TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 6, pp. 1619-1642, December 2015
DOI: 10.11650/tjm.19.2015.5337
This paper is available online at http://journal.taiwanmathsoc.org.tw

BIRATIONAL MAPS OF 3-FOLDS

Jungkai Alfred Chen

Abstract. We show that a 3-fold terminal flip or divisorial contraction can be factored into a sequence of flops, blow-downs to a smooth curve in a smooth 3-fold or divisorial contractions to points with minimal discrepancies.

1. Introduction

A main task of birational geometry is to find a good model inside a birational equivalence class and study the geometry of these models. This goal can be achieved by the minimal model program. The minimal model conjecture asserts that for any given nonsingular or mildly singular projective variety, there exists a minimal model or a Mori fiber space after a sequence of flips and divisorial contractions. Moreover, different minimal models are connected by a sequence of flops. Therefore divisorial contractions, flips and flops are the elementary birational maps of the minimal model program.

Together with some recent advances on the geometry of 3-folds, for example, the recent result that a 3 fold of general type has a birational m-canonical maps for $m \geq 73$, and the canonical volume $\geq \frac{1}{2660}$ (cf. [3, 4]), one might hope to build up an explicit classification theory for 3-folds similar to the theory of surfaces by using the minimal model program explicitly. It is thus natural to ask what explicit information we have about the birational maps of 3-fold minimal model program.

Moreover, explicit factorization of birational maps in dimension three is of fundamental importance in understanding new phenomena of higher dimensional algebraic geometry. There are various newly developed invariants such as derived categories,

Received September 23, 2014, accepted April 30, 2015.
Communicated by Yoichi Miyaoka.
2010 *Mathematics Subject Classification*: 14E30.
Key words and phrases: Flips, Divisorial contraction, Minimal model program.
The author was partially supported by NCTS and National Science Council of Taiwan. We are indebted to Cascini, Hacon, Hayakawa, Kawakita, Kawamata, Kollár, Mori and Reid for many useful discussion. We are grateful to the referee for various detailed suggestions. Some of this work was done during visits of the author to RIMS and Imperial College London. The author would like to thank both institutes for their hospitality.

Gromov-Witten invariants from the 1990s onward. Detailed studies of new invariants in the case of 3-folds are expected to provide fruitful examples and hence are helpful to understand higher dimensional geometry. Even though the minimal model program for 3-folds was established more than 25 years ago by Mori and others, the more detailed and explicit description of birational maps in the 3-dimensional minimal model program has only recently became available, and is still not completely satisfactory.

We give a quick tour of known results: Mori and then Cutkosky classified birational maps from a nonsingular and Gorenstein 3-fold respectively [20, 7], and Tziolas has a series of works on divisorial contractions to curves passing through Gorenstein singularities (cf. [25, 26, 27]). The recent project of Mori and Prokhorov (cf. [22, 23]) on extremal contractions provide a treatment which is valid for divisorial contractions to curves and conic bundles as well. They completely classified divisorial contractions to curves of type IA, IC, and IIB. Divisorial contractions to points are probably the best understood, due mainly to works of Kawamata, Hayakawa, Markushevich and Kawakita (cf. [8, 9, 10, 11, 12, 13, 14, 15, 16, 19]). Also, the structure of flops are studied in Kollár's article [17]. Flips are still quite mysterious except for some examples in [1, 18] and toric flips [24].

Instead of classifying birational maps completely, we work on the problem of factorizing birational maps as a composite of simplest possible transformations. Such a factorization can be very useful for comparing various invariants between birational models. It is also useful in classifying birational maps. In previous joint work with Christopher Hacon [5], we were able to factorize flips and divisorial contractions to curves. Our previous work [2] factorizes divisorial contractions to a point of index $r > 1$ with non-minimal discrepancy $\frac{a}{r} > \frac{1}{r}$.

The purpose of this note is to complete the factorization program. We show that one can factor 3-fold birational maps in minimal model program into some simplest and explicit ones by combing previous work [2, 5] and considering divisorial contraction to a point of index $r = 1$.

Definition 1.1. A birational map $f\colon X \dashrightarrow Y$ between 3-folds is *factorizable* if it admits a factorization into a sequence of birational maps:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = Y,$$

such that each map $X_{i-1} \dashrightarrow X_i$ is one of the following

(1) a divisorial contraction (or its inverse) to a point $P_i \in X_i$ with minimal discrepancy;

(2) a blowdown to a smooth curve in a smooth neighborhood;

(3) a flop.

Theorem 1.2. (Main Theorem). *A 3-fold divisorial contraction $f\colon X \to W$ (resp. flip $\phi\colon X \dashrightarrow X^+$) is factorizable.*

Remark 1.3. Given a divisorial contraction to a point $f\colon X \to W \ni P$ with exceptional divisor E, we can write $K_X = f^*K_W + aE$. We say that the contraction f has discrepancy a.

Given P a terminal singularity of index r, the minimal discrepancy among all divisorial contractions to P is $\dfrac{1}{r}$ by [19] and [16]. If $P \in W$ is a nonsingular point, then the minimal discrepancy among all contractions to P is 2 by [11].

The key observation is that for any more complicated divisorial contraction $X \to W$ (resp. flip $X \dashrightarrow X^+$), there exists one or more singular points of index $r > 1$ on X. By choosing $Q \in X$ a point of higher index and choosing a divisorial extraction $Y \to X$ from the point $Q \in X$ with minimal discrepancy $\frac{1}{r}$, we prove that there exists a diagram of birational maps:

(1.1)
$$\begin{tikzcd}
Y \arrow[rr, dotted] \arrow[d, "g"'] & & Y^\sharp \arrow[d, "g^\sharp"] \\
X \arrow[dr, "f"'] & & X^\sharp \arrow[dl, "f^\sharp"] \\
& W &
\end{tikzcd}$$

where $Y \dashrightarrow Y^\sharp$ consists of a sequence of flips and flops, $g^\sharp$ is a divisorial contraction, and $f^\sharp$ is also a divisorial contraction (resp. $f^\sharp$ is the flipped map). We thus refer to (1.1) as a *factoring diagram for $X \to W$ (resp. $X \dashrightarrow X^+$).*

The strategy of proof is as follows. If f is a weighted blowup, then the factoring diagram can be constructed using toric geometry and a few computations. This was the approach in [2]. For the remaining divisorial contractions which are not known to be weighted blowups, there is usually a unique non-Gorenstein singularity $P \in X$ of pretty high index. By choosing a divisorial contraction $g\colon Y \to X$ to P with minimal discrepancy, one verifies that the intersection numbers change rather little. Computation shows that $-K_{Y/W}$ is nef and one can thus play the so-called 2-ray game to obtain the factoring diagram.

Moreover, by considering *depth* (introduced in [5]) together with the discrepancy, one sees that $Y, Y^\sharp$ and $X^\sharp$ have milder singularities in some sense. Our result then follows by induction using the factoring diagram.

The reader may find that our work here has similar flavor to that of Corti in 3-fold Sarkisov Program, in which Corti proved that birational maps between Mori fiber spaces can be factorized into *Sarkisov links* (cf. [6]). We expect that some log version of our current work will be useful in various studies of geometry of 3-folds including the Sarkisov Program.

2. Notations and Preliminary

We always work on complex 3-folds with $\mathbb{Q}$-factorial singularities (except the image of a flipping contraction). Recall that 3-fold terminal singularities of index 1 are isolated cDV points and terminal singularities of index $r > 1$ are classified by Mori (cf. [21]).

This work can be considered as a continuation of our previous work [2, 5]. We usually adapt the constructions and notations there.

Given a 3-fold terminal singularity $P \in X$ of index $r > 1$, by [8, 9], there exists a partial resolution

$$X_n \to \cdots \to X_1 \to X_0 = X \tag{2.1}$$

such that X_n has Gorenstein singularities and each $X_{i+1} \to X_i$ is a divisorial contraction to a point $P_i \in X_i$ of index $r_i > 1$ with discrepancy $\frac{1}{r_i}$. The definition of depth was introduced in [5].

$$\text{dep}(P \in X) := \min\{n \mid X_n \to X \ni P \text{ is a partial resolution as in (2.1)}\}.$$

The following properties of depth are useful.

Proposition 2.1. *The following properties of depth holds.*

(1) *Let* $\phi\colon X \dashrightarrow X^+$ *be a flip (resp. flop). Then* $\text{dep}(X) > \text{dep}(X^+)$ *(resp.* $\text{dep}(X) = \text{dep}(X^+)$*).*

(2) *Let* $f\colon X \to W$ *be a divisorial contraction to a curve. Then* $\text{dep}(X) \geq \text{dep}(W)$. *Equality holds if and only if* $\text{dep}(X) = \text{dep}(W) = 0$.

