+t_i} = 0, \end{cases} \tag{2.1}$$

with

$$S(t) \geq 0, I(t) \geq 0, S(t) + I(t) \leq 1.$$

Obviously, this problem is equivalent to find the solutions of the following periodic boundary value problem:

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \beta(t)SI - \mu S, \\ \dfrac{dI(t)}{dt} = \beta(t)SI - (\mu+\gamma)I, \\ \triangle S\big|_{t=t_i} = -J_i(S(t), I(t))\big|_{t=t_i}, \\ \triangle I\big|_{t=t_i} = 0, \\ S(0) = S(T), \\ I(0) = I(T). \end{cases} \tag{2.2}$$

2.1. Outline of Gaines and Mawhin's Continuation Theorem

Let $L : \mathrm{dom}L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Index}L = \mathrm{dimKer}L - \mathrm{codim\ Im}L = 0$ and $\mathrm{Im}L$ is closed in Z. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$ and $X = \mathrm{Ker}L \oplus \mathrm{Ker}P$, $Z = \mathrm{Im}L \oplus \mathrm{Im}Q$. It follows that $L\big|_{\mathrm{dom}L\cap\mathrm{Ker}P} : (I - P)X \to \mathrm{Im}L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subset of X, the mapping N is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(1-Q)N : \overline{\Omega} \to X$ is compact. Since $\mathrm{Im}\ Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $\Lambda : \mathrm{Im}Q \to \mathrm{Ker}L$. The following theorem is very useful to our probrem.

Theorem 2. [9] *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on Ω. Assume that:*

(1) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \mathrm{dom}\, L$, $Lx \neq \lambda Nx$.*

(2) *for each $x \in \partial\Omega \cap \mathrm{Ker}\, L$, $QNx \neq 0$,*

(3) $\deg(\Lambda QN, \Omega \cap \mathrm{Ker}\, L, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

2.2. Notations

For non-negative integer j, let

$$C^j[0,T; t_1, \ldots, t_k] = \{x : [0,T] \to \mathbb{R} \mid x^{(m)}(t) \text{ exists for } t \neq t_1, \ldots, t_k;\ x^{(m)}(t_i^+) \text{ and } x^{(m)}(t_i) := x^{(m)}(t_i^-) \text{ exist, } i = 1, \ldots, k,\ m = 0, \ldots, j\}.$$

Define a Banach space

$$X = \{(x_1, x_2) \mid x_1, x_2 \in C[0, T; t_1, \ldots, t_k], x_1(0) = x_1(T), x_2(0) = x_2(T)\},$$

with the norm$\|(x_1, x_2)\|_X = \max_{t\in[0,T]}(|x_1(t)| + |x_2(t)|)$. Define another Banach space

$$Z = C[0, T; t_1, \ldots, t_k] \times C[0, T; t_1, \ldots, t_k] \times \mathbb{R}^k,$$

with the norm$\|(z_1, z_2, C_1, \ldots, C_k)\|_Z = \max_{t\in[0,T]}(|z_1(t)|+|z_2(t)|)+|C_1|+\cdots+|C_k|$. Let

$$\begin{aligned} L : \operatorname{dom} L &\to Z, \\ (S, I) &\to (S', I', \triangle S(t_1), \ldots, \triangle S(t_k)), \end{aligned}$$

where

$$\operatorname{dom} L = \{(S, I) \mid S, I \in C^1[0, T; t_1, \ldots, t_k], S(0) = S(T), I(0) = I(T)\}.$$

Let

$$\begin{aligned} &N : X \to Z, \\ &(S, I) \to (f_1(\cdot, S, I), f_2(\cdot, S, I), -J_1(S(t_1), I(t_1)), \ldots, -J_k(S(t_k), I(t_k))), \end{aligned}$$

where

$$f_1(t, S, I) = \mu - \beta(t)SI - \mu S,\ \ f_2(t, S, I) = \beta(t)SI - (\mu + \gamma)I.$$

