TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 5, pp. 1469-1493, October 2015
DOI: 10.11650/tjm.19.2015.5187
This paper is available online at http://journal.taiwanmathsoc.org.tw

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR DEGENERATE $p(x)$-LAPLACE EQUATIONS INVOLVING CONCAVE-CONVEX TYPE NONLINEARITIES WITH TWO PARAMETERS

Ky Ho and Inbo Sim

Abstract. We show the existence of two nontrivial nonnegative solutions and infinitely many solutions for degenerate $p(x)$-Laplace equations involving concave-convex type nonlinearities with two parameters. By investigating the order of concave and convex terms and using a variational method, we determine the existence according to the range of each parameter. Some Caffarelli-Kohn-Nirenberg type problems with variable exponents are also discussed.

1. Introduction

In this paper, we study the existence and multiplicity of solutions for the following equation

$$(1.1)\quad \begin{cases} -\operatorname{div}\left(w(x)|\nabla u|^{p(x)-2}\nabla u\right) = \lambda a(x)|u|^{q(x)-2}u + \mu b(x)|u|^{h(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $p, q, h \in C(\overline{\Omega}, (1,\infty)) =: C_+(\overline{\Omega})$, w, a, b are measurable functions on Ω that are positive a.e. in Ω, and λ, μ are real parameters. We call operators $\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ and $\operatorname{div}\left(w(x)|\nabla u|^{p(x)-2}\nabla u\right)$ $p(x)$-Laplacian and degenerate $p(x)$-Laplacian, respectively. If $p(x) \equiv p$ (constant), then we call $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ p-Laplacian.

Motivations for $p(x)$-Laplacian can be found in [7, 32], and we refer to [10] for the history of this area. There have been many studies about $p(x)$-Laplacian (see, e.g., [15, 16, 17, 22, 23, 24, 25, 28, 29]). Degenerate differential operators appear in the study of physical phenomena related to equilibrium of anisotropic continuous

Received August 20, 2014, accepted March 5, 2015.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification*: 35J20, 35J60, 35J70, 47J10, 46E35.
Key words and phrases: $p(x)$-Laplacian, Weighted variable exponent Lebesgue-Sobolev spaces, Concave-convex nonlinearities, Nonnegative solutions, Multiplicity.

media [8], and some research on such operators can be found in [11, 31]. However, degenerate $p(x)$-Laplacian have been researched relatively less.

Ambrosetti-Brezis-Cerami [3] originally considered the problem

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + |u|^{h-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < q < 2 < h < 2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N = 1, 2 \end{cases}$ and proved that

(i) there exists $\lambda^* > 0$ such that (1.2) has at least two positive solutions for $0 < \lambda < \lambda^*$, at least one positive solution for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$;

(ii) there exists $\lambda_* > 0$ such that (1.2) has infinitely many solutions in the both cases $I_\lambda(u) < 0$ and $I_\lambda(u) > 0$ for $0 < \lambda < \lambda_*$, where I_λ is an energy functional for (1.2).

Since then, many authors [9, 18, 21] have extended it to p-Laplace problems with the advantage of the first eigenvalue for $-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{p-2}u$ in Ω and $u = 0$ on $\partial\Omega$ and a Brezis-Nirenberg [5] type result on local minimization in $W_0^{1,p}$ and C_0^1.

Recently, similar problems for $p(x)$-Laplacian have also been investigated. The main difficulty in applying the same steps of p-Laplacian to $p(x)$-Laplacian is that $p(x)$-Laplacian cannot preserve the properties of the first eigenvalue as that of the p-Laplacian. This makes it impossible to use the typical sub- and supersolution method.

For $k \in C(\overline{\Omega})$, let us denote $k^- := \min_{x\in\overline{\Omega}} k(x)$ and $k^+ := \max_{x\in\overline{\Omega}} k(x)$. Fan [16] considered the following problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda a_1(x) g_1(x,u) + \mu a_2(x) g_2(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $a_i \in L^{r_i(x)}(\Omega), a_i(x) > 0$, for $x \in \Omega, g_i : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, $|g_i(x,t)| \le c_1 + c_2|t|^{q_i(x)-1}$ for $x \in \Omega$ and $t \in \mathbb{R}$, where $p, q_i, r_i \in C_+(\overline{\Omega})$ with $q_1^+ < p^- \le p^+ < q_2^-$ and $q_i(x) < \frac{r_i(x)-1}{r_i(x)} p^*(x)$ for all $x \in \overline{\Omega}, i = 1, 2$, where $p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x). \end{cases}$ Under the suitable conditions on g_i, Fan has proved that

(i) for every $\mu > 0$, there exists $\lambda_0(\mu) > 0$ such that, for $0 < \lambda \le \lambda_0(\mu)$, (1.3) has at least two positive solutions u_1 and v_1 with $I(u_1) > 0$ and $I(v_1) < 0$, respectively, where I is an energy functional for (1.3);

(ii) for every $\mu > 0$ and $\lambda \in \mathbb{R}$, (1.3) has a sequence of solutions $\{\pm u_k\}$ such that $I(\pm u_k) \to \infty$ as $k \to \infty$, and for every $\lambda > 0$ and $\mu \in \mathbb{R}$, (1.3) has a sequence of solutions $\{\pm v_k\}$ such that $I(\pm v_k) < 0$ and $I(\pm v_k) \to 0$ as $k \to \infty$.