(3) *Let* $f\colon X \to W$ *be a divisorial contraction to a point. Then* $\text{dep}(X) + 1 \geq \text{dep}(W)$.

Proof. All the statements were proved in [5, Propositions 2.15, 3.5, 3.6] except for the strict inequality for divisorial contractions to curves when $\text{dep}(X) > 0$. Recall that by [5], there is a factoring diagram

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[d, "g"] & Y^\sharp \arrow[d, "g^\sharp"] \\ X \arrow[dr, "f"'] & X^\sharp \arrow[d, "f^\sharp"] \\ & W \end{tikzcd}$$

such that $Y \to X$ is a divisorial contraction to a point Q in X of higher index point $r(Q \in X) > 1$ and discrepancy $\frac{1}{r}$, and dep$(Y) =$ dep$(X) - 1$. Moreover, $Y^\sharp \to X^\sharp$ is a divisorial contraction to a curve and $X^\sharp \to W$ is a divisorial contraction to a point.

Suppose that dep$(X) = 1$. Then dep$(W) > 0$ would imply that dep$(W) = 1$, since dep$(X) \geq$ dep(W) (cf. [5, Proposition 3.6]). Then by definition of depth, it is easy to see that W has only one quotient singularity of type $\frac{1}{2}(1,1,1)$. It follows that $X \to W$ is the weighted blowup with weights $v = \frac{1}{2}(1,1,1)$ by [16], which is absurd. We thus conclude that dep$(W) = 0 <$ dep(X).

In general dep$(X) = d > 1$. Then dep$(Y^\sharp) \leq$ dep$(Y) = d - 1$. By induction, one has dep$(X^\sharp) <$ dep$(Y^\sharp) \leq d - 1$. It follows that dep$(W) \leq$ dep$(X^\sharp) + 1 < d$ by [5, Proposition 2.15]. ■

3. Divisorial Contractions to Curves

The purpose of this section is to factorize 3-fold divisorial contraction to curves. Let $f\colon X \to W$ be a divisorial contraction to a curve $\Gamma \subset W$ such that X has at worst terminal Gorenstein singularities. By [7, 20], it is known that W is smooth near Γ and $\Gamma \subset W$ is locally a plane curve. Moreover, f is the blowup of Γ over the general point.

If Γ is a nonsingular curve, then $f\colon X \to W$ is nothing but the blowups of Γ. If the curve Γ is singular at o, then one can factorize the divisorial contraction $f\colon X \to W$ by the factoring diagram.

Proposition 3.1. *Let $\Gamma \subset W$ be a singular plane curve and suppose that $\tau =$ $\mathrm{mult}_o\Gamma \geq 2$. Then there is a factoring diagram (1.1) such that*

(1) *$Y \dashrightarrow Y^\sharp$ consists of a sequence of flops;*

(2) *$f^\sharp$ is the weighted blowup along $o \in W$ with weight $(1,1,\tau-1)$;*

(3) *$g^\sharp$ is the blowup of $X^\sharp$ of $\Gamma^\sharp$ over the general point, where $\Gamma^\sharp$ is the proper transform of Γ in $X^\sharp$;*

(4) *the induced map $\Gamma^\sharp \to \Gamma$ is isomorphic to the blowup of Γ at o;*

(5) *g is a divisorial contraction to a singular point $Q \in X$ of type cA with discrepancy 1.*

Proof. Recall that a weighted blowup for a toric variety is obtained by barycentric subdivision of its cone along a primitive vector v, with exceptional divisor corresponding to the vector v. Also a weighted blowup for a complete intersection in a toric variety is considered to be the induced map from its proper transform. For detailed description, see [2] for example.

By shrinking W, we may assume that W is an open subset in $\mathbb{C}^3$, $\Gamma = (x_3 = h(x_1, x_2) = 0) \subset W \subset \mathbb{C}^3$ and $o \in \Gamma$ is the only singular point of Γ. Instead, we consider another embedding $W \hookrightarrow \mathbb{C}^4$ that $W = (x_4 - h(x_1, x_2) = 0)$.

We consider towers of weighted blowups $\mathcal{X}_2 \stackrel{\pi_g}{\to} \mathcal{X}_1 \stackrel{\pi_f}{\to} \mathcal{X}_0$, where $\mathcal{X}_0 = \mathbb{C}^4$, π_g (resp π_f) are weighted blowup along the vector $v_2 = (1, 1, \tau - 1, \tau)$ (resp $v_1 = (0, 0, 1, 1)$). More explicitly, π_f is the blowup of $\mathcal{X}_0$ along $\Sigma := (x_3 = x_4 = 0)$ and $\mathcal{X}_1$ is covered by two affine pieces $U_3 \cup U_4$. One sees also that π_g is the weighted blowup over the origin of U_3 with weights $(1, 1, \tau - 1, 1)$.

Now $\Gamma = W \cap \Sigma$ and the given divisorial contraction $f \colon X \to W$ coincides with the induced map $\pi_{f|_X}$. On X, there is a unique singularity Q_3 of cA type locally given by $x_3x_4 - h(x_1x_2) = 0$. Moreover, let Y be the proper transform of X in $\mathcal{X}_2$. The induced map $g \colon Y \to X$, which is the weighted blowup with weights $(1, 1, \tau - 1, 1)$ over Q_3, is clearly a divisorial contraction to Q_3 with discrepancy 1.

Let $l := f^{-1}(o) \cong \mathbb{P}^1$ and l_Y be the proper transform of l in Y. It is easy to see that $l \cdot K_X = -1$ and $l_Y \cdot K_Y = 0$. We remark that there is only one singularity on Y, which is a quotient singularity of index $\tau - 1$ and is not contained in l_Y. By the same argument in [5, Theorem 3.3], one has a factoring diagram (1.1) and a tower of divisorial contractions $Y^\sharp \to X^\sharp \to W$.

On the other hand, we may consider $Y' \to X' \to W$ by weighted blowup with vector $v_2 = (1, 1, \tau - 1, \tau)$ and then $v_1 = (0, 0, 1, 1)$. By the same argument as in [2, Theorem 2.7], the tower $Y' \to X' \to W$ is isomorphic to $Y^\sharp \to X^\sharp \to W$.

Let $\Gamma^\sharp$ be the proper transform of Γ in $X^\sharp$. Computation shows that $\Gamma^\sharp \to \Gamma$ are isomorphic to the blowup over $o \in \Gamma$. Moreover, $Y^\sharp \to X^\sharp$ is the blowup of $\Gamma^\sharp$ over the general point. The only singularity on $X^\sharp$ is a quotient singularity $Q_3^\sharp$ of index $\tau - 1$ and $\Gamma^\sharp$ does not contains $Q_3^\sharp$. Therefore $\mathrm{dep}(X^\sharp) = \mathrm{dep}(Y^\sharp) = \tau - 1 = \mathrm{dep}(Y)$. It follows that $Y \dashrightarrow Y^\sharp$ consists of a sequence of flops only by Proposition 2.1. This completes the proof. ∎

4. Divisorial Contractions to Points

Divisorial contractions to points were intensively studied by Kawamata, Hayakawa, and Kawakita [8, 9, 11, 12, 13, 14, 15, 16]. We give a brief summary of the known classification.

- If $f \colon X \to W \ni P$ is a divisorial contraction to a point $P \in W$ of index $r > 1$ with discrepancy $\frac{a}{r} \geq \frac{1}{r}$, then f is completely classified. Any of these can be realized as a weighted blowup explicitly (cf. [8, 9, 14, 15, 16]).
- If $f \colon X \to W \ni P$ is a divisorial contraction to a point $P \in W$ of index $r = 1$ with discrepancy $a > 1$, then f is one of cases in the following Table A. Note that in the case Ia, f is the weighted blowup with weight $(1, m, n)$ and discrepancy $a = m + n$.

Table A.

type	$P \in W$	discrepancy	w. blowup	reference
Ia	nonsingular	$m+n$	Yes	[11, Theorem 1.1]
Ib	cA	$a \geq 1$	Yes	[14, Theorem 1.2.i]
Ic	cD	$a > 1$, odd	Yes	[14, Theorem 1.2.ii.a]
Id	cD	$a > 1$	Yes	[14, Theorem 1.2.ii.b]
IIa	cA_1	4	Yes	[12, Theorem 2.5]
IIb	$cE_{7,8}$	2	?	[14, Table 3, e9]
IIc	cE_7	2	?	[14, Table 3, e5]
IId	cA_2, cD, cE_6	3	?	[14, Table 3, e3]
IIe	$cD, cE_{6,7}$	2	?	[14, Table 3, e2]
IIf	cD	2	?	[14, Table 3, e1]
IIg	cD	4	?	[14, Table 3, e1]

The purpose of this section is to construct a factoring diagram (1.1) for divisorial contraction with non-minimal discrepancy $a > 1$ as listed in Table A. Given a divisorial contraction with non-minimal discrepancy $f\colon X \to W \ni P$. Let E be its exceptional divisor. By the classification of [7, 20], X cannot be Gorenstein. We will pick a point $Q \in X$ of index $p > 1$.