Obviously, system (2.2) can be written by $L(S, I) = N(S, I)$. By a simple calculation, we get

$$\operatorname{Ker} L = \{(x_1, x_2) \mid (x_1, x_2) \equiv C \in \mathbb{R}^2\},$$

and

$$\operatorname{Im} L = \left\{ (z_1, z_2, C_1, \ldots, C_k) \in Z : \frac{1}{T}\int_0^T z_1(\tau)d\tau + \frac{1}{T}\sum_{i=1}^k C_i = 0, \frac{1}{T}\int_0^T z_2(\tau)d\tau = 0 \right\}.$$

It is easy to see that

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2.$$

Since ImL is closed, L is Fredholm mapping of index 0. Let $P : X \to X$ be the projector given by

$$P(x_1, x_2) = \left(\frac{1}{T}\int_0^T x_1(\tau)d\tau, \frac{1}{T}\int_0^T x_2(\tau)d\tau\right), (x_1, x_2) \in X.$$

Obviously,

$$\mathrm{Im}P = \mathrm{Ker}\, L = \mathbb{R}^2.$$

Let $Q : Z \to Z$ be the projector given by

$$Q(z_1, z_2, C_1, \ldots, C_k) = \left(\frac{1}{T} \int_0^T z_1(\tau) d\tau + \frac{1}{T} \sum_{i=1}^{k} C_i, \frac{1}{T} \int_0^T z_2(\tau) d\tau, 0_{1\times k} \right).$$

Obviously,

$$\mathrm{Ker}\, Q = \mathrm{Im}L = \mathrm{Im}(I - Q).$$

Furthermore, the generalized inverse (to L) $K_p : \mathrm{Im}L \to \mathrm{Ker}\, P \cap \mathrm{dom}\, L$ exists given by

$$\begin{aligned}
&K_p(z_1, z_2, C_1, \ldots, C_k) \\
= &\left(\int_0^t z_1(\tau) d\tau + \sum_{t>t_i} C_i - \frac{1}{T} \int_0^T \int_0^t z_1(\tau) d\tau dt - \sum_{i=1}^{k} C_i, \right. \\
&\left. \int_0^t z_2(\tau) d\tau - \frac{1}{T} \int_0^T \int_0^t z_2(\tau) d\tau dt \right).
\end{aligned}$$

Then $QN : X \to Z$ read

$$QN(S, I) = \left(\frac{1}{T} \int_0^T f_1(\tau, S, I) d\tau - \frac{1}{T} \sum_{i=1}^{k} J_i(S(t_i), I(t_i)), \frac{1}{T} \int_0^T f_2(\tau, S, I) d\tau, 0_{1\times k} \right).$$

By a direct calculation, we have

$$\begin{aligned}
&K_p(I - Q)N(S, I) \\
= \;& K_p\bigg[\bigg(f_1(\cdot, S, I) - \frac{1}{T} \int_0^T f_1(\tau, S, I) d\tau + \frac{1}{T} \sum_{i=1}^{k} J_i(S(t_i), I(t_i)), \\
& f_2(\cdot, S, I) - \frac{1}{T} \int_0^T f_2(\tau, S, I) d\tau, -J_1(S(t_1), I(t_1)), \ldots, -J_k(S(t_k), I(t_k)) \bigg) \bigg] \\
= \;& \left(\int_0^t f_1 d\tau - \frac{1}{T} \int_0^T \int_0^t f_1 d\tau dt + \left(\frac{1}{2} - \frac{t}{T} \right) \left(\int_0^T f_1 d\tau + \sum_{i=1}^{k} J_i(S(t_i), I(t_i)) \right) \right. \\
& - \sum_{t>t_i} J_i(S(t_i), I(t_i)) + \sum_{i=1}^{k} J_i(S(t_i), I(t_i)), \int_0^t f_2 d\tau - \frac{1}{T} \int_0^T \int_0^t f_2 d\tau dt \\
& \left. + \left(\frac{1}{2} - \frac{t}{T} \right) \int_0^T f_2 d\tau \right).
\end{aligned}$$

