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R-EQUIVALENCE ON DEL PEZZO SURFACES OF DEGREE 4 AND CUBIC SURFACES

Zhiyu Tian

Abstract. We prove that there is a unique R-equivalence class on every del Pezzo surface of degree 4 defined over the Laurent field $K = k((t))$ in one variable over an algebraically closed field k of characteristic not equal to 2 or 5. We also prove that given a smooth cubic surface defined over $\mathbb{C}((t))$, if the induced morphism to the GIT compactification of smooth cubic surfaces lies in the stable locus (possibly after a base change), then there is a unique R-equivalence class.

1. INTRODUCTION

Given a variety X over a field K, Manin [12] defined the R-equivalence relation on the set of rational points $X(K)$. Recall that two points x, y are *directly R-connected* if there is a K-morphism $f : \mathbb{P}^1_K \to X$ such that $f(0) = x, f(\infty) = y$. Then the R-equivalence relation is the equivalence relation generated by such relations.

The R-equivalence relation on $X(K)$ is not so interesting unless X already contains lots of rational curves (at least over an algebraic closure of K). Such X are rationally connected. For a precise definition of rational connectedness, see [8].

Many people have studied the R-equivalence classes on cubic hypersurfaces [11], intersection of two quadrics [5, 7], and low degree complete intersections [14].

In this short note, we give a very simple proof of the following result.

Theorem 1.1. *Let X be a smooth del Pezzo surface of degree 4 defined over the Laurent field $K = k((t))$ in one variable over an algebraically closed field k of characteristic not equal to 2 or 5. Then there is exactly one R-equivalence class on $X(K)$.*

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Recall that a C_1 field is a field such that every hypersurface of degree d in $\mathbb{P}^n, n \geq d$ has a rational point over the field. A general question is that given a smooth separably rationally connected variety defined over a C_1 field, whether there is only one R-equivalence class on the set of rational points.

For the R-equivalence classes on a del Pezzo surface of degree 4, J.-L. Colliot-Thélène and A. N. Skorobogatov proved that there is only one R-equivalence class if the surface has a conic bundle structure with 4 degenerate fibers [5] over a field of cohomological dimension 1. Very recently, Colliot-Thélène proved that there is only one R-equivalence class if the field is C_1 and characteristic 0 [4].

The observation in this paper is that in the case of Laurent field one can use a simple geometric argument to prove the statement.

In the same spirit and using the G-equivariant techniques developed in [16], we also study the R-equivalence on cubic surfaces defined over $\mathbb{C}((t))$.

Recall that there is a moduli space M of smooth cubic surfaces defined over $\mathbb{C}$ constructed by geometric invariant theory (GIT), which is the quotient of the open subset of the Hilbert scheme parameterizing smooth cubic surfaces in $\mathbb{P}^3$ by the natural $PGL(4)$ action. Moreover there is a compactification $\overline{M}$ of M by adding semi-stable cubic surfaces, [13] p. 80. The stable cubic surfaces are cubic surfaces with at worst ordinary double points as singularities. The unique strictly semistable cubic surface is given by the equation $XYZ + W^3 = 0$, which has 3 A_2 singularities.

A cubic surface X defined over $\mathbb{C}((t))$ is equivalent to a map $\widehat{t} : \operatorname{Spec} \mathbb{C}((t)) \to M$. By the properness of $\overline{M}$, after a base change $\mathbb{C}[[s]] \supset \mathbb{C}[[t]]$, we can extend the map over $\mathbb{C}[[s]]$ to $\overline{M}$ in a unique way. For example, for the family $X^3 + Y^3 + Z^3 + tW^3 = 0$, after the base change $t = s^3$, the map to the compactification $\overline{M}$ is the constant map $s \mapsto X^3 + Y^3 + Z^3 + W^3 = 0$. Even though the original central fiber is quite singular, the new central fiber is smooth.

We could not prove that there is only one R-equivalence class for X in general. However we could give a sufficient geometric condition so that there is only one R-equivalence class.

