TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 4, pp. 1145-1160, August 2015
DOI: 10.11650/tjm.19.2015.5283
This paper is available online at http://journal.taiwanmathsoc.org.tw

EINSTEIN CONDITIONS FOR THE BASE SPACE OF ANTI-INVARIANT RIEMANNIAN SUBMERSIONS AND CLAIRAUT SUBMERSIONS

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Abstract. In this paper, we study the geometry of anti-invariant Riemannian submersions from a Kähler manifold onto a Riemannian manifold. We first determine the base space when the total space of an anti-invariant Riemannian submersion is Einstein and then we investigate new conditions for anti-invariant Riemannian submersions to be Clairaut submersions. We also focus on the geometry of Clairaut anti-invariant submersions.

1. INTRODUCTION

Immersions and submersions which are special tools in differential geometry also play a fundamental role in Riemannian geometry. O'Neill [9] and Gray [7] studied Riemannian submersions between Riemannian manifolds. The geometry of Riemannian submersions have been discussed in [5, 6]. We note that Riemannian submersions are related with physics and have their applications in the Yang-Mills theory, Kaluza-Klein theory and superstring theories. Later many researchers considered such submersions between manifolds with differentiable structures.

In [13], Watson defined almost Hermitian submersions between almost Hermitian manifolds. In this case, the vertical and horizontal distribution are invariant with respect to the almost complex structure of the total space of the submersion. In [10, 11], Sahin introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds in which the fibers are anti-invariant with respect to the almost complex structure of the total space. The geometry of anti-invariant Riemannian submersions is quite different from the geometry of almost Hermitian submersions.

In this paper, we consider anti-invariant Riemannian submersions from a Kähler manifold onto a Riemannian manifold. In Section 2, we present the basic information about Riemannian submersions which is needed for this paper. In Section 3, we give

Received September 12, 2014, accepted November 17, 2014.
Communicated by Sai-Kee Yeung.
2010 *Mathematics Subject Classification*: 53C15, 53B20.
Key words and phrases: Riemannian submersion, Einstein manifold, Anti-invariant submersion, Clairaut submersion.

curvature relations for an anti-invariant Riemannian submersion from a Kähler manifold onto a Riemannian manifold. In Section 4, we assume that total space is an Einstein manifold and the necessary and sufficient condition for an anti-invariant Riemannian submersion is obtained. In Section 5, we first obtain necessary and sufficient conditions for a curve on the total space of anti-invariant Riemannian submersions to be geodesic. Then we give a new characterization for anti-invariant Riemannian submersions to be Clairaut submersions. We also check the effect of Clairaut's condition on the geometry of fiber.

2. Preliminaries

In this section, we define almost Hermitian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions, more details see: [5] and [14].

Let (M, g) be an almost Hermitian manifold. This implies that M admits a tensor field J of type $(1,1)$ on M such that, $\forall X, Y \in \Gamma(TM)$, we have

$$J^2 = -I, \quad g(X,Y) = g(JX, JY). \tag{2.1}$$

An almost Hermitian manifold M is called Kähler manifold if

$$(\nabla_X J)Y = 0, \ \forall X, Y \in \Gamma(TM), \tag{2.2}$$

where ∇ is the Levi-Civita connection on M. For a Kähler manifold, we have

$$R(X,Y)J = JR(X,Y), \ R(JX, JY) = R(X,Y), \ \forall X, Y \in \Gamma(TM), \tag{2.3}$$

where $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ denotes the Riemannian curvature tensor field of M. We now denote by ρ and τ the Ricci tensor and the scalar curvature defined, respectively, by

$$\begin{aligned} \rho(X,Y) &= \text{tr } (Z \mapsto R(Z,X)Y), \\ \tau &= \text{tr } Q, \end{aligned} \tag{2.4}$$

where Q are the Ricci operator defined by $g(QX,Y) = \rho(X,Y)$, for X, $Y \in \Gamma(TM)$. A Riemannian manifold (M, g) is said to be Einstein if the Ricci tensor ρ satisfies $\rho = \lambda g$ for some function λ on manifold (M, g). We note that if dim$(M) > 2$, then λ is a constant.

Let $(M^m, g_{_M})$ and $(N^n, g_{_N})$ be Riemannian manifolds, where dim$(M) = m$, dim$(N) = n$ and $m > n$. A Riemannian submersion $F : M \longrightarrow N$ is a map of M onto N satisfying the following axioms:

(i) F has maximal rank.

(ii) The differential F_* preserves the lenghts of horizontal vectors.

For each $q \in N$, $F^{-1}(q)$ is an $(m-n)$ dimensional submanifold of M. The submanifolds $F^{-1}(q)$, $q \in N$, are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and $F-$related to a vector field X_* on N, i.e., $F_*X_p = X_{*F(p)}$ for all $p \in M$. Note that we denote the projection morphisms on the distributions $kerF_*$ and $(kerfF_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.5}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.6}$$

where ∇ is the Levi-Civita connection of $g_{_M}$. It is easy to see that a Riemannian submersion $F : M \longrightarrow N$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g)$ for any $E \in \Gamma(TM)$, reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_U W = \mathcal{T}_W U, \ \forall U, W \in \Gamma(kerF_*), \tag{2.7}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y], \ \forall X, Y \in \Gamma((kerF_*)^\perp). \tag{2.8}$$

On the other hand, from (2.5) and (2.6) we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.9}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X, \tag{2.10}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.11}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.12}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $V, W \in \Gamma(kerF_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic, then $\mathcal{H}\nabla_V X = \mathcal{A}_X V$.