3. The Suitable Region to Our Problem

In order to prove the existence of periodic solutions of (2.1), we consider the following auxiliary problem

$$\begin{cases} \dfrac{dS(t)}{dt} = \lambda(\mu - \beta(t)SI - \mu S), \\ \dfrac{dI(t)}{dt} = \lambda(\beta(t)SI - (\mu+\gamma)I), \\ \triangle S\big|_{t=t_i} = -\lambda J_i(S(t_i), I(t_i)), \\ \triangle I\big|_{t=t_i} = 0, \\ S(0) = S(T), \\ I(0) = I(T), \end{cases} \tag{3.1}$$

where $\lambda \in [0, 1]$ and $t \in [0, T]$. Let D be an open bounded subset of X satisfying

$$D = \{(S, I) \in X \mid S(t) > 0, I(t) > 0, S(t) + I(t) < 1\}.$$

Proposition 3. *$\overline{D}$ is an invariant region with respect to* (3.1)*. The disease free equilibrium $(S_0, I_0) = (1, 0)$ is the unique periodic solution of* (3.1) *satisfying $(S, I) \in \partial D$, $0 < \lambda \leq 1$.*

Proof. First, we will prove that $\overline{D}$ is an invariant region. In fact, it follows from model (3.1) that

$$\frac{dS}{dt}\Big|_{S=0} = \lambda\mu > 0, \ \frac{dI}{dt}\Big|_{I=0} = 0, \ \frac{d(S+I)}{dt}\Big|_{S+I=1} = -\lambda\gamma I \leq 0.$$

Since there is no impulsive motion for I and

$$S(t_i^+) = (1 - \lambda p_i(1 - e^{-\alpha I(t_i)}))S(t_i),$$

it is easy to conclude that every possible solution will remain in the region $\overline{D}$ ultimately.

Second, we will prove that the disease-free equilibrium $(S_0, I_0) = (1, 0)$ is the unique periodic solution of (3.1) satisfying $(S, I) \in \partial D$.

We assume that $(S, I) \in \partial D$ is a solution of (3.1), which means that at least one of the following conditions holds:

(i) There exists $t_0 \in [0, T]$ such that $I(t_0) = 0$.

(ii) There exists $t_0 \in [0, T]$ such that $S(t_0) = 0$.

(iii) There exists $t_0 \in [0, T]$ such that $S(t_0) + I(t_0) = 1$.

We now consider each of these three cases:

In case of (i), we have $I(t_0) = 0$ and $I'(t_0) = 0$, which implies $I \equiv 0$ and $\triangle S(t_i) = 0$, $i = 1, \dots, k$. Thus, the only possible periodic solution of $S' = \mu(1 - S)$ is $S \equiv 1$.

In case of (ii), we have $S(t_0) = 0$ and $S'(t_0) = \mu > 0$. Thus, it is easy to obtain that $S(t) < 0$ for $t < t_0$ sufficiently close to t_0, which contradicts the fact that $\overline{D}$ is an invariant region.

In case of (iii), we get

$$(S + I)'(t_0) = \mu(1 - S(t_0) - I(t_0)) - \gamma I(t_0) = -\gamma I(t_0) \le 0.$$

Because $I(t_0) = 0$ has been discussed, we only discuss $S(t_0) + I(t_0) = 1$, $(S + I)'(t_0) < 0$, which contradicts the fact that $\overline{D}$ is an invariant region. ■

Remark. If the impulsive motion doesn't influenced by I, that means $S(t_i^+) = (1 - \lambda p_i)S(t_i)$, the system $I \equiv 0$, $S' = \mu(1 - S)$ can have a nonconstant periodic solution on ∂D, which is hard to handle. In fact, if there is no infectious patients, it is often meaningless to vaccinate the susceptible people.

To use the continuity method, we need to choose an open bounded set $\Omega \subseteq D$, such that there is no solution (S, I) of (3.1) satisfying $(S, I) \in \partial\Omega$ for any $\lambda \in (0, 1)$. From the idea of G. Katriel [17], we choose Ω to be the open subset of D given by

$$\Omega = \left\{ (S, I) \in D \mid \min_{t \in [0,T]} S(t) < \delta \right\}, \tag{3.2}$$

where $\delta \in (0, 1)$ is to be fixed.

Proposition 4. *Let $\mathcal{R}_0 > 1$, $\delta \in (\frac{1}{\mathcal{R}_0}, 1)$. Then there exists no solution (S, I) of* (3.1) *satisfying $(S, I) \in \partial\Omega$, for any $\lambda \in (0, 1]$.*

Proof. Suppose $(S, I) \in \partial\Omega$. Then either $\{(S, I) \in \partial D \big| \min_{t \in [0,T]} S(t) < \delta\}$ or $\{(S, I) \in D \big| \min_{t \in [0,T]} S(t) = \delta\}$.