Theorem 1.2. *Let X be a smooth cubic surface defined over $\mathbb{C}((t))$ and let $\widehat{t}$: $\operatorname{Spec} \mathbb{C}((t)) \to M$ be the induced morphism to the GIT compactification of smooth cubic surfaces. If after a base change the morphism can be completed to a morphism $\widehat{s} : \operatorname{Spec} \mathbb{C}[[s]] \to \overline{M}^s$ to the locus of stable cubic surfaces, then there is a unique R-equivalence class on the $\mathbb{C}((t))$-points of X.*

The same statement should also holds in positive characteristic provided the characteristic is large enough (probably at least 7) such that the degree of the base change needed is divisible in the field.

2. R-Equivalence and Quadratic Field Extension

We first recall the definition of Weil restriction.

Let L/K be a finite field extension and U be a variety defined over L. The Weil restriction associates a K-variety $\mathrm{Res}_{L/K}U$ to U such that for any variety V defined over K , we have a natural isomorphism

$$\mathrm{Hom}(V, \mathrm{Res}_{L/K}U) \simeq \mathrm{Hom}(V \times_K L, U).$$

In particular there is a natural identification of the set of K-rational points of $R_{L/K}U$ with the set of L-rational points of U.

For our purpose it is useful to understand the special case when L/K is a separable field extension of degree 2. Let U be a variety defined over L and $\sigma \in \mathrm{Gal}(L/K)$ the generator of the Galois group. Denote by U^σ its conjugate over K. Then the Weil restriction $\mathrm{Res}_{L/K}U$ (after base changed to the field L) is the product $U \times_L U^\sigma$ and the Galois group action is $(u_1, u_2^\sigma) \mapsto (u_2, u_1^\sigma)$, which gives the product the structure of a variety defined over K.

We also recall the following construction in [12], Section 15, Proposition 15.1, and [10], Example 3.8, Exercise 3.11.

Construction 2.1. Let L/K be a separable quadratic field extension and $\widetilde{X}_K$ a smooth projective cubic surface over K. Denote by $\widetilde{X}_L$ the base change of $\widetilde{X}_K$ to the field L. Consider the Weil restriction $\mathrm{Res}_{L/K}\widetilde{X}_L$. There is a rational dominant map ([10] Ex. 3.11, [12], Section 15, Proposition 15.1):

$$\mathrm{Res}_{L/K}\widetilde{X}_L \dashrightarrow \widetilde{X}_K.$$

When base changed to an algebraic closure or the field L, this can be described as mapping a pair of points to the third intersection point of the cubic surface with the line spanned by the pair (if the line does not lie in the cubic surface). In particular, the fiber (after base-changed to an algebraic closure or the field L) over a point is an open subset of the blow-up of the cubic surface at the point, and thus, rational.

In general, the map is not surjective on K-rational points. However, we do have the following observation.

Lemma 2.2. *Given any K-rational point x in $\widetilde{X}_K$, there is one geometrically irreducible component of the fiber containing an open subset of a smooth rational surface.*

Proof. This follows from the geometry of the map as discussed below.

Since we are only interested in the geometry, it suffices to consider the base change of the rational map $\mathrm{Res}_{L/K}\widetilde{X}_L \dashrightarrow \widetilde{X}_K$ over a separably closed field. Then the rational map is the same as the rational map:

$$\widetilde{X} \times \widetilde{X} \dashrightarrow \widetilde{X}$$

which sends two points x, y in $\widetilde{X}$ to the third intersection point of the line spanned by them provided that the line is not contained in $\widetilde{X}$. Consider the graph $\Gamma \subset \widetilde{X} \times \widetilde{X} \times \widetilde{X}$ of the above rational map.

There is a open subvariety $W \subset \Gamma$ parameterizing all the triples (x, y, z) such that the line spanned by the three points does not lie in the cubic surface $\widetilde{X}$.