Finally, in this section, we recall definition of an anti-invariant Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold.

Definition 1. [10]. Let M be a complex $m-$dimensional almost Hermitian manifold with Hermitian metric g_M and almost complex structure J and N be a Riemannian manifold with Riemannian metric g_N. Suppose that there exists a Riemannian submersion $F : M \longrightarrow N$ such that kerF_* is anti-invariant with respect to J, i.e., $J(\ker F_*) \subseteq (\ker F_*)^\perp$. Then we say that F is an anti-invariant Riemannian submersion.

It follows from Definition 1 that $J(\ker F_*)^\perp \cap \ker F_* \neq \{0\}$. We denote the complementary orthogonal subbundle to $J(\ker F_*)$ in $(\ker F_*)^\perp$ by μ. Then we have

$$(\ker F_*)^\perp = J\ker F_* \oplus \mu. \tag{2.13}$$

It is easy to see that μ is an invariant subbundle of $(\ker F_*)^\perp$, under the endomorphism J. Thus, for $X \in \Gamma((\ker F_*)^\perp)$, we have

$$JX = BX + CX, \tag{2.14}$$

where $BX \in \Gamma(\ker F_*)$ and $CX \in \Gamma(\mu)$. If $\mu = \{0\}$, then an anti-invariant submersion is called a Lagrangian submersion.

3. Curvature Relations

In this section, we are going to obtain curvature relations of anti-invariant Riemannian submersions.

Lemma 1. [12]. *Let $F : M \longrightarrow N$ be a Lagrangian submersion from a Kähler manifold M to a Riemannian manifold N. Then the horizontal distribution $(\ker F_*)^\perp$ is integrable and totally geodesic. As a result of this, we have $A_X = 0$ for $X \in \Gamma((\ker F_*)^\perp)$.*

Proposition 2. *The Riemannian curvature R is given by*

$$\begin{aligned} R(U,V,W,W') = {} & \hat{R}(U,V,W,W') - g_1(\mathcal{T}_U W', \mathcal{T}_V W) \\ & + g(\mathcal{T}_V W', \mathcal{T}_U W), \end{aligned} \tag{3.1}$$

$$R(U,V,W,X) = g_1((\nabla_U \mathcal{T})(V,W),X) - g_1((\nabla_V \mathcal{T})(U,W),X), \tag{3.2}$$

$$\begin{aligned} R(X,Y,Z,V) = {} & -g_1((\nabla_Z \mathcal{A})(X,Y),V) - g_1(\mathcal{A}_X Y, \mathcal{T}_V Z) \\ & + g_1(\mathcal{A}_Y Z, \mathcal{T}_V X) + g_1(\mathcal{A}_Z X, \mathcal{T}_V Y), \end{aligned} \tag{3.3}$$

$$\begin{aligned} R(X,Y,Z,H) = {} & R^*(X,Y,Z,H) + 2g_1(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ & - g_1(\mathcal{A}_Y Z, \mathcal{A}_X H) + g_1(\mathcal{A}_X Z, \mathcal{A}_Y H), \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
R(X,Y,V,W) =& -g_1((\nabla_V\mathcal{A})(X,Y),W)+g_1((\nabla_W\mathcal{A})(X,Y),V)\\
&-g_1(\mathcal{A}_XV,\mathcal{A}_YW)+g_1(\mathcal{A}_XW,\mathcal{A}_YV)\\
&+g_1(\mathcal{T}_VX,\mathcal{T}_WY)-g_1(\mathcal{T}_WX,\mathcal{T}_VY),
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
R(X,V,Y,W) =& -g_1((\nabla_X\mathcal{T})(V,W),Y)-g_1((\nabla_V\mathcal{A})(X,Y),W)\\
&+g_1(\mathcal{T}_VX,\mathcal{T}_WY)-g_1(\mathcal{A}_XV,\mathcal{A}_YW),
\end{aligned}
\tag{3.6}
$$

for any $U,V,W,W'\in\Gamma(\mathcal{V})$ and $X,Y,Z,H\in\Gamma(\mathcal{H})$, where R^* is Riemannian curvature of N and $\hat{R}$ is Riemannian curvature of fiber.

We note that the curvature of [5] is having negative sign difference of ours.