In the first case, the fact that $S_0 \ge \delta$ and Proposition 3 imply that there is no solution of (3.1) on $\{(S, I) \in \partial D \big| \min_{t \in [0,T]} S(t) < \delta\}$.

In the second case, we have $I(t) > 0$ and $S(t) \ge \delta$, $\forall t \in [0, T]$. Dividing the second equation of (3.1) by I, we have

$$\frac{d \ln I(t)}{dt} = \lambda(\beta(t)S - \mu - \gamma).$$

Since there is no impulsive motion for I, after integrating over $[0, T]$, we obtain that

$$\mu + \gamma = \frac{1}{T} \int_0^T \beta(t) S dt \ge \delta\bar{\beta},$$

which is a contradiction to the assumptioin $\delta > \frac{1}{\mathcal{R}_0}$. ■

4. Existence of Positive Periodic Solutions

In this section, we will give the proof of Theorem 1 by 4 steps.

Proof. **Step 1.** N is L-compact on $\overline{\Omega}$.
First, it is easy to prove that $QN(\overline{\Omega})$ is bounded. For any $(S, I) \in \overline{\Omega}$,

$$\begin{aligned}
\|QN(S,I)\|_Z = & \max_{t\in[0,T]}(|\frac{1}{T}\int_0^T f_1(\tau,S,I)d\tau \\
& -\frac{1}{T}\sum_{i=1}^k J_i(S(t_i),I(t_i))| + |\frac{1}{T}\int_0^T f_2(\tau,S,I)d\tau|) \\
\leq & \ 2\mu + 2\bar{\beta} + \gamma + \frac{1}{T}\sum_{i=1}^k p_i \\
< & \ +\infty.
\end{aligned}$$

Second, it is easy to prove that $K_p(I-Q)N : \overline{\Omega} \to X$ is uniform bounded and equicontinuous on each $[t_i, t_{i+1}]$. Assume that $\{(S_j, I_j)\}_{j=1}^{\infty} \subset \overline{\Omega}$. Using Arzela-Ascoli theorem, there exists a uniformly convergent subsequence denoted by $K_p(I-Q)N(S_{j_1}, I_{j_1})$ on $[0, t_1]$. Using Arzela-Ascoli theorem on $[t_1, t_2]$, we have a uniformly convergent subsequence $K_p(I-Q)N(S_{j_2}, I_{j_2})$ which is also uniformly convergent on $[0, t_1]$. Repeat it again and again, we can prove that $K_p(I-Q)N(S_{j_{k+1}}, I_{j_{k+1}})$ is uniformly convergent on $[0, T]$. Thus, $K_p(1-Q)N : \overline{\Omega} \to X$ is compact.

Step 2. For each $\lambda \in (0, 1)$, $(S, I) \in \partial\Omega \cap \operatorname{dom} L$, $L(S, I) \neq \lambda N(S, I)$, which has been proved by Proposition 4.

Step 3. For each $(S, I) \in \partial\Omega \cap \operatorname{Ker} L$, $QN(S, I) \neq 0$.
If $QN(S_1, I_1) = 0$, we have

$$\tag{4.1} \begin{cases} \dfrac{1}{T}\displaystyle\int_0^T f_1(\tau,S_1,I_1)d\tau - \dfrac{1}{T}\sum_{i=1}^k J_i(S_1(t_i),I_1(t_i)) = 0, \\ \dfrac{1}{T}\displaystyle\int_0^T f_2(\tau,S_1,I_1)d\tau = 0. \end{cases}$$

Assume $(S_1, I_1) \in \operatorname{Ker} L$, we know that (S_1, I_1) is a constant vector in $\mathbb{R}^2$. Thus, (4.1) is equivalent to

$$\tag{4.2} \begin{cases} \mu - \bar{\beta}S_1I_1 - \mu S_1 - \dfrac{1}{T}\displaystyle\sum_{i=1}^k p_i \cdot (1-e^{-\alpha I_1})S_1 = 0, \\ \bar{\beta}S_1I_1 - (\mu+\gamma)I_1 = 0. \end{cases}$$