Thus, it seems that the condition $q_1^+ < p^- \leq p^+ < q_2^-$ is crucial. However, Mihailescu-Radulescu [29] considered the problem as follows:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda|u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $p, q \in C_+(\overline{\Omega})$ with $q^- < p^- < q^+$ and proved that (1.4) has a nontrivial solution (without showing a positive solution) for small $\lambda > 0$ by using the Ekeland variational principle.

Inspired by the above results, we study the degenerate $p(x)-$Laplace equations (1.1) involving concave-convex nonlinearities with singular coefficients and containing the cases $\{x \in \overline{\Omega} : q(x) < p(x)\} \neq \emptyset$, which generalizes the Mihailescu-Radulescu's condition $q^- < p^- < q^+$. Thus, the main goal of the present paper is to improve and extend the above mentioned results under looser conditions.

Furthermore, we also consider the following Caffarelli-Kohn-Nirenberg type problems

$$\begin{cases} -\operatorname{div}\left(|x|^{\theta(x)}|\nabla u|^{p(x)-2}\nabla u\right) = \frac{\lambda}{|x|^{\xi(x)}}|u|^{q(x)-2}u + \frac{\mu}{|x|^{\delta(x)}}|u|^{h(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $p, q, h \in C_+(\overline{\Omega})$ and $\theta, \xi, \delta \in C(\overline{\Omega})$ such that $\xi(x) \geq 0, \delta(x) \geq 0$ for all $x \in \overline{\Omega}$. Many authors studied (1.5) in relation to Caffarelli-Kohn-Nirenberg inequalities in the case of constant exponents (see, e.g., [1, 4, 6, 12, 19]). In the case of variable exponents and no degeneracy, the problem involving subcritical Sobolev-Hardy exponents when $\theta \equiv 0$ was first studied in [16]. In [30], the authors studied a Caffarelli-Kohn-Nirenberg type problem with degeneracy and variable exponents but no singular terms, $\xi = \delta \equiv 0$.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries about the weighted variable exponent Lebesgue-Sobolev spaces and some properties of functionals on these spaces. In this section, we also obtain a compact imbedding from the weighted variable exponent Sobolev spaces into weighted variable exponent Lebesgue spaces. In Section 3, we show the existence of two nontrivial nonnegative solutions for (1.1) using the Ekeland variational principle and the Mountain Pass theorem. In Section 4, we show the existence of infinitely many solutions for (1.1) with positive and negative energies using Fountain Theorem and Dual Fountain Theorem, respectively. Section 5 is devoted to Caffarelli-Kohn-Nirenberg type problems for variable exponents case with degeneracy.

2. Abstract Framework and Preliminary Results

In this section, we only review the weighted variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(w,\Omega)$ and $W^{1,p(x)}(w,\Omega)$, which were studied in [22, 25]; for the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, we refer to [13, 14, 26] and the references therein.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. For an arbitrary weight w that is measurable and positive a.e. in Ω and $p \in C_+(\overline{\Omega})$, we define the weighted variable exponent Lebesgue space as

$$L^{p(x)}(w,\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega w(x)|u(x)|^{p(x)}dx < \infty \right\}.$$

Then $L^{p(x)}(w,\Omega)$ is a normed space with norm

$$|u|_{L^{p(x)}(w,\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega w(x)\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}dx \leq 1 \right\}.$$

When $w(x) \equiv 1$, we have $L^{p(x)}(w,\Omega) \equiv L^{p(x)}(\Omega)$ and use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(w,\Omega)}$. Denote by $L^{p(x)}_+(\Omega)$ the set of all $u \in L^{p(x)}(\Omega)$ satisfying $u(x) > 0$ for a.e. $x \in \Omega$.

The following propositions will be used in the next sections.

Proposition 2.1. [13, 26] *The space $L^{p(x)}(\Omega)$ is a separable and uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) |u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)} \leq\ 2|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)}.$$

Define the modular $\rho : L^{p(x)}(w,\Omega) \to \mathbb{R}$ as

$$\rho(u) = \int_\Omega w(x)|u(x)|^{p(x)}dx, \quad \forall u \in L^{p(x)}(w,\Omega).$$

Proposition 2.2. [25] *For all $u \in L^{p(x)}(w,\Omega)$, we have*

(i) $|u|_{L^{p(x)}(w,\Omega)} < 1\ (= 1, > 1)$ *if and only if* $\rho(u) < 1\ (= 1, > 1)$, *respectively;*

(ii) *If* $|u|_{L^{p(x)}(w,\Omega)} > 1$, *then* $|u|^{p^-}_{L^{p(x)}(w,\Omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(w,\Omega)}$;

(iii) *If* $|u|_{L^{p(x)}(w,\Omega)} < 1$, *then* $|u|^{p^+}_{L^{p(x)}(w,\Omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(w,\Omega)}$.