Let p_i be the projection of W to the i-th factor. We consider the projection to the first factor. Denote by F the fiber of W over z. Then in the first case the morphism $p_1 : F \to \widetilde{X}$ is birational and is an isomorphism over the complement of the point z. The preimage of each point $x \neq z$, the preimage in F is a unique point (x, y, z). The fiber of $F \to \widetilde{X}$ over z are points of the form (z, y, z), which corresponds to tangent lines of $\widetilde{X}$ at z. So the fiber over z is a $\mathbb{P}^1$. In the second case, the morphism is birational and is an isomorphism over the complement of the lines containing z. For each point x, not contained in the lines that contain the point z, there is a unique point (x, y, z) in F which is mapped to x. Thus we have proved the following,

- For a point $z \in \widetilde{X}$ which is not contained in a line, the fiber of the morphism $p_3 : W \to \widetilde{X}$ over z is isomorphic to the blow-up of the surface $\widetilde{X}$ at z.
- For a point $z \in \widetilde{X}$ which is contained in a line, the fiber of the morphism p_3 over z is isomorphic to the complement of the lines containing z

Moreover, the fiber dominates the first (or the second) factor via the projection. So we find the geometrically integral rational surface over a point. ∎

Assume K is C_1. Then there is at least one rational point in a birational modification of this irreducible component of the fiber ([3], Proposition 2). Thus given any K-rational point x in $\widetilde{X}_K$, there is a K-rational point of $\mathrm{Res}_{L/K}\widetilde{X}_L$ which is mapped to x. This allows us to prove the following.

Lemma 2.3. *Let K be a C_1-field of characteristic 0 or the Laurent field $k(\!(t)\!)$ over an algebraically closed field k. And let $\widetilde{X}_K$ be a smooth cubic surface defined over K and L/K a separable degree 2 field extension. Finally let $\widetilde{X}_L$ be the base change of $\widetilde{X}_K$ to L. If there is a unique R-equivalence class on $\widetilde{X}_L(L)$, then there is a unique R-equivalence class on $\widetilde{X}_K(K)$.*

Proof. The previous construction 2.1 shows that $\widetilde{X}_K$ is dominated by a variety (i.e. $\mathrm{Res}_{L/K}\widetilde{X}_L$) whose rational points are R-connected. Moreover, the fiber over any rational point in $\widetilde{X}_K(K)$ is an open subset of a rational surface (which is isomorphic to the blow-up of the cubic surface at the rational point over the field L).

Over a Laurent field (or more generally over any large fields), being R-connected is the same as being directly R-connected [9]. Thus any two points in $\widetilde{X}_L$, hence also any two points in $\mathrm{Res}_{L/K}\widetilde{X}_L$, are directly R-connected by a single rational curve. Since the fiber in $\mathrm{Res}_{L/K}\widetilde{X}_L$ over any rational point in $\widetilde{X}_K$ contains a geometrically irreducible component which has an open subset isomorphic to an open subset of a smooth (geometrically) rational surface, and the field K is a large field and C_1, the set of K-rational points in the open subset of $\mathrm{Res}_{L/K}\widetilde{X}_L$ where the rational map to $\widetilde{X}_K$ is defined maps surjectively to the set of rational points $\widetilde{X}_K(K)$. It follows that any two K-rational points in $\widetilde{X}_K(K)$ are directly R-connected.

We use the following argument in the case of characteristic 0 C_1-field. We first resolve the indeterminacy by $\overline{\mathrm{Res}} \to \mathrm{Res}_{L/K}\widetilde{X}_L$. Since the set of rational points modulo R-equivalence is a birational invariant in characteristic 0 (Proposition 10, [6] p.195), there is only one R-equivalence class on $\overline{\mathrm{Res}}(K)$. By previous discussion, there is a geometrically irreducible component of the fiber over any rational point on $\widetilde{X}_K$, which is geometrically rational. A resolution of singularities of this irreducible component is a smooth projective (geometrically) rational surface over a C_1-field. In particular, there is a K-rational point on the resolution ([3], Proposition 2), which is mapped to a K-rational point of the irreducible component. Thus the set of rational points of $\overline{\mathrm{Res}}(K)$ maps surjectively to $\widetilde{X}_K(K)$. Then one can deduce R-equivalence of any two K-rational points in $\widetilde{X}_K$ by lifting them to $\overline{\mathrm{Res}}(K)$. ∎

3. Proof of Theorem 1.1

To begin the proof, first notice the following fact.