For any divisor D on X passing through Q, we set $D_W = f_*D$ and $D_Y = g_*^{-1}D$ to be the proper transform of D on W and Y respectively. Let E_Y denotes the proper transform of E on Y. We have

$$f^*D_W = D + \frac{c_0}{n}E, \quad g^*D = D_Y + \frac{q_0}{p}F, \quad g^*E = E_Y + \frac{\mathfrak{q}}{p}F$$

for some $c_0, q_0, \mathfrak{q} \in \mathbb{Z}_{>0}$.

Proposition 4.1. [2, Proposition 2.4] *Let $f\colon X \to W$ be a divisorial contraction to a point $P \in W$ of index n with discrepancy $\frac{a}{n}$ and E the exceptional divisor of f. Let $g\colon Y \to X$ be a divisorial contraction to a point $Q \in E$ of index p with discrepancy $\frac{b}{p}$. Suppose that there is a divisor D on X such that $D \cap E$ is irreducible. Then $-K_{Y/W}$ is nef if the following inequalities hold:*

$$\begin{cases} T(f,g,D) := \frac{-ac_0}{n^2}E^3 + \frac{q_0\mathfrak{q}b}{p^3}F^3 \leq 0; \\ bc_0 - aq_0 \leq 0. \end{cases}$$

In [15, Theorem 1.5], Kawakita give an affirmative answer to the General Elephant Conjecture. In particular, let $f\colon X \to W$ be a divisorial contraction, then a general element $S_X \in |-K_X|$ is normal and has only Du Val singularities.

Proposition 4.2. [2, Proposition 2.5] *Let $f\colon X \to W$ be a divisorial contraction to a point with exceptional divisors E and let $g\colon Y \to X$ be a divisorial contraction* *to a point $Q \in E \subset X$ of index p with discrepancy $\frac{1}{p}$. Let F be the exceptional divisor of g. Suppose that $-K_{Y/W}$ is nef and there is an irreducible curve $l \subset S_X \cap E$ such that $l_Y \cdot K_Y < 0$. Then we have a factoring diagram (1.1) such that*

(1) *$\phi\colon Y \dashrightarrow Y^\sharp$ is a sequence of flips and flops;*

(2) *$g^\sharp$ is a divisorial contraction contracting $E_{Z^\sharp}$;*

(3) *$f^\sharp$ contracts $F_{Y^\sharp}$ to the point $P \in W$.*

We will need the following variant. The proof is almost the same as [2, Corollary 2.6].

Corollary 4.3. *Let $f\colon X \to W$ be a divisorial contraction to a point with exceptional divisor E and let $g\colon Y \to X$ be a divisorial contraction to a point $Q \in E \subset X$ of index p with discrepancy $\frac{1}{p}$. Let F be the exceptional divisor of g. Suppose that $l_Y \cdot K_Y \le 0$ for any irreducible curve $l \subset S_X \cap E$ and $T(f,g) := \frac{-a^2}{n^2}E^3 + \frac{\mathfrak{q}}{p^3}F^3 < 0$. Then we have a factoring diagram (1.1) as in Proposition 4.2.*

An immediate but useful consequence is the following:

Corollary 4.4. *Keep the notation as in Corollary 4.3. Suppose that $Q \in E$ is the only non-Gorenstein point on E, which is of index $p > 1$. Suppose furthermore that $\frac{\mathfrak{q}}{p^3}F^3 < \frac{1}{p}$. Then there exists a factoring diagram (1.1) as in Proposition 4.2.*

Proof. Suppose that $[S_X \cap E] = [\sum c_i l_i]$ as 1-cycle for some $c_i \in \mathbb{Z}_{>0}$. Note that $l_{i,Y} \cdot K_Y \ge l_i \cdot K_X$ for all i. Hence for all i,

$$\begin{aligned} l_{i,Y} \cdot K_Y &= l_i \cdot K_X + (l_{i,Y} \cdot K_Y - l_i \cdot K_X) \\ &\le l_i \cdot K_X + \sum_i c_i(l_{i,Y} \cdot K_Y - l_i \cdot K_X) \\ &\le \frac{-1}{p} + \frac{\mathfrak{q}}{p^3}F^3 < 0. \end{aligned}$$

By Corollary 4.3, there exists a factoring diagram. ∎

We remark that once there is a factoring diagram, then the induced map $f^\sharp\colon X^\sharp \to W$ is a divisorial contraction to $P \in W$ with exceptional divisor $F_{X^\sharp}$ and discrepancy $\mathfrak{a} := \frac{a\mathfrak{q}+n}{p} \in \mathbb{Z}_{>0}$.

We now study the divisorial contraction to a Gorenstein point with non-minimal discrepancies case by case (cf. Table A).

Case Ia. Suppose that $P \in W$ is nonsingular.
By [11], f is the weighted blowup of weight $(1, m, n)$ with $(m, n) = 1$, $1 \le m < n$, and the discrepancy is $a = m + n$.

On X, the highest index point, say Q, is a terminal quotient singularity of type $\frac{1}{n}(1,m,-1)$. Let $g\colon Y \to X$ be the Kawamata blowup, which is the weighted blowup of weights $\frac{1}{n}(t,1,n-t)$, where t is the minimal positive integer satisfying $mt = ns+1$. Clearly $1 \le t < n, 0 \le s < m$.

Pick $D = f_*^{-1}\mathrm{div}(x_2)$. Then $l = D \cap E$ is clearly irreducible. Since $c_0 = m, q_0 = 1$ and $\mathfrak{q} = n-t$, one has

$$T(f,g,D) = -\frac{m+n}{n} + \frac{1}{nt} < 0.$$

Hence we have the factoring diagram by Proposition 4.2. By Theorem 2.7 of [2], one sees that both $f^\sharp, g^\sharp$ are weighted blowups. Indeed, the factoring diagram fits into the following diagram.

$$\begin{tikzcd}
Y \arrow[rr, dotted] \arrow[d, "\frac{1}{n}"', "\text{wt}=w_2"] & & Y^\sharp \arrow[d, "s+t"', "\text{wt}=w_2'"] \\
Q_3 \in X \arrow[d, "m+n"', "\text{wt}=w_1"] & & X^\sharp \ni Q_1^\sharp \arrow[d, "m+n-s-t"', "\text{wt}=w_1'"] \\
W \arrow[rr] & & W
\end{tikzcd}$$

where

$$\begin{aligned} w_1 &= (1,m,n), & w_1' &= (1, m-s, n-t), \\ w_2 &= \frac{1}{n}(t,1,n-t), & w_2' &= (1,s,t). \end{aligned}$$

Remark 4.5. In the case that $m=1$, then $Y^\sharp \to X^\sharp$ is a weighted blowup with weights $w_2' = (1,0,1)$. Hence it is the blowup along a smooth curve in $X^\sharp$. Notice also that $\mathrm{dep}(Y^\sharp) = \mathrm{dep}(X^\sharp) = n-2 = \mathrm{dep}(Y)$. Therefore, $Y \dashrightarrow Y^\sharp$ consists of a sequence of flops in this situation. By induction on n, it follows in particular that a weighted blowup over a smooth point of weights $(1,1,n)$ is factorizable.

Case Ib. This contraction is described in [11, Theorem 1.2.i]. In fact, the factoring diagram is described in [2, Subsection 3.5] with $n=1$. We give a brief review for the reader's convenience. The equation of $P \in W$ is given by

$$(\varphi : x_1x_2 + g(x_3,x_4) = 0) \subset \mathbb{C}^4.$$

The map f is given by weighted blowup with weight $v_1 = (r_1, r_2, a, 1)$. We may write $r_1 + r_2 = da$ for some $d > 0$ with the term $x_3^d \in \varphi$. Moreover, $(a, r_1) = (a, r_2) = 1$. Hence, there exist $0 < s_i^* < r_i$ and $0 < a_i < a$ so that

$$\begin{cases} 1 + a_1 r_1 = s_1^* a; \\ 1 + a_2 r_2 = s_2^* a. \end{cases}$$

Note that $as_2^* = 1 + a_2 r_2 = 1 + a_2(ad - r_1)$. Therefore, $a(s_2^* - a_2 d) = 1 - a_2 r_1$. By $(a, r_1) = 1$ and comparing it with $as_1^* = 1 + a_1 r_1$, we have $a_1 = -a_2 + ta$ for some $t \in \mathbb{Z}$. Since $0 < a_1 + a_2 < 2a$, it follows that $a_1 + a_2 = a$.