Consequently,

$$|u|^{p^-}_{L^{p(x)}(w,\Omega)} - 1 \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(w,\Omega)} + 1,\ \forall u \in L^{p(x)}(w,\Omega).$$

Proposition 2.3. [22] *If* $u, u_n \in L^{p(x)}(w,\Omega)$ *(*$n = 1, 2, \ldots$*), then the following statements are equivalent:*

(i) $\lim_{n\to\infty} |u_n - u|_{L^{p(x)}(w,\Omega)} = 0$*;*

(ii) $\lim_{n\to\infty} \rho(u_n - u) = 0$.

The weighted variable exponent Sobolev space $W^{1,p(x)}(w,\Omega)$ is defined by

$$W^{1,p(x)}(w,\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(w,\Omega)\},$$

with norm

$$\|u\|_{W^{1,p(x)}(w,\Omega)} = |u|_{L^{p(x)}(\Omega)} + \big||\nabla u|\big|_{L^{p(x)}(w,\Omega)}.$$

$W_0^{1,p(x)}(w,\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(w,\Omega)$ with respect to norm $\|u\|_{W^{1,p(x)}(w,\Omega)}$.

To assure basic properties of the weighted variable exponent Sobolev spaces, we assume that the weight w satisfies the following:

$(w0)$ w is measurable and positive a.e. in Ω, $w \in L^1_{loc}(\Omega)$ and $w^{-s(\cdot)} \in L^1(\Omega)$ for some $s \in C(\overline{\Omega})$ satisfying $s(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right)$ for all $x \in \overline{\Omega}$.

Proposition 2.4. [23, 25] *Assume that* $(w0)$ *holds. Then* $W^{1,p(x)}(w,\Omega)$ *is a separable reflexive Banach space.*

For s given in $(w0)$ and $x \in \overline{\Omega}$, let us denote

$$p_s(x) := \frac{p(x)s(x)}{1+s(x)} \quad \text{and} \quad p_s^*(x) := \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{if } p_s(x) < N, \\ +\infty & \text{if } p_s(x) \ge N. \end{cases}$$

We then have the following compact imbedding result.

Proposition 2.5. [25] *Assume that* $(w0)$ *holds. If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p_s^*(x)$ *for all* $x \in \overline{\Omega}$*, then we obtain the continuous compact imbedding*

$$W^{1,p(x)}(w,\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega).$$

Using Proposition 2.5 and the similar argument as in [16, Theorem 2.1], we obtain the following result, which is the key for our following proofs.

Proposition 2.6. *Assume that* $(w0)$ *holds and that* $d \in L_+^{\gamma(x)}(\Omega)$ *for some* $\gamma \in C_+(\overline{\Omega})$*. Then we have the following continuous compact imbedding*

$$W^{1,p(x)}(w,\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(d,\Omega),$$

for any $r \in C(\overline{\Omega})$ *such that* $1 \le r(x) < \frac{\gamma(x)-1}{\gamma(x)} p_s^*(x)$ *for all* $x \in \overline{\Omega}$.

The next corollary is true on a stronger condition for the weight w.

Corollary 2.7. *Let d, γ be as in Proposition 2.6. Assume that w satisfies:*

($w1$) w is measurable and positive a.e. in Ω, $w \in L^1_{loc}(\Omega)$ and $w^{-s} \in L^1(\Omega)$ for any constant $s > \max\{\frac{N}{p^-}, \frac{1}{p^- - 1}\}$.

Then we have the continuous compact imbedding

$$W^{1,p(x)}(w,\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(d,\Omega),$$

for any $r \in C(\overline{\Omega})$ such that $1 \le r(x) < \frac{\gamma(x)-1}{\gamma(x)} p^*(x)$ for all $x \in \overline{\Omega}$.

Proof. By the continuity of r, γ, p, and the compactness of $\overline{\Omega}$, the condition $1 \le r(x) < \frac{\gamma(x)-1}{\gamma(x)} p^*(x), \forall x \in \overline{\Omega}$ implies that

$$1 \le r(x) < \frac{\gamma(x)-1}{\gamma(x)} p^*_{s_0}(x), \forall x \in \overline{\Omega} \tag{2.1}$$

for some constant $s_0 > \max\{\frac{N}{p^-}, \frac{1}{p^- - 1}\}$. Indeed, in the case $p^- \ge N$, we fix $0 < \alpha_0 < \frac{\gamma^- - 1}{\gamma^-} \frac{N}{2r^+}$ and then take $s_0 > \max\{1, \frac{1}{p^- - 1}\}$ such that $\frac{Ns_0}{s_0+1} > N - \alpha_0$. For any $x \in \overline{\Omega}$, if $p_{s_0}(x) \ge N$, then (2.1) is obvious. If $p_{s_0}(x) < N$, we have

$$\frac{\gamma(x)-1}{\gamma(x)} p^*_{s_0}(x) \ge \frac{\gamma^- - 1}{\gamma^-} \frac{N\frac{s_0}{s_0+1}}{N - \frac{Ns_0}{s_0+1}} \ge \frac{\gamma^- - 1}{\gamma^-} \frac{N}{2\alpha_0} > r^+ \ge r(x).$$