Lemma 3. *Let F be an anti-invariant Riemannian submersion from a Kähler manifold (M,J,g_M) onto a Riemannian manifold (N,g_N). Then for $X\in\Gamma(\mathcal{H})$ and $U,V\in\Gamma(\mathcal{V})$, we have the following relations:*

$$B\mathcal{T}_UV = \mathcal{T}_UJV, \tag{3.7}$$

$$B\mathcal{A}_XU = \mathcal{A}_XJU. \tag{3.8}$$

Proof. From (2.2) and (2.10), we have

$$J\nabla_UV=\mathcal{H}\nabla_UJV+\mathcal{T}_UJV.$$

Using (2.9), we get

$$J\mathcal{T}_UV+J\hat{\nabla}_UV=\mathcal{H}\nabla_UJV+\mathcal{T}_UJV.$$

Then (2.14) implies that

$$B\mathcal{T}_UV+C\mathcal{T}_UV+J\hat{\nabla}_UV=\mathcal{H}\nabla_UJV+\mathcal{T}_UJV.$$

Taking the vertical and horizontal parts of this equation, we obtain (3.7). The other assertion can be obtained in a similar way. ∎

Lemma 4. *Let F be an anti-invariant Riemannian submersion from a Kähler manifold (M,g_1,J) onto a Riemannian manifold (N,g). Then we have the following curvature relations:*

$$
\begin{aligned}
R(U,V,W,W') =& R^*(JU,JV,JW,JW')+2g_1(B\mathcal{A}_{JU}V,B\mathcal{A}_{JW}W')\\
&-g_1(B\mathcal{A}_{JV}W,B\mathcal{A}_{JU}W')+g_1(B\mathcal{A}_{JU}W,B\mathcal{A}_{JV}W'),
\end{aligned}
\tag{3.9}
$$

$$\begin{aligned}
R(X,U,V,W) = {} & R^*(CX,JU,JV,JW) + 2g_1(B\mathcal{A}_{CX}U, B\mathcal{A}_{JV}W) \\
& -g_1(B\mathcal{A}_{JU}V, B\mathcal{A}_{CX}W) + g_1(B\mathcal{A}_{CX}V, B\mathcal{A}_{JU}W) \\
& +g_1((\nabla_{JU}\mathcal{A})(JV,JW),BX) + g_1(B\mathcal{A}_{JV}W, B\mathcal{T}_{BX}U) \\
& -g_1(B\mathcal{A}_{JW}U, B\mathcal{T}_{BX}V) + g_1(B\mathcal{A}_{JU}V, B\mathcal{T}_{BX}W),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
R(X,U,Y,W) = {} & R^*(CX,JU,CY,JW) + 2g_1(B\mathcal{A}_{CX}U, B\mathcal{A}_{CY}W) \\
& -g_1((\nabla_{JU}\mathcal{T})(BX,BY),JW) - g_1(\mathcal{A}_{JU}CY, B\mathcal{A}_{CX}W) \\
& -g_1((\nabla_{BX}\mathcal{A})(JU,JW),BY) + g_1(\mathcal{A}_{CX}CY, B\mathcal{A}_{JU}W) \\
& -g_1(\mathcal{A}_{JU}BX, \mathcal{A}_{JW}BY) + g_1(B\mathcal{T}_{BX}U, B\mathcal{T}_{BY}W) \\
& +g_1((\nabla_{JU}\mathcal{A})(CY,JW),BX) + g_1(B\mathcal{A}_{CY}W, B\mathcal{T}_{BX}U) \\
& -g_1(B\mathcal{A}_{JW}U, \mathcal{T}_{BX}CY) + g_1(B\mathcal{A}_{CY}U, B\mathcal{T}_{BX}W) \\
& +g_1((\nabla_{JW}\mathcal{A})(CX,JU),BY) + g_1(B\mathcal{A}_{CX}U, B\mathcal{T}_{BY}W) \\
& -g_1(B\mathcal{A}_{JU}W, \mathcal{T}_{BY}CX) + g_1(B\mathcal{A}_{CX}W, B\mathcal{T}_{BY}U)
\end{aligned} \tag{3.11}$$

for any $U,V,W,W' \in \Gamma(\mathcal{V})$ and $X,Y \in \Gamma(\mathcal{H})$, where R^* is Riemannian curvature of N.

Proof. From (2.3) we have $R(U,V,W,W') = R(JU,JV,W,W')$. Also from equation (2.1) we get $R(U,V,W,W') = g(JR(JU,JV)W,JW')$. Using again (2.3) we obtain $R(U,V,W,W') = R(JU,JV,JW,JW')$. Now using (3.4) and (3.8) in this equation we derive (3.9). By using (3.7), (3.8) and Proposition 2, the other curvature relations can be obtained in a similar way. ■

Lemma 5. *Let F be an anti-invariant Riemannian submersion from a Kähler manifold (M,g_1,J) onto a Riemannian manifold (N,g). Then the Ricci tensor ρ is given by*

$$\begin{aligned}
& \rho(U,X) \\
= {} & -3\sum_{i}^{r+s} g_1(\mathcal{A}_{E_i}JU, \mathcal{A}_{E_i}CX) + \sum_{i}^{r+s} g_1((\nabla_{E_i}\mathcal{A})(E_i,JU),BX) \\
& -2\sum_{i}^{r+s} g_1(\mathcal{A}_{JU}E_i, \mathcal{T}_{BX}E_i) + \sum_{i}^{r} g_1((\nabla_{BX}\mathcal{T})(u_i,u_i),JU) \\
& +\sum_{i}^{r} g_1((\nabla_{JU}\mathcal{T})(u_i,u_i),CX) + \sum_{i}^{r} g_1((\nabla_{u_i}\mathcal{A})(JU,CX),u_i) \\
& +\sum_{i}^{r} g_1(\mathcal{A}_{JU}u_i, \mathcal{A}_{CX}u_i) - \sum_{i}^{r} g_1((\nabla_{u_i}\mathcal{T})(BX,u_i),JU) \\
& -\sum_{i}^{r} g_1(\mathcal{T}_{u_i}JU, \mathcal{T}_{u_i}CX) + \rho^*(JU,CX),
\end{aligned} \tag{3.12}$$