We claim that there are exactly two solutions: $(1,0)$ and (S^*,I^*) in $KerL$ satisfying

$$\begin{cases} S^* = \dfrac{\mu+\gamma}{\bar{\beta}}, \\ \mu\left(\dfrac{\bar{\beta}}{\mu+\gamma}-1\right) - \bar{\beta}I^* - \dfrac{1}{T}\displaystyle\sum_{i=1}^{k} p_i(1-e^{-\alpha I^*}) = 0. \end{cases}$$

By the definition of Ω, we know that $(1,0) \notin \overline{\Omega}$. Denote

$$G(I) = \mu\left(\frac{\bar{\beta}}{\mu+\gamma}-1\right) - \bar{\beta}I - \frac{1}{T}\sum_{i=1}^{k} p_i(1-e^{-\alpha I}).$$

We have

$$G(0) = \mu\left(\frac{\bar{\beta}}{\mu+\gamma}-1\right) > 0,\ G(1) = \frac{-\gamma\bar{\beta}}{\mu+\gamma} - \mu - \frac{1}{T}\sum_{i=1}^{k} p_i(1-e^{-\alpha}) < 0.$$

Since

$$\frac{\partial G(I)}{\partial I} = -\bar{\beta} - \frac{1}{T}\sum_{i=1}^{k} p_i\alpha e^{-\alpha I} < 0,$$

$G(I)$ is a monotonous function with respect to I. Thus, (S^*,I^*) is the unique solution of $QN(S,I) = 0 in \operatorname{Ker} L \cap \overline{\Omega}$. Because

$$S^* = \frac{\mu+\gamma}{\bar{\beta}} = \frac{1}{\mathcal{R}} < \delta,$$

we have $(S^*,I^*) \notin \operatorname{Ker} L \cap \partial\Omega$. Thus, for each $(S,I) \in \partial\Omega \cap KerL$, $QN(S,I) \neq 0$.

Step 4. There exists an isomorphism $\Lambda : \mathrm{Im}Q \to \operatorname{Ker} L$ such that

$$\begin{aligned} &\Lambda Q(z_1,z_2,C_1,\ldots,C_k) \\ &= \Lambda\left(\frac{1}{T}\int_0^T z_1(\tau)d\tau + \frac{1}{T}\sum_{i=1}^{k} C_i, \frac{1}{T}\int_0^T z_2(\tau)d\tau, 0_{1\times k}\right) \\ &= \left(\frac{1}{T}\int_0^T z_1(\tau)d\tau + \frac{1}{T}\sum_{i=1}^{k} C_i, \frac{1}{T}\int_0^T z_2(\tau)d\tau\right). \end{aligned}$$

We will prove that $\deg(\Lambda QN, \Omega\cap\operatorname{Ker} L, 0) \neq 0$. From the discussion in Step 3, we know that (S^*,I^*) is the unique solution of $\Lambda QN(S,I) = 0$ in $\Omega\cap\operatorname{Ker} L$.

A direct calculation shows that

$$
\begin{aligned}
&\deg(\Lambda QN(S,N),\Omega\cap\operatorname{Ker}L,(0,0))\\
=\ &\deg((\mu-\bar{\beta}SI-\mu S-\frac{1}{T}\sum_{i}^{k}p_i(1-e^{-\alpha I})S,\bar{\beta}SI-(\mu+\gamma)I),\Omega\cap\operatorname{Ker}L,(0,0))\\
=\ &\operatorname{Sign}\begin{vmatrix}-\bar{\beta}I^*-\mu-\frac{1}{T}\sum\limits_{i=1}^{k}p_i(1-e^{-\alpha I^*}) & -\bar{\beta}S^*-\frac{1}{T}\sum\limits_{i=1}^{k}p_i\alpha e^{-\alpha I^*}S^*\\ \bar{\beta}I^* & \bar{\beta}S^*-\mu-\gamma\end{vmatrix}\\
=\ &\operatorname{Sign}\begin{vmatrix}-\bar{\beta}I^*-\mu-\frac{1}{T}\sum\limits_{i=1}^{k}p_i(1-e^{-\alpha I^*}) & -\mu-\gamma-\frac{1}{T}\sum\limits_{i=1}^{k}p_i\alpha e^{-\alpha I^*}\frac{\mu+\gamma}{\beta}\\ \bar{\beta}I^* & 0\end{vmatrix}\\
=\ &1\neq 0.
\end{aligned}
$$

Thus, by theorem 2, the equation $L(S,I)=N(S,I)$ has at least one solution on $\operatorname{dom}L\cap\overline{\Omega}$.