Suppose that $r_1 > 1$. We have the following factoring diagram.

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[d, "\frac{1}{r_1}"', "\text{wt}=w_2"] & Y^\sharp \arrow[d, "a_1"', "\text{wt}=w_2'"] \\ Q_1 \in X \arrow[d, "a"', "\text{wt}=w_1"] & X^\sharp \ni Q_4^\sharp \arrow[d, "a_2"', "\text{wt}=w_1'"] \\ W \arrow[r] & W \end{tikzcd}$$

where

$$\begin{aligned} w_1 &= (r_1, r_2, a, 1), & w_1' &= (r_1 - s_1^*, r_2 - a_1 d + s_1^*, a_2, 1) \\ w_2 &= \frac{1}{r_1}(r_1 - s_1^*, d, 1, s_1^*), & w_2' &= (s_1^*, a_1 d - s_1^*, a_1, 1). \end{aligned}$$

Suppose that $r_2 > 1$. We have the following factoring diagram.

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[d, "\frac{1}{r_2}"', "\text{wt}=w_2"] & Y^\sharp \arrow[d, "a_2"', "\text{wt}=w_2'"] \\ Q_2 \in X \arrow[d, "a"', "\text{wt}=w_1"] & X^\sharp \ni Q_4^\sharp \arrow[d, "a_1"', "\text{wt}=w_1'"] \\ W \arrow[r] & W \end{tikzcd}$$

where

$$\begin{aligned} w_1 &= (r_1, r_2, a, 1), & w_1' &= (r_1 + s_2^* - a_2 d, r_2 - s_2^*, a_1, 1) \\ w_2 &= \frac{1}{r_2}(d, r_2 - s_2^*, 1, s_2^*), & w_2' &= (a_2 d - s_2^*, s_2^*, a_2, 1). \end{aligned}$$

Case Ic. This contraction is described in [11, Theorem 1.2.ii.a] and the discussion is parallel to that in [2, Subsection 3.2]. The local equation of $P \in W$ is given by

$$(\varphi : x_1^2 + x_2^2 x_4 + x_1 q(x_3^2, x_4) + \lambda x_2 x_3^2 + \mu x_3^3 + p(x_2, x_3, x_4) = 0) \subset \mathbb{C}^4,$$

f is the weighted blowup with weights $v_1 = (r+1, r, a, 1)$, where $2r+1 = ad$ and both a, d are odd. Notice that $wt_{v_1}(\varphi) = 2r+1$ and we have that $x_3^d \in p(x_2, x_3, x_4)$ otherwise $Q_3 \in X$ is singular of index a.

There are two quotient singularities Q_1, Q_2 of index $r+1, r$ respectively. We take $g\colon Y \to X$ the weighted blowup with weights $w_2 = \frac{1}{r}(d, r-d, 1, d)$ over Q_2. Then

$$E^3 = \frac{2r+1}{ar(r+1)}, \quad F^3 = \frac{r^2}{d(r-d)}, \quad \mathfrak{q} = r-d, \quad \mathfrak{a} = a-2.$$

In this case, we pick $S = f_*^{-1}\mathrm{div}(x_3) \in |-K_X|$, then $S \cap E$ is irreducible. Now

$$T(f,g) = \frac{1}{r}\left(-\frac{a(2r+1)}{r+1} + \frac{1}{d}\right) < 0.$$

Therefore there exists a factoring diagram by Proposition 4.2.

$$\begin{tikzcd}
Y \arrow[r, dotted] \arrow[d, "\frac{1}{r}"', "\mathrm{wt}=w_2"] & Y^\sharp \arrow[d, "2"', "\mathrm{wt}=w_2'"] \\
Q_2 \in X \arrow[d, "a"', "\mathrm{wt}=w_1"] & X^\sharp \ni Q_4^\sharp \arrow[d, "a-2"', "\mathrm{wt}=w_1'"] \\
W \arrow[r] & W
\end{tikzcd}$$

where

$$\begin{aligned} w_1 &= v_1 = (r+1, r, a, 1), \quad & w_1' &= v_2 = (r+1-d, r-d, a-2, 1), \\ w_2 &= \frac{1}{r}(d, r-d, 1, d), & w_2' &= (d, d, 2, 1). \end{aligned}$$

Case Id. In the case (1.2.ii.b), the local equation of $P \in W$ is given by

$$(P \in W) \cong o \in \left(\begin{array}{l} \varphi_1 : x_1^2 + x_2x_5 + p(x_2, x_3, x_4) = 0 \\ \varphi_2 : x_2x_4 + x_3^d + q(x_3, x_4)x_4 + x_5 = 0 \end{array} \right) \subset \mathbb{C}^5,$$

f is a weighted blowup with weights $v_1 = (r+1, r, a, 1, r+2)$, where $r+1 = ad$. There are quotient singularities Q_2, Q_5 of index $r, r+2$ respectively. We take $g\colon Y \to X$ the weighted blowup with weights $w_2 = \frac{1}{r+2}(d, 2d, 1, r-d+2, d)$ over Q_5. Then

$$E^3 = \frac{2r+2}{ar(r+2)}, \quad F^3 = \frac{(r+2)^2}{d(r-d+2)}, \quad \mathfrak{q} = d, \quad \mathfrak{a} = 1.$$

We pick $D = f_*^{-1}\text{div}(x_2)$. It is easy to check that $E \cap D$ is irreducible but non-reduced. We have $c_0 = r, q_0 = 2d$, hence $c_0 - aq_0 < 0$ and moreover

$$T(f,g,D) = \frac{1}{r+2}\left(-(2r+2) + \frac{2d}{r-d+2}\right) < 0.$$

There exists a factoring diagram by Proposition 4.2.

$$\begin{tikzcd}
Y \arrow[r, dotted] \arrow[d, "\frac{1}{r+2}"', "\text{wt}=w_2"] & Y^\sharp \arrow[d, "a-1"', "\text{wt}=w_2'"] \\
Q_5 \in X \arrow[d, "a"', "\text{wt}=w_1"] & X^\sharp \ni Q_4^\sharp \arrow[d, "1"', "\text{wt}=w_1'"] \\
X \arrow[r] & X
\end{tikzcd}$$

where

$$\begin{aligned}
w_1 &= v_1 = (r+1, r, a, 1, r+2),\\
w_2 &= \frac{1}{r+2}(d, 2d, 1, r-d+2, d),\\
w_1' &= v_2 = (d, d, 1, 1, d),\\
w_2' &= (r-d+1, r-d, 2, a-1, 1, r-d+2).
\end{aligned}$$

Case IIa. This contraction is described in [12, Theorem 1.1.(2)]. The local equation of $P \in W$ is given by

$$(\varphi : x_1x_2 + x_3^2 + x_4^3 = 0) \subset \mathbb{C}^4,$$

and f is the weighted blowup with weights $v_1 = (1,5,3,2)$.

There is a unique singularity Q_2 on E, which is a quotient singularities of index 5. We take $g\colon Y \to X$ the weighted blowup with weights $w_2 = \frac{1}{5}(4,1,2,3)$ over Q_2. Thus $\mathfrak{q} = 1$, $\mathfrak{a} = 1$ and $\frac{\mathfrak{q}}{5^3}F^3 = \frac{1}{30} < \frac{1}{5}$. Therefore there exists a factoring diagram by Corollary 4.4.

$$\begin{tikzcd}
Y \arrow[r, dotted] \arrow[d, "\frac{1}{5}"', "\text{wt}=w_2"] & Y^\sharp \arrow[d, "3"', "\text{wt}=w_2'"] \\
Q_2 \in X \arrow[d, "4"', "\text{wt}=w_1"] & X^\sharp \ni Q_1^\sharp \arrow[d, "1"', "\text{wt}=w_1'"] \\
W \arrow[r] & W
\end{tikzcd}$$

where

$$\begin{aligned} w_1 &= v_1 = (1,5,3,2), \quad w_1' = v_2 = (1,1,1,1), \\ w_2 &= \frac{1}{5}(4,1,2,3), \qquad w_2' = (1,4,2,1). \end{aligned}$$

Case IIb. f is of type e9 with discrepancy 2. This case was studied in [13]. We summarize some results in [13]. There are two singularities Q_1, Q_2 of type $\frac{1}{5}(1,1,-1)$ and $\frac{1}{3}(1,1,-1)$ respectively. Pick any general elephant $S \in |-K_X|$, then $[S \cap E] = 2[l]$, where $l \cong \mathbb{P}^1$ and l passes through both Q_1, Q_2 [13, Lemma 5.1]. We may assume that, near Q_1, $S = \operatorname{div}(x)$, $E = \operatorname{div}(y^2)$ (after coordinate change) and $l = (x = y = 0)$. Now $E^3 = \frac{1}{15}$ and $l \cdot E = \frac{-1}{15}$.