$$
\begin{aligned}
\rho(U,V) \;=\; & -3\sum_i^{r+s} g_1(B\mathcal{A}_{E_i}U, B\mathcal{A}_{E_i}V) + \sum_i^{r} g_1((\nabla_{JU}\mathcal{T})(u_i,u_i), JV) \\
& + \sum_i^{r} g_1((\nabla_{u_i}\mathcal{A})(JU,JV), u_i) - \sum_i^{r} g_1(B\mathcal{T}_{u_i}U, B\mathcal{T}_{u_i}V) \\
& + \sum_i^{r} g_1(\mathcal{A}_{JU}u_i, \mathcal{A}_{JV}u_i) + \rho^*(JU,JV),
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
& \rho(X,Y) \\
= & \sum_i^{r+s} g_1((\nabla_{E_i}\mathcal{A})(E_i,CX), BY) + \sum_i^{r+s} g_1((\nabla_{E_i}\mathcal{T})(BX,BY), E_i) \\
& -2\sum_i^{r+s} g_1(\mathcal{A}_{CX}E_i, \mathcal{T}_{BY}E_i) - 3\sum_i^{r+s} g_1(\mathcal{A}_{E_i}CX, \mathcal{A}_{E_i}CY) \\
& -\sum_i^{r+s} g_1(\mathcal{T}_{BX}E_i, \mathcal{T}_{BY}E_i) + \sum_i^{r+s} g_1(\mathcal{A}_{E_i}BX, \mathcal{A}_{E_i}BY) \\
& -\sum_i^{r+s} g_1((\nabla_{E_i}\mathcal{A})(CY,E_i), BX) + \sum_i^{r} g_1((\nabla_{BY}\mathcal{T})(u_i,u_i), CX) \\
& -\sum_i^{r} g_1((\nabla_{u_i}\mathcal{T})(BY,u_i), CX) + \sum_i^{r} g_1((\nabla_{u_i}\mathcal{A})(CX,CY), u_i) \\
& -\sum_i^{r} g_1(\mathcal{T}_{u_i}CX, \mathcal{T}_{u_i}CY) + \sum_i^{r} g_1(\mathcal{A}_{CX}u_i, \mathcal{A}_{CY}u_i) \\
& -\sum_i^{r} g_1(\mathcal{T}_{u_i}u_i, \mathcal{T}_{BX}BY) + \sum_i^{r} g_1(\mathcal{T}_{BX}u_i, \mathcal{T}_{BY}u_i) \\
& -\sum_i^{r} g_1((\nabla_{u_i}\mathcal{T})(BX,u_i), CY) + \sum_i^{r} g_1((\nabla_{BX}\mathcal{T})(u_i,u_i), CY) \\
& +\sum_i^{r} g_1((\nabla_{CX}\mathcal{T})(u_i,u_i), CY) + \rho^*(CX,CY) + \hat{\rho}(BX,BY) \\
& -2\sum_i^{r+s} g_1(\mathcal{A}_{CY}E_i, \mathcal{T}_{BX}E_i),
\end{aligned} \tag{3.14}
$$

for $X, Y \in \Gamma(\mathcal{H})$, $U, V \in \Gamma(\mathcal{V})$, where $\{u_1, ..., u_r\}$, $\{E_1, ..., E_{r+s}\}$ and $\{\mu_1, ..., \mu_s\}$ are orthonormal frames of $(\ker F_*)$, $J(\ker F_*) \oplus \mu$ and μ, respectively, ρ^* is the Ricci tensor of N and $\hat{\rho}$ is the Ricci tensor of any fiber.

Proof. We see that for $X \in J\ker F_*$, CX is zero and for $X \in \mu$, BX is zero. Thus Lemma 5 comes from (3.9)-(3.11). ∎

From (3.13)-(3.14) the scalar curvature τ is given by

$$\begin{aligned}\tau = \ & \tau^* + \hat{\tau} - 3\sum_{i,j}^{r+s} g_1(\mathcal{A}_{E_i}E_j, \mathcal{A}_{E_i}E_j) + 2\sum_{j}^{r+s}\sum_{i}^{r} g_1((\nabla_{E_j}\mathcal{T})(u_i,u_i), E_j) \\ & - 2\sum_{j}^{r+s}\sum_{i}^{r} g_1(\mathcal{T}_{u_i}E_j, \mathcal{T}_{u_i}E_j) + 2\sum_{j}^{r+s}\sum_{i}^{r} g_1(\mathcal{A}_{E_j}u_i, \mathcal{A}_{E_j}u_i) \\ & - \sum_{i,j}^{r} g_1(\mathcal{T}_{u_i}u_i, \mathcal{T}_{u_j}u_j) + \sum_{i,j}^{r} g_1(\mathcal{T}_{u_i}u_j, \mathcal{T}_{u_i}u_j),\end{aligned} \tag{3.15}$$

where τ^* is the scalar curvature of N and $\hat{\tau}$ is the scalar curvature of any fiber.