From Proposition 4, we know that for each $\lambda=1$, $(S,I)\in\operatorname{dom}L\cap\partial\Omega$, $L(S,I)\neq N(S,I)$. Thus, $L(S,I)=N(S,I)$ has at least one solution in $\operatorname{dom}L\cap\Omega$. ∎

5. Simulation

In this section, we present some numerical examples to illustrate the effectiveness of pulse vaccination strategy. Furthermore, we show how the various parameters influence the solutions of our SIR model.

With the period $T=2\pi$ of the forcing representing one year, we take $\gamma=14\frac{2\pi}{365}$ corresponding to a two-week infectious period. We set $\bar{\beta}=4\gamma$, $\mu=\frac{0.5}{2\pi}$ and $\beta(t)=\bar{\beta}(1+0.6\cos(t))$. Assume that there are three impulsive point at fixed time $\frac{\pi}{2}$, $\frac{2\pi}{2}$ and $\frac{3\pi}{2}$ with $p_i=0.2$, $i=1,2,3$. Let $[0,2\pi]$ be divided into $k=200$ intervals equally. Given the initial point $(S^{**},I^{**})=(\frac{\mu+\gamma}{\beta},\frac{\mu}{\mu+\gamma}-\frac{\mu}{\beta})$, which is the endemic equilibrium of SIR model without periodic transmission rate and pulse vaccination. The periodic solutions of system (2.2) can be solved by the Newton iteration method in which we set $S(t_i+1)-S(t_i)=p_i(1-e^{-\alpha I(t_i)})S(t_i)$ at fixed time $\frac{\pi}{2}$, $\frac{2\pi}{2}$ and $\frac{3\pi}{2}$.

In Fig. 1, we make eight steps of the Newton iteration to get the approximate infective population of system (2.2) with with both $p_i=0$ (the surface at the bottom) and $p_i=0.2$ (the surface at the top). Obviously, the infective population of system (2.2) with pulse is lower than the infective population of system (2.2) without pulse in Fig. 1. Thus, it is very effective to lower the infective population by pulse vaccination strategy. The solutions in both case are locally stable and the error is about 10^{-10}.

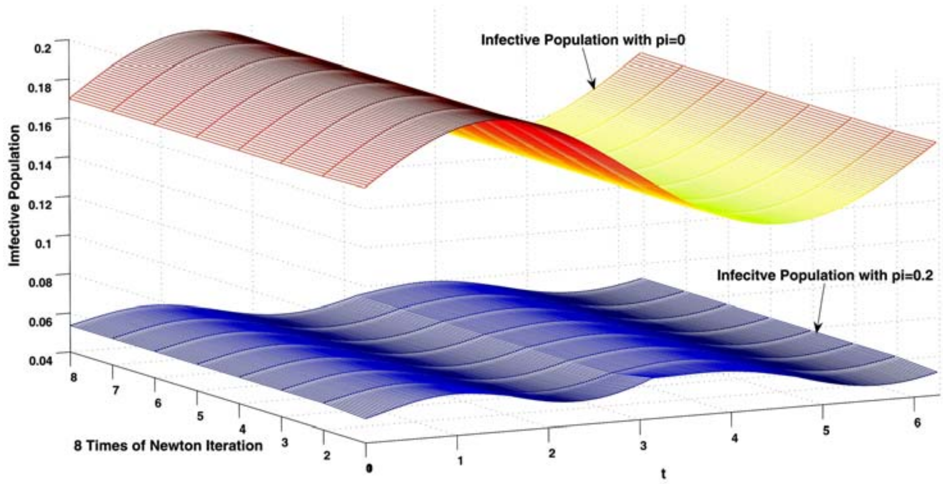

Fig. 1. Infective population with both $p_i = 0$ and $p_i = 0.2$.