Let $g\colon Y \to X$ be the Kawamata blowup over Q_1 with weights $\frac{1}{5}(1,1,4)$. One sees that $\mathfrak{q} = 2$, $\mathfrak{a} = 1$. Notice that

$$2l_Y \cdot K_Y = 2l \cdot K_X + \frac{2}{5^3}F^3 = \frac{-2}{15} + \frac{2}{20} < 0.$$

By Proposition 4.2, there exists a factoring diagram.

$$\begin{tikzcd} Y \arrow[rr, dotted] \arrow[d, "g"', "\frac{1}{5}"] & & Y^\sharp \arrow[d, "g^\sharp"] \\ X \arrow[dr, "f"', "a=2"] & & X^\sharp \arrow[dl, "f^\sharp", "1"'] \\ & W & \end{tikzcd}$$

where $f^\sharp$ is a divisorial contraction with exceptional divisor $F_{X^\sharp}$ and discrepancy $\mathfrak{a} = 1$.

Case IIc. f is of type e5 with discrepancy 2.
There is only one singularity $Q \in X$, which is of type $\frac{1}{7}(1,1,6)$. Let $g\colon Y \to X$ be the weighted blowup of weights $\frac{1}{7}(1,1,6)$ over Q and let $\mu\colon Z \to Y \to X \ni Q$ be the economic resolution by further weighted blowups. Clearly,

$$\begin{cases} K_Z = \mu^* K_X + \displaystyle\sum_{j=1}^{6} \frac{j}{7} F_j; \\ \mu^* E = E_Z + \displaystyle\sum_{j=1}^{6} \frac{q_j}{7} F_j, \end{cases}$$

for some q_j, where $F_1 = F$ is the exceptional divisor of g. Hence

$$K_Z = \mu^* f^* K_W + 2E_Z + \sum_{j=1}^{6} a_j F_{j,Z}$$

with $a_j = \frac{2q_j+j}{7} \in \mathbb{Z}$.

Suppose that E is given by $(\phi : \sum c_{\alpha\beta\gamma}x^\alpha y^\beta z^\gamma = 0) \subset \mathbb{C}^3/\frac{1}{7}(1,1,6)$ locally around Q. Then

$$q_j := \min\{\alpha j + \beta j + \gamma(7-j) | x^\alpha y^\beta z^\gamma \in \phi\} \geq \min\{j, 7-j\}.$$

By [19], there must exists an exceptional divisor with discrepancy 1 centering at $P \in W$. Since $Z \to W$ is a Gorenstein partial resolution, the exceptional divisor with discrepancy 1 must appear in Z, that is, among $\{F_{j,Z}\}_{j=1,\ldots,6}$. One can verify that F_1 is the only exceptional divisor with discrepancy 1 and $\mathfrak{q} = q_1 = 3$. Hence $\frac{\mathfrak{q}}{p^3}F^3 = \frac{1}{14} < \frac{1}{7}$. By Corollary 4.4, we have a factoring diagram so that $f^\sharp \colon X^\sharp \to W$ is a divisorial contraction contracting $F_{X^\sharp}$ with discrepancy $\mathfrak{a} = 1$.

Case IId. f is of type e3 with discrepancy 3.
There is only one singularity $Q \in X$, which is of type $cAx/4$ with axial weight 2. More precisely, $Q \in X$ is given by

$$(\varphi : x^2 + y^2 + f(z,u) = 0) \subset \mathbb{C}^4/\tfrac{1}{4}(1,3,1,2),$$

such that $u^3 \in \varphi$ and $wt_{\frac{1}{4}(1,2)}f(z,u) = \frac{6}{4}$. By [8, Theorem 7.4], there is a unique divisorial contraction $g \colon Y \to X$ over Q with discrepancy $\frac{1}{4}$, which is the weighted blowup of weights $\frac{1}{4}(5,3,1,2)$. Take economic resolution $\nu \colon Z \to Y$ over the unique higher index point, which is a quotient singularity of index 5, and let $\mu \colon Z \xrightarrow{g \circ \nu} X$. Then we ends up with

$$\begin{cases} K_Z = \mu^* K_X + \dfrac{1}{4}F + \displaystyle\sum_{j=1}^{4} \frac{b_j}{4}F_j; \\ \mu^* E = E_Z + \dfrac{\mathfrak{q}}{4}F + \displaystyle\sum_{j=1}^{4} \frac{q_j}{4}F_j, \end{cases}$$

where F_j are ν-exceptional divisors and $(b_1,b_2,b_3,b_4) = (2,2,3,4)$. Hence

$$K_Z = (f \circ \mu)^* K_W + \mathfrak{a}F + \sum_{j=1}^{4} a_j F_j,$$

where $\mathfrak{a} = \frac{1+3\mathfrak{q}}{4}$ and $a_j = \frac{b_j + 3q_j}{4}$. Since $a_j := \frac{b_j+3q_j}{4} > 1$ for all j, it follows that F is the only exceptional divisor with discrepancy 1 over $P \in W$ and hence $\mathfrak{q} = 1$ and $\mathfrak{a} = 1$. Thus $\frac{\mathfrak{q}}{p^3}F^3 = \frac{1}{20} < \frac{1}{4}$. By Corollary 4.4, we have a factoring diagram such that $f^\sharp \colon X^\sharp \to W$ is a divisorial contraction with exceptional divisor $F_{X^\sharp}$ and discrepancy $\mathfrak{a} = 1$.

Case IIe. f is of type e2 with discrepancy 2.

There is a unique higher index point $Q \in X$ of type cA/r or $cD/3$ with axial weight 2.

Subcase 1. Q is of type $cD/3$.

Let $\mu\colon Z \to X$ be a common resolutions of Q dominating all divisorial contractions with minimal discrepancies over Q. We have

$$K_Z = \mu^* K_X + \sum_{j=1}^{N} \frac{1}{3} F_j + \sum \frac{c_l}{3} G_l,$$

where $\{F_j\}_{j=1,\ldots,N}$ is the set all all exceptional divisors with discrepancy $\frac{1}{3}$ over Q and $c_l \geq 2$. Suppose that $\mu^* E = E_Z + \sum \frac{q_j}{3} F_j + \sum \frac{t_l}{3} G_l$, then

$$K_X = \mu^* f^* K_W + 2E_Z + \sum_{j=1}^{N} a_j F_j + \sum b_l G_l,$$

where $a_j = \frac{2q_j+1}{3}$ and $b_l = \frac{2t_l+c_l}{3} > 1$. Since there exists an exceptional divisor with discrepancy 1 over $P \in W$, we may assume that $a_1 = 1$.

By [10, Section 9], a $cD/3$ point can be classified as $cD/3$-1, $cD/3$-2 and $cD/3$-3. Unless $Q \in X$ is of type $cD/3$-3 and Equation $*$ holds (cf. [10, p. 549]), we know that any exceptional divisor with minimal discrepancy $\frac{1}{3}$ over a $cD/3$ point is obtained by a divisorial contraction. Hence there is a divisorial contraction $g\colon Y \to X$ with exceptional divisor $F = F_1$ and discrepancy $\frac{1}{3}$. We thus have $\mathfrak{q} = 1$ and $\mathfrak{a} = 1$.

It is also straightforward to check that $\frac{\mathfrak{q}}{3^3} F^3 = \frac{1}{12}$ for any such divisorial contraction with discrepancy $\frac{1}{3}$. By Corollary 4.4, we have a factoring diagram such that $f^\sharp\colon X^\sharp \to W$ is a divisorial contraction with exceptional divisor $F_{X^\sharp}$ and discrepancy 1.

In the remaining situation that $Q \in X$ is of type $cD/3$-3 and Equation $*$ holds (cf. [10, p. 549]), then there is only one divisorial contraction $g\colon Y \to X$, which is a weighted blowup with weights $v_1 = \frac{1}{3}(5,4,1,6)$. There is another valuation with discrepancy $\frac{1}{3}$ given by the weighted blowup with weights $v_2 = \frac{1}{3}(2,4,1,3)$. We claim that we have $\mathfrak{a} = a_1 = 1$.

To this end, we write $K_Z = \mu^* K_X + \frac{1}{3} F_1 + \frac{1}{3} F_2 + \sum \frac{c_l}{3} G_l$, and

$$K_Z = \mu^* f^* K_W + 2E_Z + a_1 F_1 + a_2 F_2 + \sum b_l G_l,$$

where F_i corresponds to the valuation with weights v_i for $i = 1, 2$.