Thus, (2.1) holds. In the case $p^- < N$, let $0 < \alpha_0 < \min\{\frac{\gamma^- - 1}{\gamma^-} \frac{p^-}{2r^+}, N - p^-\}$. Then the set $\Omega_1 = \{x \in \overline{\Omega} : p(x) \le N - \frac{\alpha_0}{2}\}$ is nonempty compact. Thus,

$$0 < \beta := \max_{x \in \Omega_1} \frac{\gamma(x) r(x)}{(\gamma(x)-1) p^*(x)} < 1.$$

We will show that (2.1) holds for s_0 such that $s_0 > \max\{\frac{N}{p^-}, \frac{1}{p^- - 1}\}$ and $s_0 > \max\{\frac{2\beta N}{(1-\beta)\alpha_0}, \frac{2(N-\alpha_0)}{\alpha_0}\}$. In fact, for any $x \in \overline{\Omega}$, it is clear if $p_{s_0}(x) \ge N$. If $N - \alpha_0 < p_{s_0}(x) < N$,

$$\frac{\gamma(x)-1}{\gamma(x)} p^*_{s_0}(x) = \frac{\gamma(x)-1}{\gamma(x)} \frac{Np(x)\frac{s_0}{s_0+1}}{N - p_{s_0}(x)} > \frac{\gamma^- - 1}{\gamma^-} \frac{p^-}{2\alpha_0} > r^+ \ge r(x).$$

For the last case $p_{s_0}(x) \le N - \alpha_0$, we find that $p(x) \le \frac{s_0+1}{s_0}(N - \alpha_0) < N - \frac{\alpha_0}{2}$ and thus, $x \in \Omega_1$. Therefore, this yields $\frac{\gamma(x) r(x)}{(\gamma(x)-1) p^*(x)} \le \beta$. The proof will be complete if we can show that $\beta < \frac{p^*_{s_0}(x)}{p^*(x)} = \frac{s_0(N-p(x))}{s_0(N-p(x))+N}$, equivalently, $\frac{N\beta}{(1-\beta)(N-p(x))} < s_0$.

This is true since $s_0 > \frac{2N\beta}{(1-\beta)\alpha_0} > \frac{N\beta}{(1-\beta)(N-p(x))}$. Thus, we can always choose $s_0 > \max\{\frac{N}{p^-}, \frac{1}{p^- -1}\}$ such that (2.1) holds. Since w satisfies $(w1)$, w also satisfies $(w0)$ for $s(x) \equiv s_0$. Applying Proposition 2.6, we obtain the continuous compact imbedding $W^{1,p(x)}(w,\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(d,\Omega)$. ■

Let $X := W_0^{1,p(x)}(w,\Omega)$ and on X, we hereafter use an equivalent norm $\|u\| = \big||\nabla u|\big|_{L^{p(x)}(w,\Omega)}$ (see [25, Corollary 2.12]).

Definition 2.8. We say that $u \in X$ is a (weak) solution of (1.1) if

$$\int_\Omega w(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi dx = \lambda \int_\Omega a(x)|u|^{q(x)-2}u\varphi dx + \mu \int_\Omega b(x)|u|^{h(x)-2}u\varphi dx$$

for all $\varphi \in X$.

Next, we shall give the differentiability in several variational settings. Define $\Psi : X \to \mathbb{R}$ by $\Psi(u) = \int_\Omega F(x,u)dx$, where $F(x,t) = \int_0^t f(x,\xi)d\xi$. Using a standard argument as in [22, The proof of Lemma 3.1] with Proposition 2.6, we obtain the following.

Proposition 2.9. *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that*

$$|f(x,t)| \le d(x)\left(|m(x)| + |t|^{r(x)-1}\right) \quad \textit{for a.e. } x \in \Omega, \forall t \in \mathbb{R}, \tag{2.2}$$

where d, r are as in Proposition 2.6, and $m \in L^{r'(x)}(d,\Omega)$ with $1/r(x) + 1/r'(x) = 1$ for all $x \in \overline{\Omega}$. Then Ψ is sequentially weakly continuous and is of $C^1(X,\mathbb{R})$, and

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u)v\, dx, \quad \textit{for any} \quad u, v \in X.$$