4. Einstein Metrics on the Total Space of an Anti-invariant Submersion

In this section, we assume that total space is an Einstein manifold. Then we have the following proposition.

Proposition 6. *Let F be an anti-invariant Riemannian submersion with totally geodesic fibers from a Kähler manifold (M, g_1, J) onto a Riemannian manifold (N, g). Then, (M, g_1) is Einstein if and only if the following relations hold:*

$$\begin{aligned} (i)\ & -\frac{\tau}{m}g_1(U,V) + \rho^*(JU,JV) - 3\sum_{i}^{r+s} g_1(B\mathcal{A}_{E_i}U, B\mathcal{A}_{E_i}V) \\ & + \sum_{i}^{r} g_1((\nabla_{u_i}\mathcal{A})(JU,JV), u_i) + \sum_{i}^{r} g_1(\mathcal{A}_{JU}u_i, \mathcal{A}_{JV}u_i) = 0, \\ (ii)\ & -\frac{\tau}{m}g_1(X,Y) + \sum_{i}^{r+s} g_1((\nabla_{E_i}\mathcal{A})(E_i, CX), BY) - 3\sum_{i}^{r+s} g_1(\mathcal{A}_{E_i}CX, \mathcal{A}_{E_i}CY) \\ & + \sum_{i}^{r+s} g_1(\mathcal{A}_{E_i}BX, \mathcal{A}_{E_i}BY) - \sum_{i}^{r+s} g_1((\nabla_{E_i}\mathcal{A})(CY, E_i), BX) + \hat{\rho}(BX, BY) \\ & + \sum_{i}^{r} g_1((\nabla_{u_i}\mathcal{A})(CX, CY), u_i) + \sum_{i}^{r} g_1(\mathcal{A}_{CX}u_i, \mathcal{A}_{CY}u_i) + \rho^*(CX, CY) = 0, \\ (iii)\ & \rho^*(JU, CX) - 3\sum_{i}^{r+s} g_1(\mathcal{A}_{E_i}JU, \mathcal{A}_{E_i}CX) + \sum_{i}^{r+s} g_1((\nabla_{E_i}\mathcal{A})(E_i, JU), BX) \\ & + \sum_{i}^{r} g_1((\nabla_{u_i}\mathcal{A})(JU, CX), u_i) + \sum_{i}^{r} g_1(\mathcal{A}_{JU}u_i, \mathcal{A}_{CX}u_i) = 0, \end{aligned} \tag{4.1}$$

where ρ^* and $\hat{\rho}$ are the Ricci tensor of N and the Ricci tensor of the fiber, respectively, and $m = \dim(M)$, $r = \dim(\ker F_*)$, $s = \dim(\mu)$.

From the above Proposition, we have the following theorem.

Theorem 7. *Let $F : M \longrightarrow N$ be a Lagrangian submersion with totally geodesic fibers from a Kähler manifold (M, g_1, J) to a Riemannian manifold (N, g). Then, (M, g_1, J) is Einstein if and only if the fibers and the base space (N, g) are Einstein.*

Proof. From Proposition 6 and (3.15), taking account into Lemma 1, the following relations hold

$$-\frac{\tau^* + \hat{\tau}}{m} g_1(U, V) + \rho^*(JU, JV) = 0, \quad (4.2)$$
$$-\frac{\tau^* + \hat{\tau}}{m} g_1(X, Y) + \hat{\rho}(BX, BY) = 0.$$

From (4.2), for $U, V \in \Gamma(\mathcal{V})$ we have

$$\rho^*(JU, JV) = \hat{\rho}(U, V). \quad (4.3)$$

This completes the proof. ∎

5. Clairaut Anti-invariant Riemannian Submersions

If θ is the angle between the velocity vector of a geodesic and a meridian, and r is the distance to the axis of a surface of revolution, Clairaut's relation states that $r \sin\theta$ is constant. In the submersion theory, this notion was defined by Bishop. According to his definition, a submersion $F : M \to N$ to be a Clairaut submersion if there is a function $r : M \to \mathbb{R}^+$ such that for every geodesic, making angles θ with the horizontal subspaces, $r \sin\theta$ is constant. He found the following characterization.

Theorem 8. [3]. *Let $F : (M, g_1) \to (B, g)$ be a Riemannian submersion with connected fibers. Then F is a Clairaut submersion with $r = e^f$ if and only if each fiber is totally umbilical and has the mean curvature vector field $H = -\mathrm{grad} f$.*

Clairaut submersions have been studied in Lorentizan spaces and timelike, spacelike and null geodesics of Lorentzian Clairaut submersion with one-dimensional fibers have been investigated in details [1]. Such submersions have been further generalized [2, 4].

As we have seen above, the origin of the notion of Clairaut submersion comes from geodesic on a surface. Therefore we are going to find necessary conditions for a curve on the total space to be geodesic.