Let $(\phi = 0) \subset \mathbb{C}^3/\frac{1}{3}(2,1,1,0)$ be the local equation of E near Q. We know that there exists a divisor with discrepancy 1. Suppose that $a_2 = 1$, then $q_2 = 1$ and $\frac{q_2}{3} = wt_{v_2}(\phi) = \frac{1}{3}$. One sees that ϕ contains z. It follows that $\frac{q_1}{3} = wt_{v_1}(\phi) = \frac{1}{3}$ and hence $q_1 = 1$ and $a_1 = 1$ holds. This proves the claim.

Now we have $\frac{\mathfrak{q}}{3^3}F^3 = \frac{1}{10}$. By Corollary 4.4 again, we have a factoring diagram such that $f^\sharp\colon X^\sharp \to W$ is a divisorial contraction with exceptional divisor $F_{X^\sharp}$ and discrepancy 1.

Subcase 2. Q is of type cA/r.

After coordinate changes, we may assume that local equation near Q is given by $(\varphi : xy + z^{tr} + u^2 = 0) \subset \mathbb{C}^4/\frac{1}{r}(1,-1,2,r)$ for some $t \geq 2$. Set $r = 2k+1$. Let $Y \to X$ be the weighted blowup with weights $v_1 := \frac{1}{2k+1}(k+1, 3k+1, 1, 2k+1)$ with exceptional divisor F. There are quotient singularities R_1, R_2 of index $k+1, 3k+1$. Let $Z \to Y$ be the economic resolution of R_1, R_2. Then we have

$$\begin{aligned} K_Z = \mu^* K_X + \frac{1}{2k+1}F + \sum_{j=1}^{k} \frac{2j}{2k+1}F_j \\ + \sum_{i=1}^{k}\left(\frac{2i+1}{2k+1}G_{0i} + \frac{2i}{2k+1}G_{1i} + \frac{2i-1}{2k+1}G_{2i}\right). \end{aligned}$$

More explicitly, the resolution over R_1 is obtained by weighted blowups of weights $\frac{1}{k+1}(j, 2k+2-2j, j, k+1-j)$ for $1 \leq j \leq k$. Over Q these weights corresponds to vectors $\frac{1}{2k+1}(j, 4k+2-j, 2j, 2k+1)$. Similarly, the resolution over R_2 is obtained by weighted blowups of weights $\frac{1}{3k+1}(2i, 3k+1-i, 3i, i)$, $\frac{1}{3k+1}(2k+2i, 2k+1-i, 3i-1, k+i)$, and $\frac{1}{3k+1}(4k+2i, k+1-i, 3i-2, 2k+i)$ for $1 \leq i \leq k$. Over Q, these weights corresponds to vectors

$$\begin{cases} \frac{1}{2k+1}(k+1+i, 3k+1-i, 2i+1, 2k+1), \\ \frac{1}{2k+1}(2k+1+i, 2k+1-i, 2i, 2k+1), \\ \frac{1}{2k+1}(3k+1+i, k+1-i, 2i-1, 2k+1). \end{cases}$$

for $1 \leq i \leq k$ respectively.

Suppose that E is given by $(\phi : \sum c_{\alpha\beta\gamma\delta}x^\alpha y^\beta z^\gamma u^\delta = 0) \subset \mathbb{C}^4/\frac{1}{r}(1,-1,2,r)$ locally around Q. We write $\mu^*E = E_Z + \frac{\mathfrak{q}}{2k+1}F + \sum_{j=1}^k \frac{q_j}{2k+1}F_j + \sum_{i=1}^k(\frac{t_{0i}}{2k+1}G_{0i} + \frac{t_{1i}}{2k+1}G_{1i} + \frac{t_{2i}}{2k+1}G_{2i})$ and hence

$$K_Z = \mu^* f^* K_W + 2E_Z + \mathfrak{a}F + \sum_{j=1}^{k} a_j F_j + \sum_{i=1}^{k}(b_{0i}G_{0i} + b_{1i}G_{1i} + b_{2i}G_{2i}),$$

with $\mathfrak{a} := \frac{2\mathfrak{q}+1}{2k+1}$, $a_j := \frac{2q_j+2j}{2k+1}$, $b_{0i} := \frac{2t_{0i}+2i+1}{2k+1}$, $b_{1i} := \frac{2t_{1i}+2i}{2k+1}$, $b_{2i} := \frac{2t_{2i}+2i-1}{2k+1}$. There exists an exceptional divisor with discrepancy 1. Hence either $\mathfrak{a}$, b_{0i} or $b_{2i} = 1$ for some i because a_j and b_{1i} are even.

Claim. $\mathfrak{a} = 1$.

Suppose that $b_{0i} = 1$ for some i. Then $t_{0i} = k - i$. Since

$$t_{01} = \min\{\alpha(k+1+i) + \beta(3k+1-i) + \gamma(2i+1) + \delta(2k+1) \mid x^\alpha y^\beta z^\gamma u^\delta \in \phi\}.$$

It follows that ϕ contains z^γ with $\gamma(2i+1) = k - i$. Hence

$$\frac{\mathfrak{q}}{2k+1} = wt_{v_1}\phi \le \frac{k-i}{2k+1} \le \frac{k-1}{2k+1}$$

and $\mathfrak{a} < 1$, a contradiction.

Suppose that $b_{2i} = 1$ for some i. Then similarly, one sees that ϕ contains z^γ with $\gamma(2i-1) = k - i + 1$. This leads to the same contradiction unless $b_{21} = 1$ and ϕ contains z^k. Hence $\mathfrak{q} = k$ and $\mathfrak{a} = 1$.

Now $\frac{\mathfrak{q}}{(2k+1)^3}F^3 = \frac{2k}{(k+1)(3k+1)(2k+1)} < \frac{1}{2k+1}$. By Corollary 4.4, there is a factoring diagram such that $f^\sharp$ is a divisorial contraction with discrepancy $\mathfrak{a} = 1$.

Case IIf. f is of type e1 with discrepancy 2.

In this case, there is a unique higher point Q of type $\frac{1}{r}(1,-1,4)$.

Subcase 1. $r = 4k+3$.

Let $Y \to X$ be the Kawamata blowup along Q with weights $\frac{1}{4k+3}(k+1, 3k+2, 1)$. Suppose that the local equation of E near Q is given by $(\phi : \sum c_{\alpha\beta\gamma}x^\alpha y^\beta z^\gamma = 0)$. Let $\mu\colon Z \to X$ be the economic resolution over Q, which factors through Y. Then we have

$$\begin{cases} K_Z = \mu^* K_X + \displaystyle\sum_{j=1}^{4k+2} \frac{j}{4k+3} F_j; \\ \mu^* E = E_Z + \displaystyle\sum_{j=1}^{4k+2} \frac{q_j}{4k+3} F_j, \end{cases}$$

where $F_1 = F$ and

$$q_j := \min\{\alpha\overline{(k+1)j} + \beta\overline{(3k+2)j} + \gamma j \mid x^\alpha y^\beta z^\gamma \in \phi\}.$$

We have $K_Z = g^* f^* K_W + 2E_Z + \sum_{j=1}^{4k+2} a_j F_j$ with $a_j = \frac{2q_j + j}{4k+3} \in \mathbb{Z}$. Note that $a_j \equiv j \pmod 2$ and $a_j = 1$ for some j.

Claim. $a_1 \le 3$.

Suppose on the contrary that $a_1 \ge 5$. For all monomial $x^\alpha y^\beta z^\gamma \in \phi$, we have

$$q_1 = \alpha(k+1) + \beta(3k+2) + \gamma \ge 10k + 7. \qquad \dagger 1$$

If $a_j = 1$ for some j, then

$$q_j = \begin{cases} 2k-2s+1 = (k+s+1)\alpha + (3k-s+2)\beta + (4s+1)\gamma, & \text{if } j = 4s+1; \\ 2k-2s = (3k+s+3)\alpha + (k-s)\beta + (4s+3)\gamma, & \text{if } j = 4s+3, \end{cases}$$

for some $x^\alpha y^\beta z^\gamma \in \phi$, which is a contradiction to †1.

Notice that if $a_1 = 3$, i.e., $q_1 = 6k+4$, then $y^2 \in \phi$ and $a_j = 1$ if and only if $j = 4s+3$ with $s < k$. In this case, there are exactly $k-1$ exceptional divisors with discrepancy 1. Hence $k \geq 2$ in this situation. Also, if $a_1 = 1$, then $q_1 = 2k+1$. Thus in any event,

$$\frac{\mathfrak{q}}{(4k+3)^3}F^3 = \frac{2\mathfrak{q}}{(k+1)(3k+2)(4k+3)} \leq \frac{4}{3(4k+3)}.$$

For any $l \subset S \cap E$, one has $l \cdot E \geq \frac{1}{4k+3}$ and hence $l \cdot K_X \leq \frac{-2}{4k+3}$. Therefore, $l_Y \cdot K_Y < 0$ for all i. Hence there exists a factoring diagram by Corollary 4.3. The resulting divisorial contraction $f^\sharp \colon X^\sharp \to W$ is a divisorial contraction with discrepancy 1 or 3.