In Fig. 2, we make eight steps of the Newton iteration to get the approximate susceptible population of system (2.2) with both $p_i = 0$ (the smooth surface) and $p_i = 0.2$ (the nonsmooth surface). Obviously, the susceptible population of system (2.2) with $p_i = 0.2$ has periodic and impulsive properties. The solutions in both case are locally stable and the error is about 10^{-10}.

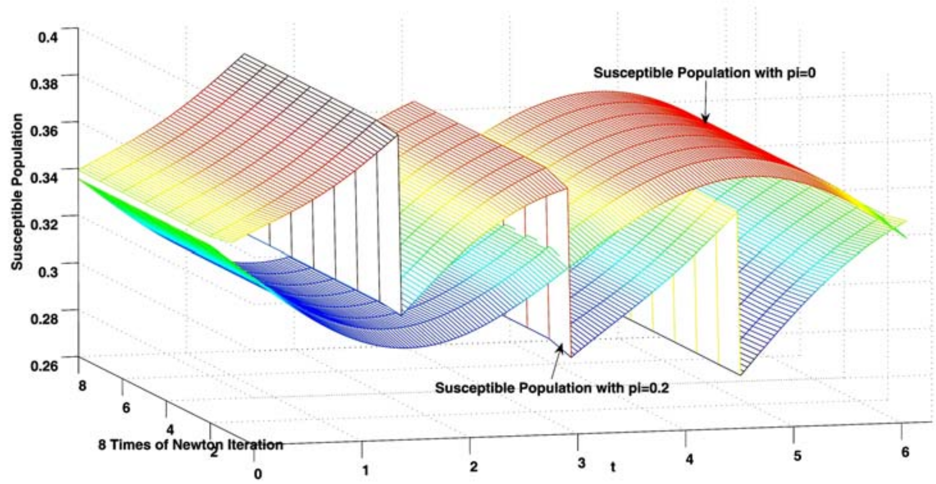

Fig. 2. Susceptible population with both $p_i = 0$ and $p_i = 0.2$.

Assume T, β, γ, μ and t_i are defined above. We make eight steps of the Newton iteration to get the approximate infective population of system (2.2) with $p_i = 0.15$,

$p_i = 0.2$ and $p_i = 0.25$. Fig. 3 shows that the effectiveness of pulse vaccination strategy increase with p_i. The solutions here are locally stable and the error is about 10^{-10}. When $p_i = 0.3$, the solution will be unstable and our program will be out of work.

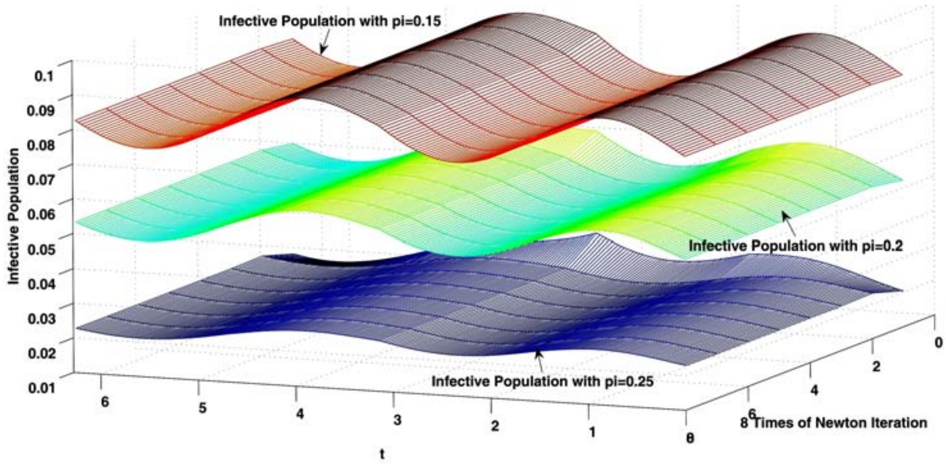

Fig. 3. Infective population with different p_i.