Lemma 3.1. *Let $\widetilde{X}$ be a smooth cubic surface with a line defined over a the Laurent field $K = k(\!(t)\!)$ over an algebraically closed field k of characteristic not equal to 2 or 5. Then there is a sequence of degree 2 Galois field extensions $K = K_0 \subset K_1 \subset \cdots \subset K_n$ such that the base change $\widetilde{X}_n$ is rational over K_n.*

Proof. We use the following construction: projection from the line to get a conic bundle structure of $\widetilde{X}$ over $\mathbb{P}^1_K$ with five degenerate fibers.

First consider the case the five degenerate fibers form an irreducible cycle defined over K. There are two further possibilities according to the field of definitions of the 10 lines in the degenerate fibers. These lines are defined either in a degree 5 (separable) field extension $K' = k(\!(s)\!), s^5 = t$ or a degree 10 (separable) field extension $K' = k(\!(s)\!), s^{10} = t$. In the first case the 10 lines form two irreducible cycles over K, each consisting of 5 lines in distinct fibers. And we can contract one set of 5 lines in distinct fibers of the conic fibration over K. Thus we get a rational surface over K. In the second case we can make the contraction after a (separable) degree 2 field extension.

If there is a degree 2 cycle of the singular fibers over K, we first make a degree 2 field extension so that the two singular fibers are both defined over the field. Then after a possible degree 2 field extension, we may assume that all the four lines are defined over the field. Then we can contract two disjoint lines and get a del Pezzo surface of degree 5 with a rational point. Thus the surface is rational over the field.

For the case where there is just one singular fiber defined over K and the other 4 singular fibers are conjugate over K, we know there is a tower of two separable degree 2 field extension $K \subset K_1 \subset K_2$ so that each of the singular fiber cycle is defined over K_2. Then up to making a further degree 2 field extension, all the fibers are defined over the field. Then the base change is rational since we can contract 2 disjoint lines and a del Pezzo surface of degree 5 with rational points is rational. ∎

Now we can finish the proof of the theorem.

Proof of Theorem 1.1. The Laurent field K is a large field. In particular, the set of rational points is Zariski dense once we have a point [15]. In any case there is a general point x not on a line. Blow up at the point x. Then we have a smooth cubic surface $\widetilde{X}$ with a line corresponding to the exceptional divisor. It suffices to show that there is a unique R-equivalence class on $\widetilde{X}$.

Projection from the line gives a conic bundle structure on $\widetilde{X}$ with five degenerate fibers.

By Lemma 3.1, there is a sequence of separable degree 2 field extensions $K = K_0 \subset K_1 \subset \cdots \subset K_n$ such that $\widetilde{X}_n = \widetilde{X} \times_K K_n$ is rational.

The R-equivalence on $\widetilde{X}_n$ is trivial. We use Lemma 2.3 to finish the proof. ∎

4. Proof of Theorem 1.2

We first discuss the simultaneous resolution of an ordinary double point. The general result can be found, for example in [1].

For any family of surfaces with at worst du Val singularities over $\operatorname{Spec} \mathbb{C} [\![s]\!]$, there is a simultaneous resolution after a base change [1], i.e. a diagram as the following:

$$\begin{tikzcd}
\widetilde{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spec} \mathbb{C} [\![s']\!] \arrow[r] & \operatorname{Spec} \mathbb{C} [\![s']\!] \arrow[r] & \operatorname{Spec} \mathbb{C} [\![s]\!]
\end{tikzcd}$$

such that the right square is a pull-back diagram and the family $\widetilde{\mathcal{X}} \to \operatorname{Spec} \mathbb{C} [\![s']\!]$ is a smooth proper family. In general $\widetilde{\mathcal{X}}$ is only an algebraic space.