Consequently, Ψ^+ is also sequentially weakly continuous and is of $C^1(X,\mathbb{R})$, and

$$\langle (\Psi^+)'(u), v\rangle = \int_\Omega f^+(x,u)v\, dx, \quad \textit{for any} \quad u, v \in X,$$

where $\Psi^+(u) = \int_\Omega F^+(x,u)dx$ with

$$F^+(x,t) = \int_0^t f^+(x,\xi)d\xi, \;\; f^+(x,t) = \begin{cases} f(x,t) & \textit{if } x \in \Omega, t \ge 0, \\ 0 & \textit{if } x \in \Omega, t < 0. \end{cases}$$

Proof. We only show the sequentially weak continuity of Ψ. Let $u_n \rightharpoonup u$ (weakly) in X. By virtue of the compact imbedding $X \hookrightarrow\hookrightarrow L^{r(x)}(d,\Omega)$, we have $u_n \to u$ in $L^{r(x)}(d,\Omega)$. Hence, up to a subsequence, we have

$$\begin{cases} u_n(x) \to u(x) & \text{for a.e. } x \in \Omega, \\ d(x)|u_n(x)-u(x)|^{r(x)} \le g(x) & \text{for a.e. } x \in \Omega, \end{cases} \tag{2.3}$$

for some $g \in L^1(\Omega)$. Therefore, it follows that $F(x,u_n(x)) \to F(x,u(x))$ a.e in Ω. Using Young's inequality we deduce from (2.2) that

$$\begin{aligned} |F(x,t)| \le d(x)\left(|m(x)||t| + \frac{|t|^{r(x)}}{r(x)}\right) &\le d(x)\left(\frac{|m(x)|^{r'(x)}}{r'(x)} + \frac{|t|^{r(x)}}{r(x)} + \frac{|t|^{r(x)}}{r(x)}\right) \\ &\le d(x)|m(x)|^{r'(x)} + 2d(x)|t|^{r(x)} \end{aligned}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. By this and (2.3), we obtain

$$\begin{aligned} |F(x,u_n(x))| &\le d(x)|m(x)|^{r'(x)} + 2d(x)|u_n|^{r(x)} \\ &\le d(x)|m(x)|^{r'(x)} + 2^{r^+}d(x)|u|^{r(x)} + 2^{r^+}d(x)|u_n-u|^{r(x)} \\ &\le d(x)|m(x)|^{r'(x)} + 2^{r^+}d(x)|u|^{r(x)} + 2^{r^+}g(x). \end{aligned}$$

Thus, the dominated convergence theorem implies that

$$\lim_{n\to\infty}\int_\Omega F(x,u_n)dx = \int_\Omega F(x,u)dx, \quad \text{i.e.,} \quad \lim_{n\to\infty}\Psi(u_n) = \Psi(u).$$

For Ψ^+, note that f^+ is also a Carathéodory function, and

$$|f^+(x,t)| \le d(x)\left(|m(x)| + |t|^{r(x)-1}\right) \quad \text{for a.e. } x \in \Omega, \forall t \in \mathbb{R}.$$

Thus, we also have the same conclusion for Ψ^+ as Ψ. ∎

Throughout this paper, we assume that $p \in C_+(\overline{\Omega})$, the weight w satisfies $(w0)$, and the exponent functions q,h and the coefficient functions a,b satisfy the following.

(H) $q,h \in C_+(\overline{\Omega})$, $a \in L_+^{\alpha(x)}(\Omega), b \in L_+^{\beta(x)}(\Omega)$ where $\alpha,\beta \in C_+(\overline{\Omega})$ such that

$$q(x) < \frac{\alpha(x)-1}{\alpha(x)}p_s^*(x) \text{ and } h(x) < \frac{\beta(x)-1}{\beta(x)}p_s^*(x), \ \forall x \in \overline{\Omega}.$$

As we will see later, if $(w0)$ is replaced by $(w1)$, the condition (H) can be replaced by the following weaker condition, which was introduced in [16]:

(H^*) $q,h \in C_+(\overline{\Omega})$, $a \in L_+^{\alpha(x)}(\Omega), b \in L_+^{\beta(x)}(\Omega)$ where $\alpha,\beta \in C_+(\overline{\Omega})$ such that

$$q(x) < \frac{\alpha(x)-1}{\alpha(x)}p^*(x) \text{ and } h(x) < \frac{\beta(x)-1}{\beta(x)}p^*(x), \ \forall x \in \overline{\Omega}.$$

For $u \in X$, we set $u^+ = \max\{u, 0\}$. Define $J, J_1 : X \to \mathbb{R}$ corresponding to the problem (1.1) by

$$J(u) = \int_\Omega \frac{w(x)}{p(x)} |\nabla u|^{p(x)} dx - \lambda \int_\Omega \frac{a(x)}{q(x)} |u|^{q(x)} dx - \mu \int_\Omega \frac{b(x)}{h(x)} |u|^{h(x)} dx,$$

$$J_1(u) = \int_\Omega \frac{w(x)}{p(x)} |\nabla u|^{p(x)} dx - \lambda \int_\Omega \frac{a(x)}{q(x)} (u^+)^{q(x)} dx - \mu \int_\Omega \frac{b(x)}{h(x)} (u^+)^{h(x)} dx.$$

Then, by Lemma 3.1 in [22] and Proposition 2.9, we deduce that J, J_1 are sequentially weakly lower semicontinuous and are of $C^1(X, \mathbb{R})$. Obviously, a critical point of J (resp. J_1) is a solution (resp. a nonnegative solution) of (1.1). Moreover, by Proposition 2.6 and the $(S_+)-$property of the degenerate $p(x)-$Laplacian (see [22, Lemma 3.2]), we obtain the following.