Subcase 2. $r = 4k+1$.

Similarly, let $Y \to X$ be the Kawamata blowup along Q with weights $\frac{1}{4k+1}(3k+1, k, 1)$ and $\mu \colon Z \to X$ be the economic resolution over Q, which factors through Y.

Thus we have $K_Z = g^* f^* K_W + 2E_Z + \sum_{j=1}^{4k} a_j F_j$ with $a_j = \frac{2q_j + j}{4k+1} \in \mathbb{Z}$ and

$$q_j := \min\{\alpha\overline{(3k+1)j} + \beta\overline{kj} + \gamma j \mid x^\alpha y^\beta z^\gamma \in \phi\}.$$

Note that $a_j \equiv j \pmod 2$ and $a_j = 1$ for some j.

Claim. $a_1 = 1$.

Suppose on the contrary that $a_1 \geq 3$. For all monomial $x^\alpha y^\beta z^\gamma \in \phi$, we have

$$q_1 = \alpha(3k+1) + \beta k + \gamma \geq 6k+1. \qquad \dagger 2$$

Suppose that $a_j = 1$, it is straightforward to see that

$$q_j = \begin{cases} 2k-2s+1 = (k+s)\alpha + (3k-s+1)\beta + (4s-1)\gamma, & \text{if } j = 4s-1; \\ 2k-2s = (3k+s+1)\alpha + (k-s)\beta + (4s+1)\gamma, & \text{if } j = 4s+1, \end{cases}$$

for some $x^\alpha y^\beta z^\gamma \in \phi$, which is a contradiction to †2.

Now $\mathfrak{a} = a_1 = 1$, $\mathfrak{q} = 2k$ and thus

$$\frac{\mathfrak{q}}{(4k+1)^3}F^3 = \frac{4}{(3k+1)(4k+1)} \leq \frac{1}{4k+1}.$$

For any $l \subset S \cap E$, one has $l \cdot E \geq \frac{1}{4k+1}$ and hence $l \cdot K_X \leq \frac{-2}{4k+1}$. Therefore, $l_Y \cdot K_Y < 0$ for all i. Hence there exists a factoring diagram by Corollary 4.3. The resulting map $f^\sharp \colon X^\sharp \to W$ is a divisorial contraction with discrepancy 1.

Case IIg. f is of type e1 with discrepancy 4.

In this case, there is a unique higher index point Q of type $\frac{1}{r}(1,-1,8)$. One can work out this case similar to Case IIf.

Subcase 1. $r=8k+7$.

Let $Y\to X$ be the Kawamata blowup along Q with weights $\frac{1}{8k+7}(k+1,7k+6,1)$ and $\mu\colon Z\to X$ be the economic resolution over Q, which factors through Y. Suppose that the local equation of E near Q is given by $(\phi:\sum c_{\alpha\beta\gamma}x^\alpha y^\beta z^\gamma=0)$. Thus we have $K_Z=\mu^*f^*K_W+4E_Z+\sum_{j=1}^{8k+6}a_jF_j$ with $a_j=\frac{4q_j+j}{8k+7}\in\mathbb{Z}$ and

$$q_j:=\min\{\alpha\overline{(k+1)j}+\beta\overline{(7k+6)j}+\gamma j\mid x^\alpha y^\beta z^\gamma\in\phi\}.$$

Note that $a_j\equiv -j \pmod 4$ and $a_j=1$ for some j.

Claim. $a_1=3$ or 7.[1]

Suppose on the contrary that $a_1\geq 11$. For all monomial $x^\alpha y^\beta z^\gamma\in\phi$, we have

$$q_1\geq \alpha(k+1)+\beta(7k+6)+\gamma\geq 22k+19. \qquad \dagger 3$$

Suppose that $a_j=1$, it is straightforward to see that

$$q_j=\begin{cases} 2k-2s+1=(3k+s+3)\alpha+(5k-s+4)\beta+(8s+3)\gamma, & \text{if } j=8s+3;\\ 2k-2s=(7k+s+1)\alpha+(k-s)\beta+(8s+7)\gamma, & \text{if } j=8s+7,\end{cases}$$

for some $x^\alpha y^\beta z^\gamma\in\phi$, which is a contradiction to †3.

Now $\mathfrak{q}\leq 14k+12$ and thus

$$\frac{\mathfrak{q}}{(8k+7)^3}F^3=\frac{2\mathfrak{q}}{(k+1)(7k+6)(8k+7)}\leq\frac{4}{(k+1)(8k+7)}.$$

For any $l_i\subset S\cap E$, one has $l_i\cdot E\geq\frac{1}{8k+7}$ and hence $l_i\cdot K_X\leq\frac{-4}{8k+7}$. Therefore, $l_{i,Y}\cdot K_Y\leq 0$ for all i and strictly <0 for some i. Hence there exists a factoring diagram by Proposition 4.3. The resulting map $f^\sharp\colon X^\sharp\to W$ is a divisorial contraction with discrepancy 3 or 7.

Subcase 2. $r=8k+5$.[2]

Similar argument shows that $a_1=1$ or 5 (since $a_1\equiv 1 \pmod 4$) and there exists a factoring diagram by Corollary 4.3. The resulting map $f^\sharp\colon X^\sharp\to W$ is a divisorial contraction with discrepancy 1 or 5.

[1] If $a_1=7$, then $y^2\in\phi$ and $a_j=1$ if and only if $j=8s+3$ with $s<k$. In this case, there are exactly $k-1$ exceptional divisors with discrepancy 1.

[2] If $a_1=5$, then $y^2\in\phi$ and $a_j=1$ if and only if $j=8s+5$ with $s<k$. In this case, there are exactly $k-1$ exceptional divisors with discrepancy 1.

Subcase 3. $r = 8k + 3$.

Similar argument shows that $a_1 = 3$ (since $a_1 \equiv -1 \pmod 4$) and there exists a factoring diagram by Proposition 4.3. The resulting map $f^\sharp\colon X^\sharp \to W$ is a divisorial contraction with discrepancy 3.

Subcase 4. $r = 8k + 1$.

Similar argument shows that $a_1 = 1$ (since $a_1 \equiv 1 \pmod 4$) and there exists a factoring diagram by Proposition 4.3. The resulting map $f^\sharp\colon X^\sharp \to W$ is a divisorial contraction with discrepancy 1.

5. Proof of the Main Theorem

Proof. We prove by induction on depth and discrepancies.

1. Suppose first that $\mathrm{dep}(X) = 0$, that is, X has at worst Gorenstein terminal singularities. By the classification of Mori and Cutkosky [7, 20], f cannot be a flipping contraction.

If $f\colon X \to W$ is a divisorial contraction to a point then f is a divisorial contraction with minimal discrepancy (cf. [7, 20]).

If $f\colon X \to W$ be a divisorial contraction to a curve, then f is a blowup along a lci curve in a smooth neighborhood by the classification of Mori and Cutkosky again. By Proposition 3.1 and Remark 4.5, f is factorizable.

2. Let $f\colon X \to W$ be a divisorial contraction to a curve Γ with $\mathrm{dep}(X) = d > 0$. By [5], there is a factoring diagram

$$\begin{tikzcd}
Y \arrow[rr, dotted] \arrow[d, "g"'] & & Y^\sharp \arrow[d, "g^\sharp"] \\
X \arrow[dr, "f"'] & & X^\sharp \arrow[dl, "f^\sharp"] \\
& W &
\end{tikzcd}$$

satisfying:

(1) $Y \to X$ is a divisorial contraction to a highest index point of index $r > 1$ with discrepancy $\frac{1}{r}$;

(2) $Y \to Y^\sharp$ is a sequence of flips and flops;

(3) $g^\sharp\colon Y^\sharp \to X^\sharp$ is divisorial contraction to the proper transform of Γ;

(4) $f^\sharp$ is a divisorial contraction to a point.

Note that dep$(Y) = d-1$, and dep$(Y^\sharp) \le$ dep$(Y) = d-1$. Therefore by Proposition 2.1,

$$\text{dep}(X^\sharp) \le \min(0, \text{dep}(Y^\sharp) - 1) < d.$$

It follows that $X \to W$ can be factored into

$$X \dashrightarrow Y \dashrightarrow Y^\sharp \to X^\sharp \to W$$

so that each map is factorizable by induction on depth.