Lemma 9. *Let F be a anti-invariant Riemannian submersion from a Kähler manifold (M, g_1, J) onto a Riemannian manifold (N, g). If $\alpha : I \to M$ is a regular curve and $X(t)$, $V(t)$ denote the horizontal and vertical components of its tangent vector field, then α is a geodesic on M if and only if*

$$\bar{\nabla}_{\dot{\alpha}} CX + \bar{\nabla}_{\dot{\alpha}} JV + \mathcal{A}_X BX + \mathcal{T}_V BX = 0 \quad (5.1)$$

$$\bar{\nabla}_{\dot{\alpha}} BX + \mathcal{V}\nabla_{\dot{\alpha}} JV + \mathcal{A}_X CX + \mathcal{T}_V CX = 0 \quad (5.2)$$

where $\bar{\nabla}$ is the Schouten connection associated with the mutually orthogonal distributions $\mathcal{V}$ and $\mathcal{H}$.

Proof. Since $\nabla_{\dot\alpha}\dot\alpha = -J\nabla_{\dot\alpha}J\dot\alpha$, using (2.14), we get

$$\begin{aligned}\nabla_{\dot\alpha}\dot\alpha = -J(&\mathcal{V}\nabla_{\dot\alpha}BX(t) + \mathcal{H}\nabla_{\dot\alpha}BX(t)\\ &+\mathcal{V}\nabla_{\dot\alpha}CX(t) + \mathcal{H}\nabla_{\dot\alpha}CX(t) + \nabla_{\dot\alpha}JV(t)).\end{aligned}$$

Then nonsingular J implies that α is geodesic if and only if

$$\mathcal{V}\nabla_{\dot\alpha}BX(t) + \mathcal{H}\nabla_{\dot\alpha}BX(t) + \mathcal{V}\nabla_{\dot\alpha}CX(t) + \mathcal{H}\nabla_{\dot\alpha}CX(t) + \nabla_{\dot\alpha}JV(t) = 0.$$

Thus using (2.9)∼(2.12) and the Schouten connection $\bar\nabla_X W' = \mathcal{V}\nabla_X \mathcal{V}W' + \mathcal{H}\nabla_X \mathcal{H}W'$ for $X, W' \in \Gamma(TM)$, we obtain (5.1) and (5.2). ■

Theorem 10. *Let F be an anti-invariant Riemannian submersion from a Kähler manifold (M, J, g_1) onto a Riemannian manifold (N, g). Then F is a Clairaut anti-invariant submersion with $r = e^f$ if and only if*

$$g_1(V, V)g_1(X, gradf) - g_1(\mathcal{A}_Z BZ + \mathcal{T}_V BZ + \bar\nabla_{\dot\alpha(t)}CZ, JV) = 0,$$

where $Z(t)$ and $V(t)$ denote the horizontal and vertical components of $\dot\alpha(t) = X$. Moreover if F is a Clairaut anti-invariant submersion with e^f then at least one of the following statements is true (a) f is constant on $J(\ker F_*)$ (b) the fibers are one dimensional (c)

$$\mathcal{A}_{JW}JY = Y(f)W$$

for $Y \in \Gamma(\mu)$ and $W \in \Gamma(\ker F_*)$ such that JW is basic.

Proof. For a geodesic $\alpha(t)$ on M, putting $a = \| \dot\alpha(t) \|^2$ which is constant, one obtains

$$g_{1\alpha(t)}(Z(t), Z(t)) = a\cos^2\theta,\ g_{1\alpha(t)}(V(t), V(t)) = a\sin^2\theta. \tag{5.3}$$

Differentiating the second expression, we have

$$\frac{d}{dt}g_{1\alpha(t)}(V(t), V(t)) = 2g_{1\alpha(t)}(\nabla_{\dot\alpha(t)}V(t), V(t)) = 2a\sin\theta\cos\theta\frac{d\theta}{dt}.$$

Hence we get

$$g_{1\alpha(t)}(\mathcal{H}\nabla_{\dot\alpha(t)}JV(t), JV(t)) = a\sin\theta\cos\theta\frac{d\theta}{dt}.$$

Using (5.1), we derive

$$-g_{1\alpha(t)}(\bar\nabla_{\dot\alpha}CZ(t) + \mathcal{A}_{Z(t)}BZ(t) + \mathcal{T}_{V(t)}BZ(t), JV(t)) = a\sin\theta\cos\theta\frac{d\theta}{dt}. \tag{5.4}$$

On the other hand, F is a Clairaut submersion with e^f if and only if

$$\frac{d}{dt}(e^f \sin\theta) = 0.$$

Multiplying with the nonzero factor $a\sin\theta(t)$ and using (5.3) and (5.4), it follows that F is a Clairaut submersion if and only if

$$g_1(V,V)\frac{df(\alpha(t))}{dt} - g_1(\mathcal{A}_Z BZ + \mathcal{T}_V BZ + \bar{\nabla}_{\dot{\alpha}(t)} CZ, JV) = 0$$

which gives the first part of the theorem. Now suppose that F is a Clairaut anti-invariant submersion with $r = e^f$, then from Bishop's theorem, the fibers of F are totally umbilical with mean curvature vector field $H = -\mathrm{grad} f$. Thus we have

$$\mathcal{T}_U V = -g_1(U,V)\mathrm{grad} f,$$

for $U, V \in \Gamma(\ker F_*)$. Multiplying this expression with JW for $W \in \Gamma(\ker F_*)$ and using (2.2) and (2.9), we get

$$g_1(\nabla_U JV, W) = g_1(U,V)g_1(\mathrm{grad} f, JW).$$

Since ∇ is a metric connection, using again(2.9), we derive

$$g_1(U,W)g_1(\mathrm{grad} f, JV) = g_1(U,V)g_1(\mathrm{grad} f, JW). \tag{5.5}$$