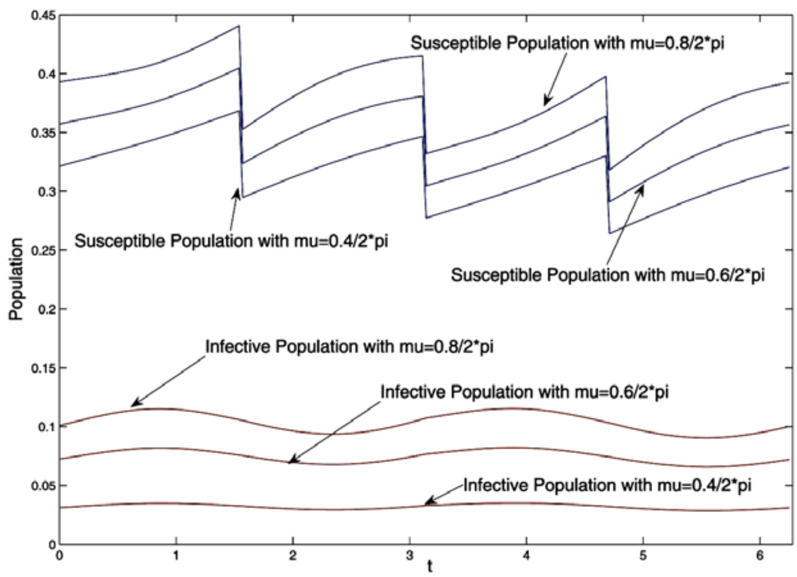

Fig. 4. Population with different μ.

Assume T, β, γ and t_i are defined above, $p_i = 0.2$. We make ten steps of the Newton iteration to get the approximate infective population and susceptible population of system (2.2) with $\mu = \frac{0.4}{2\pi}$, $\frac{0.6}{2\pi}$ and $\frac{0.8}{2\pi}$. Fig. 4 shows that the impact on the infective population and susceptible population by μ. The solutions here are locally stable and the error is about 10^{-13}.

Assume T, γ and t_i are defined above, $p_i = 0.2$ and $\mu = \frac{0.5}{2\pi}$. We make ten steps of the Newton iteration to get the approximate infective population and susceptible population of system (2.2) with $\bar{\beta} = 2\gamma$, 4γ, 6γ and 8γ. Fig. 5 shows that the impact on the infective population and susceptible population by $\bar{\beta}$. The solutions here are locally stable and the error is about 10^{-13}.

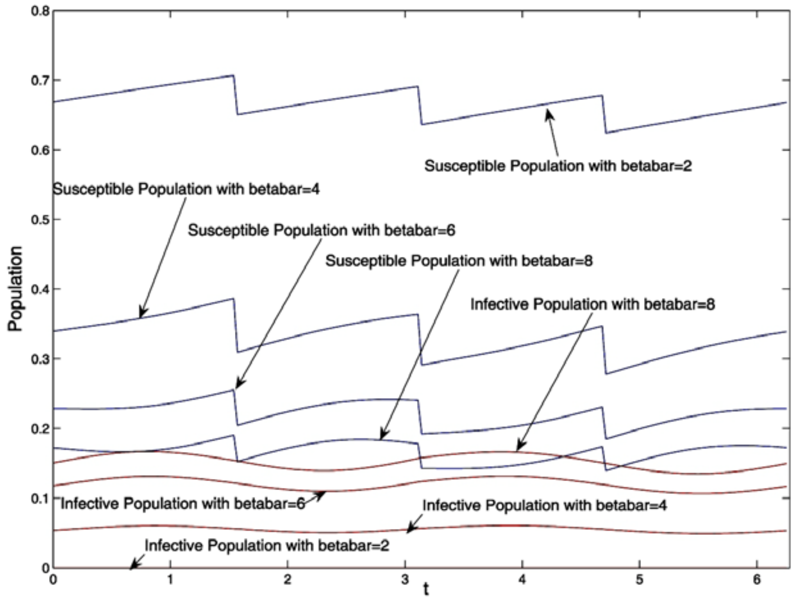

Fig. 5. Population with different $\bar{\beta}$.

6. Conclusion

We obtain the existence of positive periodic solutions of seasonally forced SIR models with impulse vaccination by Mawhin's coincidence degree method. Some numerical simulations are presented to illustrate the effectiveness of such pulse vaccination strategy.

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