Proposition 2.10. *$J', J_1' : X \to X^*$ are $(S_+)-$operators.*

Proof. It is sufficient to prove for J_1' since the proof for J' is similar. Let $\{u_n\}$ be a sequence in X such that $u_n \rightharpoonup u$ as $n \to \infty$, and $\limsup_{n\to\infty} \langle J_1'(u_n), u_n - u\rangle \le 0$. By Proposition 2.6, this implies that $u_n \to u$ in both $L^{q(x)}(a, \Omega)$ and $L^{h(x)}(b, \Omega)$ as $n \to \infty$. By Proposition 2.9, we find that

$$\begin{aligned} \langle J_1'(u_n), u_n - u\rangle &= \int_\Omega w(x) |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla (u_n - u) dx \\ &\quad -\lambda \int_\Omega a(x) (u_n^+)^{q(x)-1} (u_n - u) dx \\ &\quad -\mu \int_\Omega b(x) (u_n^+)^{h(x)-1} (u_n - u) dx. \end{aligned} \tag{2.4}$$

Meanwhile, we estimate

$$\begin{aligned} \left| \int_\Omega a(x) (u_n^+)^{q(x)-1} (u_n - u) dx \right| &\le \int_\Omega a^{\frac{1}{q'(x)}} |u_n|^{q(x)-1} a^{\frac{1}{q(x)}} |u_n - u| dx \\ &\le 2 \left| a^{\frac{1}{q'(x)}} |u_n|^{q(x)-1} \right|_{L^{q'(x)}(\Omega)} \left| a^{\frac{1}{q(x)}} |u_n - u| \right|_{L^{q(x)}(\Omega)} \\ &\le 2 \left(1 + \int_\Omega a(x) |u_n|^{q(x)} dx \right)^{\frac{1}{(q')^-}} |u_n - u|_{L^{q(x)}(a,\Omega)} \\ &\le 2 \left(2 + |u_n|^{q^+}_{L^{q(x)}(a,\Omega)} \right)^{\frac{1}{(q')^-}} |u_n - u|_{L^{q(x)}(a,\Omega)}. \end{aligned}$$

Combining this and the fact that $u_n \to u$ in $L^{q(x)}(a, \Omega)$, we infer

$$\lim_{n\to\infty} \int_\Omega a(x) (u_n^+)^{q(x)-1} (u_n - u) dx = 0.$$

Similarly, we obtain

$$\lim_{n\to\infty}\int_\Omega b(x)(u_n^+)^{h(x)-1}(u_n-u)dx=0.$$

Hence, (2.4) implies that

$$\limsup_{n\to\infty}\int_\Omega w(x)|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(u_n-u)dx=\limsup_{n\to\infty}\langle J_1'(u_n),u_n-u\rangle\le 0.$$

By the $(S_+)-$property of the degenerate $p(x)-$Laplacian (see [22, Lemma 3.2]), we find that $u_n\to u$ (strongly) in X as $n\to\infty$. ∎

3. Existence of Two Nontrivial Nonnegative Solutions

In this section, we show the existence of two nontrivial nonnegative solutions for (1.1). More restrictions can make it positive (see Remark 3.4). The following are the main results of this section. We emphasize that are not only many cases not considered in [16] to be treated, but also the condition on p,q,h is relaxed.

Theorem 3.1. *Assume that* $(w0)$ *and* (H) *hold.*

(i) *If* $p^+<h^-$ *and* $\{x\in\overline{\Omega}:q(x)<p(x)\}\ne\emptyset$, *then for each given* $\mu\in\mathbb{R}$, *there exists* $\lambda^*=\lambda^*(\mu)>0$ *such that, for any* $\lambda\in(0,\lambda^*)$, (1.1) *has a nontrivial nonnegative solution* $\underline{u}$ *with* $J(\underline{u})<0$.

(ii) *If* $q^+<p^-\le p^+<h^-$, *then for each given* $\mu>0$, *there exists* $\lambda^{**}=\lambda^{**}(\mu)>0$ *such that, for any* $\lambda<\lambda^{**}$, (1.1) *has a nontrivial nonnegative solution* $\overline{u}$ *with* $J(\overline{u})>0$.