Taking $U = W$, interchanging the role of U and V in (5.5), we obtain

$$g_1(V,V)g_1(\mathrm{grad} f, JU) = g_1(V,U)g_1(\mathrm{grad} f, JV). \tag{5.6}$$

Using (5.5) with $W = U$ and (5.6), we have

$$g_1(\mathrm{grad} f, JU) = \frac{(g_1(U,V))^2}{\| U \|^2 \| V \|^2} g_1(\mathrm{grad} f, JU). \tag{5.7}$$

On the other hand, from (2.2) and (2.9), we find

$$g_1(\nabla_V JW, JY) = -g_1(V,W)g_1(\mathrm{grad} f, Y)$$

for $V, W \in \Gamma(\mathcal{V})$ and $Y \in \Gamma(\mu)$. Since μ is invariant with respect to J and $[V, JW]$ is belong to $\mathcal{V}$, by using the fact that $\mathcal{H}\nabla_V JW = \mathcal{A}_{JW} V$ and (2.10), we get

$$g_1(\mathcal{A}_{JW} V, JY) = -g_1(V,W)g_1(\mathrm{grad} f, Y).$$

Since $\mathcal{A}_{JW}$ is skew-symmetric with respect to g_1 and since $\mathcal{A}_{JW} JY$, V and W are vertical and $\mathrm{grad} f$ is horizontal, and above equation holds for all vertical V and W, hence we derive

$$\mathcal{A}_{JW} JY = Y(f)W. \tag{5.8}$$

Now if $\mathrm{grad} f \in \Gamma(J(\mathcal{V}))$, then (5.7) and the equality case of Schwarz inequality implies that either f is constant on $J(\mathcal{V})$ or the fibers are one dimensional. If $\mathrm{grad} f \in \Gamma(\mu)$, then (5.8) implies (c). This completes the proof. ∎

From Theorem 10, we have the following results.

Corollary 11. *Let F be a Clairaut anti-invariant submersion from a Kähler manifold (M, g_1, J) onto a Riemannian manifold (N, g) with $r = e^f$ and $\dim(\mathcal{V}) > 1$. Then the fibers of F are totally geodesic if and only if $\mathcal{A}_{JW} JX = 0$ for $W \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mu)$.*

For a Lagrangian submersion we have the following.

Corollary 12. *Let F be a Clairaut Lagrangian submersion from a Kähler manifold (M, g_1, J) onto a Riemannian manifold (N, g) with $r = e^f$. Then either the fibers of F are totally geodesic or they are one dimensional.*

In fact, this case is valid for general case of Lagrangian submersions with totally umbilical fibers, We omit its proof, which is similar to the proof of Theorem 4. We recall that a Riemannian submersion with the condition

$$\mathcal{T}_U W = g_1(U, W) H$$

is called a Riemannian submersion with totally umbilical fibres, where $U, W \in \Gamma(\mathcal{V})$, H is the mean curvature vector field of the fiber.

Proposition 13. *Let F be a Lagrangian submersion from Kähler manifold (M, g_1, J) onto a Riemannian manifold (N, g) with totally umbilical fibers with $\dim(\mathcal{V}) > 1$. Then the fibers are totally geodesic.*

We note that Proposition 13 was already given in [12] as Proposition 5.4.

Lemma 14. *For an anti-invariant Riemannian submersion, we have*

$$g_1(\mathcal{T}_V U, X) = -g_1(\mathcal{T}_V BX, JU) + g_1(\mathcal{A}_{CX} JU, V),$$

for $U, V \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$. Observe that if F is a Clairaut anti-invariant submersion, we have

$$\mathcal{A}_{JU} CX = X(f)U + JU(f)BX,$$

and if $\dim(\mathcal{V}) > 1$, we get

$$\mathcal{A}_{JU} CX = X(f)U,$$

for basic CX.

From Lemma 14, we also have the following expressions.

Lemma 15. *Let F be a Clairaut anti-invariant submersion from a Kähler manifold (M, g_1, J) onto a Riemannian manifold (N, g) with $r = e^f$ and $\dim(\mathcal{V}) > 1$. Then we have*

$$\sum_{i=1}^{r+s} g_1(\mathcal{A}_{CY}E_i, \mathcal{T}_{BX}E_i) = \sum_{j=1}^{s} g_1(\mathcal{A}_{CY}\mu_j, \mathcal{T}_{BX}\mu_j), \tag{5.9}$$

$$\sum_{i=1}^{r+s} g_1(\mathcal{A}_{E_i}CX, \mathcal{A}_{E_i}CY) = rX(f)Y(f) + \sum_{j=1}^{s} g_1(\mathcal{A}_{\mu_j}CX, \mathcal{A}_{\mu_j}CY), \tag{5.10}$$

for $X, Y \in \Gamma(\mathcal{H})$ such that CX and CY are basic .