For the case of interest to us, namely ordinary double points, things can be done very explicitly. For the family

$$xy + z^2 = s^2$$

over $\operatorname{Spec} \mathbb{C} [\![s]\!]$, there are exactly two simultaneous resolutions by blowing up $x = z - s = 0$ or $x = z + s$=0. The two simultaneous resolutions are the classical example of an “Atiyah flop”.

Every family of surfaces whose generic fiber is smooth and whose central fiber has only ordinary double points as singularities has a simultaneous resolution after a ramified degree 2 base change. A simultaneous resolution exists without the need of base change if and only if locally around each singularity, the family is given by

$$xy + z^2 = s^{2k}, \quad k \in \mathbb{N}.$$

The family (locally) can be thought of as obtained from a degree k base change from the family $xy + z^2 = s^2$. There are again two simultaneous resolutions (by blowing-up $x = z - s^k = 0$ or $x = z + s^k = 0$, or by pulling back the simultaneous resolutions of the family $xy + z^2 = s^2$), which are also related by a flop.

As a motivating example, consider the family $xy + z^2 = s^2$ as obtained from a degree 2 base change of the family $xy + z^2 = s$, with the natural $\mathbb{Z}/2\mathbb{Z}$ action $s \mapsto -s$. The action does not extend to any of the two resolutions since it does not preserve the blow-up center. However, an index 2 subgroup, i.e. the trivial subgroup consisting of the identity, admits an action on both of the resolutions. This is not a coincidence, as shown by the following.

Lemma 4.1. *Let $\mathcal{X} \to S$ be a family of surfaces over $S = \operatorname{Spec} \mathbb{C}[\![s]\!]$ with a G-action on both $\mathcal{X}$ and S, compatible with the fibration structure. Assume furthermore that there is a simultaneous resolution over the base S. Then the action of a subgroup, whose index is an power of 2, always lifts to the resolution.*

Proof. For each ordinary double points fixed by the G-action, there are exactly two different simultaneous resolutions for one single ordinary double point obtained by blowing up some subvariety (in the étale topology) and glue as an algebraic space. The G action on $\mathcal{X}$ either preserves the blow-up center or permutes the two centers. If the action preserves the center, the action extends to an action of a simultaneous resolution near the ordinary double point. Otherwise the index 2 subgroup which preserves the center acts on the resolution (In the example $xy + z^2 = s^2$, this is the trivial subgroup.).

In general we need to take care of the existence of several ordinary double points, some of which are fixed by the action while others are permuted. If the ordinary double point is a fixed point of G, this is exactly the case discussed above. If the ordinary double point has a non-trivial orbit, then we choose one resolution for one point in the orbit and use the group action to determine resolution at all the other orbits. So all we can say is the existence of a lifting of the action of a subgroup in G whose index is a power of 2. ■

Now we return to the proof of Theorem 1.2.

By assumption, after a degree l base change $t = s^l$, there is a projective family $\mathcal{X} \to \operatorname{Spec} \mathbb{C}[\![s]\!]$ such that

(1) The generic fiber of the family is isomorphic to the base change of X to $\operatorname{Spec} \mathbb{C}(\!(s)\!)$.

(2) The central fiber $\mathcal{X}_0$ is a cubic surface with at worst ordinary double points as singularities.

(3) The Galois group $G = \mathbb{Z}/l\mathbb{Z}$ acts on $\mathcal{X}$ and the projection to $\operatorname{Spec}\mathbb{C}\llbracket s\rrbracket$ is G-equivariant.

Any $\mathbb{C}(\!(t)\!)$-points of X induces a G-equivariant section of the family.