(iii) *If* $q^+=p^-\le p^+<h^-$, *then for each given* $\mu>0$, *there exists* $\lambda^{***}=\lambda^{***}(\mu)>0$ *such that, for any* $\lambda\in(-\lambda^{***},\lambda^{***})$, (1.1) *has a nontrivial nonnegative solution* $\overline{u}$ *with* $J(\overline{u})>0$.

In the case of a fixed parameter in front of the concave term, we have the following.

Theorem 3.2. *Assume that* $(w0)$ *and* (H) *hold.*

(i) *If* $q^+<p^-$, *then for each given* $\lambda>0$, *there exists* $\mu^*=\mu^*(\lambda)>0$ *such that, for any* $\mu\in[0,\mu^*)$, (1.1) *has a nontrivial nonnegative solution* $\underline{u}$ *with* $J(\underline{u})<0$.

(ii) *If* $q^+<p^-$ *and* $\{x\in\overline{\Omega}:q(x)<h(x)\}\ne\emptyset$, *then for each given* $\lambda>0$, *there exists* $\mu^{**}=\mu^{**}(\lambda)>0$ *such that, for any* $\mu<\mu^{**}$, (1.1) *has a nontrivial nonnegative solution* $\underline{u}$ *with* $J(\underline{u})<0$.

(iii) *If* $q^+<p^-\le p^+<h^-$, *then for each given* $\lambda\in\mathbb{R}$, *there exists* $\mu^{***}=\mu^{***}(\lambda)>0$ *such that, for any* $\mu\in(0,\mu^{***})$, (1.1) *has a nontrivial nonnegative solution* $\overline{u}$ *with* $J(\overline{u})>0$.

(iv) *If* $\max\{p^+, q^+\} < h^-$, *then for any* $\lambda \leq 0$ *and* $\mu > 0$, (1.1) *has a nontrivial nonnegative solution* $\overline{u}$ *with* $J(\overline{u}) > 0$.

These results immediately yield the existence of two nontrivial nonnegative solutions for (1.1).

Corollary 3.3. *Assume that* $(w0)$ *and* (H) *hold.*

(i) *If* $q^+ \leq p^- \leq p^+ < h^-$ *and* $\{x \in \overline{\Omega} : q(x) < p(x)\} \neq \emptyset$, *then for each given* $\mu > 0$, *there exists* $\overline{\lambda} = \overline{\lambda}(\mu) > 0$ *such that, for any* $\lambda \in (0, \overline{\lambda})$, (1.1) *has two nontrivial nonnegative solutions* $\underline{u}$ *and* $\overline{u}$ *with* $J(\underline{u}) < 0 < J(\overline{u})$.

(ii) *If* $q^+ < p^- \leq p^+ < h^-$, *then for each given* $\lambda > 0$, *there exists* $\overline{\mu} = \overline{\mu}(\lambda) > 0$ *such that, for any* $\mu \in (0, \overline{\mu})$, (1.1) *has two nontrivial nonnegative solutions* $\underline{u}$ *and* $\overline{u}$ *with* $J(\underline{u}) < 0 < J(\overline{u})$.

It is worth noticing the positivity of the solutions for (1.1).

Remark 3.4. If we consider (1.1) with $w \equiv 1, p \in C^1(\overline{\Omega})$, (H^*) holds, and the parameters satisfy either $\lambda \geq 0, \mu \geq 0$ or $\lambda \geq 0, \mu < 0, b \in L^\infty(\Omega), p(x) \leq h(x), \forall x \in \overline{\Omega}$, then by the strong maximum principle for $p(x)-$Laplacian (see [17, Proposition 3.1]), all nontrivial nonnegative solutions are positive.

To apply the Ekeland variational principle and the Mountain Pass Theorem to show the proofs of Theorems 3.1-3.2, we first establish some geometric structures of J_1.

Lemma 3.5. *Assume that* $(w0)$ *and* (H) *hold.*

(i) *If* $p^+ < h^-$, *then for each given* $\mu \in \mathbb{R}$, *there exists* $\lambda^0 = \lambda^0(\mu) > 0$ *such that, for any* $\lambda < \lambda^0$, *there exist* $r, \rho > 0$ *such that* $J_1(u) \geq \rho$ *if* $\|u\| = r$.

(ii) *If* $q^+ < p^-$, *then for each given* $\lambda \in \mathbb{R}$, *there exists* $\mu^0 = \mu^0(\lambda) > 0$ *such that, for any* $\mu < \mu^0$, *there exist* $\overline{r}, \overline{\rho} > 0$ *such that* $J_1(u) \geq \overline{\rho}$ *if* $\|u\| = \overline{r}$.