3. Let $f\colon X \to W$ be a flipping contraction. By [5], there is a factoring diagram as above so that $f^\sharp\colon X^\sharp = X^+ \to W$ is the flipped contraction. Similarly, each map of

$$X \dashrightarrow Y \dashrightarrow Y^\sharp \to X^\sharp = X^+$$

is factorizable by induction on depth.

4. Let $f\colon X \to W$ be a divisorial contraction to a point $P \in W$ of index r with dep$(X) = d$ and discrepancy $\frac{1}{r}$. Nothing to do.

5. Let $f\colon X \to W$ be a divisorial contraction to a point $P \in W$ of index $r > 1$ with dep$(X) = d$ and discrepancy $\frac{a}{r} > \frac{1}{r}$. By [2], there is a factoring diagram satisfying:

(1) $Y \to X$ is a divisorial contraction to a highest index point of index $r > 1$ with discrepancy $\frac{1}{r}$;

(2) $Y \to Y^\sharp$ is a sequence of flips and flops;

(3) $f^\sharp$ is a divisorial contraction with discrepancy $\frac{a'}{r} < \frac{a}{r}$;

(4) $g^\sharp$ is divisorial contraction to a point Q of index r with discrepancy $\frac{a''}{r} < \frac{a}{r}$ and $a'' + a' = a$ if $P \in W$ is not of type $cE/2$;

(5) $g^\sharp$ is divisorial contraction to a point Q of index 3 with discrepancy $\frac{1}{3}$ if $P \in W$ is of type $cE/2$.

Notice that dep$(Y^\sharp) \le$ dep$(Y) = d-1$ and dep$(X^\sharp) \le$ dep$(Y^\sharp) + 1 \le d$. By induction on depth, both $Y \dashrightarrow Y^\sharp$ and $Y^\sharp \to X^\sharp$ are factorizable. If dep$(X^\sharp) <$ dep(X), then we are done by induction. If dep$(X^\sharp) =$ dep(X), then we may proceed by induction on a which measures the discrepancy.

6. Let $f\colon X \to W$ be a divisorial contraction to a point $P \in W$ of index 1 with dep$(X) = d$ and discrepancy $a > 1$.

6.1 If $P \in W$ is a non-singular point, then by the study of Case Ia, f is factorizable by induction on a.

6.2 $P \in W$ is of type cA.

By the studies in Case Ib, IIa, and IId, there exists a factoring diagram such that $f^\sharp\colon X^\sharp \to W$ has discrepancy $a_1 < a$ (Case Ib) or 1 (Case IIa, IId). Moreover dep$(X^\sharp) \le d$. Therefore, $f^\sharp$ is factorizable by induction on discrepancy a hence so is $f\colon X \to W$ because $Y \dashrightarrow Y^\sharp \to X^\sharp$ has dep $< d$.

6.3 $P \in W$ is of type cD or cE and the discrepancy a is odd.

This could be Case Ic, Id, IId. There exists a factoring diagram such that $f^\sharp\colon X^\sharp \to W$ has discrepancy $a_2 < a$ (Case Ic) or 1 (Case Id, IId). Similarly f is factorizable by induction on a and on depth.

6.4 $P \in W$ is of type cD or cE and the discrepancy a is even.

This could be Case Id, IIb, IIc, IIe, IIf, and IIg. There exists a factoring diagram such that $f^\sharp\colon X^\sharp \to W$ has odd discrepancy a_1 (Case IIf, IIg) or 1 (other cases). Therefore, f is factorizable by 6.3 and induction on depth. ■

References

[1] G. Brown, *Flips arising as quotients of hypersurfaces,* Math. Proc. Cambridge Philos. Soc. **127(1)** (1999), 13–31.
`http://dx.doi.org/10.1017/s0305004198003351`

[2] J. A. Chen, *Factoring threefold divisorial contractions to points*, Ann Scoula Ecole Norm. Sup Pisa, (5) **13(2)** (2014), 435–463.

[3] J. A. Chen and M. Chen, *Explicit birational geometry of threefolds of general type, I*, Ann. Sci. Éc. Norm. Supér **43** (2010), 365–394.

[4] ———, *Explicit birational geometry of threefolds of general type, II*, J. of Diff. Geom. **86** (2010), 237–271.

[5] J. A. Chen and C. D. Hacon, *Factoring 3-fold flips and divisorial contractions to curves*, J. reine angew. Math. **657** (2011), 173–197.
`http://dx.doi.org/10.1515/crelle.2011.056`

[6] A. Corti, *Factoring birational maps of 3-folds after Sarkisov*, Journal of Algebraic Geometry **4** (1995), 223–254.

[7] S. D. Cutkosky, *Elementary contractions of Gorenstein 3-folds*, Math. Ann. **280(3)** (1988), 521–525. `http://dx.doi.org/10.1007/bf01456342`

[8] T. Hayakawa, *Blowing ups of 3-dimensional terminal singularities*, Publ. Res. Inst. Math. Sci. **35(3)** (1999), 515–570.
`http://dx.doi.org/10.2977/prims/1195143612`

[9] ———, *Blowing ups of 3-dimensional terminal singularities. II*, Publ. Res. Inst. Math. Sci. **36(3)** (2000), 423–456.
`http://dx.doi.org/10.2977/prims/1195142953`

[10] ———, *Divisorial contractions to 3-dimensional terminal singularities with discrepancy one*, J. Math. Soc. Japan **57(3)** (2005), 651–668.
`http://dx.doi.org/10.2969/jmsj/1158241927`

[11] M. Kawakita, *Divisorial contractions in dimension three which contract divisors to smooth points*, Invent. Math. **145(1)** (2001), 105–119.
`http://dx.doi.org/10.1007/s002220100144`

[12] ______, *Divisorial contractions in dimension three which contract divisors to compound A_1 points*, Compo. Math. **133** (2002), 95–116.

[13] ______, *General elephants of three-fold divisorial contractions*, J. Amer. Math. Soc. **16(2)** (2002), 331–362. http://dx.doi.org/10.1090/s0894-0347-02-00416-2

[14] ______, *Three-fold divisorial contractions to singularities of higher indices*, Duke Math. J. **130(1)** (2005), 57–126. http://dx.doi.org/10.1215/s0012-7094-05-13013-7

[15] ______, *Supplement to classification of three-fold divisorial contractions*, Nagoya Math. J. **206** (2012), 67–73. http://dx.doi.org/10.1215/00277630-1548493

[16] Y. Kawamata, *Divisorial contractions to 3-dimensional terminal quotient singularities*, Higher-dimensional complex varieties (Trento, 1994), 241–246, de Gruyter, Berlin, 1996.

[17] J. Kollár, *Flops*, Nagoya Math. J. **113** (1989), 15–36.

[18] J. Kollár and S. Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc. **5(3)** (1992), 533–703. http://dx.doi.org/10.2307/2152704

[19] D. Markushevich, *Minimal discrepancy for a terminal cDV singularity is 1*, J. Math. Sci. Tokyo **3** (1996), 445–456.

[20] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math. **116** (1982), 133–176. http://dx.doi.org/10.1007/bfb0093587

[21] ______, *On 3-dimensional terminal singularities*, Nagoya Math. J. **98** (1985), 43–66.

[22] S. Mori and Y. Prokhorov, *3-fold extremal contractions of type (IA)*, Kyoto J. Math. **51(2)** (2011), 393–438. http://dx.doi.org/10.1215/21562261-1214393

[23] ______, *Threefold Extremal Contractions of Types (IC) and (IIB)*, Proceedings of the Edinburgh Mathematical Society **57** (2014), no. 1, 231–252. http://dx.doi.org/10.1017/s0013091513000850

[24] M. Reid, *Decomposition of toric morphisms*, Arithmetic and geometry, Vol. II, 395–418, Progr. Math. **36**, Birkhaüser Boston, Boston, MA, 1983. http://dx.doi.org/10.1007/978-1-4757-9286-7_15

[25] N. Tziolas, *Terminal 3-fold divisorial contractions of a surface to a curve. I*, Compositio Math. **139(3)** (2003), 239–261. http://dx.doi.org/10.1023/b:comp.0000018135.42305.7a

[26] ______, *Three dimensional divisorial extremal neighborhoods*, Math. Ann. **333(2)** (2005), 315–354. http://dx.doi.org/10.1007/s00208-005-0676-9

[27] ______, *$\mathbb{Q}$-Gorenstein deformations of nonnormal surfaces*, Amer. J. Math. **131(1)** (2009), 171–193. http://dx.doi.org/10.1353/ajm.0.0039

Jungkai Alfred Chen
Department of Mathematics
National Taiwan University
Taipei, 106, Taiwan
and
National Center for Theoretical Sciences
No. 1 Sec. 4 Roosevelt Rd.
National Taiwan University
Taipei, 106, Taiwan
E-mail: jkchen@ntu.edu.tw