We may assume that there is already a simultaneous resolution $\widetilde{\mathcal{X}} \to \mathcal{X} \to \operatorname{Spec}\mathbb{C}\llbracket s\rrbracket$, possibly after making a further ramified base change. By Lemma 4.1, the action of a subgroup G_1, whose index is an power of 2, always lifts to the resolution.

The subgroup G_1 determines a subfield extension $\mathbb{C}(\!(t)\!) \subset \mathbb{C}(\!(t_1)\!) \subset \mathbb{C}(\!(s)\!)$, where $\mathbb{C}(\!(t)\!) \subset \mathbb{C}(\!(t_1)\!)$ a sequence of quadratic field extensions and the Galois group of $\mathbb{C}(\!(s)\!)/\mathbb{C}(\!(t_1)\!)$ is isomorphic to G_1.

Let X_1 be the base change of X to the field $\mathbb{C}(\!(t_1)\!)$.

The group G_1 acts on both $\widetilde{\mathcal{X}}$ and $\mathcal{X}$, and the projections to $\operatorname{Spec}\mathbb{C}\llbracket s\rrbracket$ are G_1-equivariant. Thus any two $\mathbb{C}(\!(t_1)\!)$-rational points of X_1 induce two G_1-equivariant sections $\widehat{s}_1$ and $\widehat{s}_2$ of the family $\widetilde{\mathcal{X}} \to \operatorname{Spec}\mathbb{C}\llbracket s\rrbracket$, whose intersection points with the central fiber $\widetilde{\mathcal{X}}_0$ are fixed points of the action of G_1 on $\widetilde{\mathcal{X}}_0$. Denote the two points by x and y.

By Theorem 1.4 in [16], there is a G_1-equivariant very free curve $f : \mathbb{P}^1 \to \widetilde{\mathcal{X}}_0$ such that $f(0) = x, f(\infty) = y$, where the G_1 action on $\mathbb{P}^1$ is $z \mapsto \zeta z$, and ζ is a primitive r-th root of unity. Consider the relative Hom-scheme

$$\operatorname{Hom}(\mathbb{P}^1 \times \operatorname{Spec}\mathbb{C}\llbracket s\rrbracket, \widetilde{\mathcal{X}}, 0 \times \operatorname{Spec}\mathbb{C}\llbracket s\rrbracket \mapsto \widehat{s}_1, \infty \times \operatorname{Spec}\mathbb{C}\llbracket s\rrbracket \mapsto \widehat{s}_1).$$

Here we choose the same G_1-action on $\mathbb{P}^1$ as above. Then there is a natural G_1-action on the relative Hom-scheme such that the projection to the base is G_1-equivariant. The very free curve f gives a G_1-fixed point of the relative Hom-scheme and the morphism to the base is smooth at this point. Then by Lemma 4.2 below, there is a G_1-equivariant section of the relative Hom-scheme, which gives a family of rational curves connecting the two G_1-equivariant sections. Then the two $\mathbb{C}(\!(t_1)\!)$-rational points are R-equivalent.

So after a number of quadratic field extension $\mathbb{C}(\!(t_1)\!)/\mathbb{C}(\!(t)\!)$, there is a unique R-equivalence class on the base change X_1 of X. Then the theorem follows from Lemma 2.3.

Lemma 4.2. [16] *Let X and Y be two $\mathbb{C}$-schemes with a G-action and $f : X \to Y$ be a finite type G-equivariant morphism. Let $x \in X$ be a fixed point, and $y = f(x)$ (hence also a fixed point). Assume that f is smooth at x. Then there exists a G-equivariant section $s : \operatorname{Spec}\widehat{\mathcal{O}}_{y,Y} \to X$.*

For the proof see Corollary 2.2, [16].

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Zhiyu Tian
Mathematisches Institut der Universität Bonn
Endenicher Allee 60
Bonn 53115
Germany
E-mail: zhtian@math.univ-bonn.de
Current Address:
Institut Fourier
100 Rue des mathématiques, BP 74
38402, St Martin d'Hères
France
E-mail: zhiyu.tian@ujf-grenoble.fr