Proof. For an anti-invariant Riemannian submersion, we can write

$$\sum_{i=1}^{r+s} g_1(\mathcal{A}_{CY}E_i, \mathcal{T}_{BX}E_i) = \sum_{i=1}^{r} g_1(\mathcal{A}_{CY}Ju_i, \mathcal{T}_{BX}Ju_i) + \sum_{j=1}^{s} g_1(\mathcal{A}_{CY}\mu_j, \mathcal{T}_{BX}\mu_j).$$

Since F is a Clairaut submersion, $\mathcal{A}$ is anti-symmetric on $\mathcal{H}$ and $\dim(\mathcal{V}) > 1$, from Lemma 14, we have

$$\sum_{i=1}^{r+s} g_1(\mathcal{A}_{CY}E_i, \mathcal{T}_{BX}E_i) = -\sum_{i=1}^{r} Y(f)g_1(u_i, \mathcal{T}_{BX}Ju_i) + \sum_{j=1}^{s} g_1(\mathcal{A}_{CY}\mu_j, \mathcal{T}_{BX}\mu_j).$$

Also since $\mathcal{T}$ is anti-symmetric with respect to g_1, using Theorem 8, we obtain

$$\begin{aligned}\sum_{i=1}^{r+s} g_1(\mathcal{A}_{CY}E_i, \mathcal{T}_{BX}E_i) =& -\sum_{i=1}^{r} Y(f)g_1(BX, u_i)g_1(grad f, Ju_i)\\ &+\sum_{j=1}^{s} g_1(\mathcal{A}_{CY}\mu_j, \mathcal{T}_{BX}\mu_j).\end{aligned}$$

Then we get (5.9) since f is constant on $J\mathcal{V}$ from Theorem 10. In a similar way, we obtain (5.10). ∎

From (5.10), (5.9) and Lemma 5, we have the following result which characterizes the fibers.

Proposition 16. *Let F be a Clairaut anti-invariant submersion from a Kähler manifold (M, g_1, J) onto a Riemannian manifold (N, g) with $r = e^f$ and $\dim(\mathcal{V}) > 1$.*

Then we have

$$\begin{aligned}\rho(X,X) \leq & \sum_{i}^{r+s}\{2g_1((\nabla_{E_i}\mathcal{A})(E_i,CX),BX)+g_1((\nabla_{E_i}\mathcal{T})(BX,BX),E_i)\}\\ & + \parallel \mathrm{trace}\mathcal{A}_{(.)}BX \parallel^2_{J\mathcal{V}\oplus\mu} -4\sum_{i}^{s} g_1(\mathcal{A}_{\mu_i}CX,\mathcal{T}_{BX}\mu_i)\\ & +\sum_{i}^{r}\{2g_1((\nabla_{BX}\mathcal{T})(u_i,u_i),CX)+g_1((\nabla_{CX}\mathcal{T})(u_i,u_i),CX)\\ & -2g_1((\nabla_{u_i}\mathcal{T})(BX,u_i),CX)+g_1((\nabla_{u_i}\mathcal{A})(CX,CX),u_i)\\ & -g_1(\mathcal{T}_{u_i}u_i,\mathcal{T}_{BX}BX)\}+\parallel \mathrm{trace}\mathcal{A}_{CX}(.)\parallel^2_{\mathcal{V}}+\parallel \mathrm{trace}\mathcal{T}_{BX}(.)\parallel^2_{\mathcal{V}}\\ & -3\parallel \mathrm{trace}\mathcal{A}_{(.)}CX\parallel^2_{\mu}+\rho^*(CX,CX)+\hat{\rho}(BX,BX),\end{aligned}$$

for $X \in \Gamma(\mathcal{H})$ such that CX is basic. The equality case is satisfied if and only if f is constant on the horizontal distribution, that is, F has totally geodesic fibers. In the equality case, it takes the following form

$$\begin{aligned}\rho(X,X) = & \sum_{i}^{r+s}2g_1((\nabla_{E_i}\mathcal{A})(E_i,CX),BX)+\parallel \mathrm{trace}\mathcal{A}_{CX}(.)\parallel^2_{\mathcal{V}}\\ & +\sum_{i}^{r}g_1((\nabla_{u_i}\mathcal{A})(CX,CX),u_i)+\parallel \mathrm{trace}\mathcal{A}_{(.)}BX\parallel^2_{J\mathcal{V}\oplus\mu}+\rho^*(CX,CX)\\ & -3\parallel \mathrm{trace}\mathcal{A}_{(.)}CX\parallel^2_{\mu}+\hat{\rho}(BX,BX).\end{aligned}$$

Also from Theorem 7 and Corollary 12, we have

Corollary 17. *Let F be a Clairaut Lagrangian submersion from a Kähler manifold (M,g_1,J) onto a Riemannian manifold (N,g) with $r=e^f$ and* $\dim M>2$. *Then (M,g_1,J) is Einstein if and only if the fibers and the base space (N,g) are Einstein.*

ACKNOWLEDGMENTS

Research of J. H. Park was supported by Samsung Research Fund, Sungkyunkwan University, 2014. Research of J. Lee was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education(2013R1A6A3A01059965). The authors would like to express their thanks to the referee, for his helpful comments and valuable suggestions on this paper.

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