Proof. By Proposition 2.6, there are two constants $C_a > 1, C_b > 1$ such that

$$|u|_{L^{q(x)}(a,\Omega)} \leq C_a\|u\|, |u|_{L^{h(x)}(b,\Omega)} \leq C_b\|u\|, \forall u \in X. \tag{3.1}$$

Proof of (i). For each given $\mu \in \mathbb{R}$, fix r_1 such that $0 < r_1 < \min\left\{\frac{1}{C_a}, \frac{1}{C_b}\right\}$ and $\frac{|\mu| C_b^{h^-}}{h^-} r_1^{h^-} \leq \frac{1}{2p^+} r_1^{p^+}$. By Proposition 2.2 and (3.1), for $\|u\| \leq r_1 (< 1)$, we have

$$
\begin{aligned}
J_1(u) &\geq \frac{1}{p^+}\int_\Omega w(x)|\nabla u|^{p(x)}dx - \frac{\max\{\lambda,0\}}{q^-}\int_\Omega a(x)|u|^{q(x)}dx \\
&\quad -\frac{|\mu|}{h^-}\int_\Omega b(x)|u|^{h(x)}dx \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\max\{\lambda,0\}}{q^-}|u|^{q^-}_{L^{q(x)}(a,\Omega)} - \frac{|\mu|}{h^-}|u|^{h^-}_{L^{h(x)}(b,\Omega)} \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\max\{\lambda,0\}C_a^{q^-}}{q^-}\|u\|^{q^-} - \frac{|\mu|C_b^{h^-}}{h^-}\|u\|^{h^-} \\
&\geq \frac{1}{2p^+}\|u\|^{p^+} - \frac{\max\{\lambda,0\}C_a^{q^-}}{q^-}\|u\|^{q^-}.
\end{aligned} \tag{3.2}
$$

Thus, for $u \in X$ with $\|u\| = r_1$, we have

$$
J_1(u) \geq \frac{1}{2p^+}r_1^{p^+} - \frac{\max\{\lambda,0\}C_a^{q^-}}{q^-}r_1^{q^-} = \frac{C_a^{q^-}}{q^-}r_1^{q^-}\left(\frac{q^-}{2p^+C_a^{q^-}}r_1^{p^+-q^-} - \max\{\lambda,0\}\right).
$$

Therefore, if we take $\lambda^0 = \frac{q^-}{2p^+C_a^{q^-}}r_1^{p^+-q^-} > 0$, then for any $\lambda < \lambda^0$, we have $r = r_1 > 0, \rho = \frac{C_a^{q^-}}{q^-}r_1^{q^-}(\lambda^0 - \max\{\lambda,0\}) > 0$ satisfying $J_1(u) \geq \rho$ if $\|u\| = r$.

Proof of (ii). Let $\lambda \in \mathbb{R}$ be given. Once again, using Proposition 2.2 and (3.1), for all $u \in X$, we have

$$
\begin{aligned}
\textstyle\int_\Omega a(x)|u(x)|^{q(x)}dx &\leq \max\left\{|u|^{q^-}_{L^{q(x)}(a,\Omega)}, |u|^{q^+}_{L^{q(x)}(a,\Omega)}\right\} \\
&\leq \max\left\{C_a^{q^-}\|u\|^{q^-}, C_a^{q^+}\|u\|^{q^+}\right\}.
\end{aligned} \tag{3.3}
$$

Similarly, we get

$$
\int_\Omega b(x)|u(x)|^{h(x)}dx \leq \max\left\{C_b^{h^-}\|u\|^{h^-}, C_b^{h^+}\|u\|^{h^+}\right\}. \tag{3.4}
$$

Setting $r_2 = \max\left\{1, \left(\frac{2p^+C_a^{q^+}|\lambda|}{q^-}\right)^{\frac{1}{p^- - q^+}}\right\}$ and taking into account (3.3)-(3.4) and (3.1), for $\|u\| = r_2$, we have

$$
\begin{aligned}
J_1(u) &\geq \frac{1}{p^+}\int_\Omega w(x)|\nabla u|^{p(x)}dx - \frac{|\lambda|}{q^-}\int_\Omega a(x)|u|^{q(x)}dx \\
&\quad -\frac{\max\{\mu,0\}}{h^-}\int_\Omega b(x)|u|^{h(x)}dx \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{|\lambda|C_a^{q^+}}{q^-}\|u\|^{q^+} - \frac{\max\{\mu,0\}C_b^{h^+}}{h^-}\|u\|^{h^+} \\
&\geq \frac{1}{2p^+}\|u\|^{p^-} - \frac{\max\{\mu,0\}C_b^{h^+}}{h^-}\|u\|^{h^+} \\
&= \frac{C_b^{h^+}r_2^{h^+}}{h^-}\left(\frac{h^- r_2^{p^- - h^+}}{2p^+C_b^{h^+}} - \max\{\mu,0\}\right).
\end{aligned} \tag{3.5}
$$

Therefore, if we take $\mu^0 = \frac{h^- r_2^{p^- - h^+}}{2p^+C_b^{h^+}} > 0$, then for any $\mu < \mu^0$, we have $\overline{r} = r_2 > 0, \overline{\rho} = \frac{C_b^{h^+}r_2^{h^+}}{h^-}(\mu^0 - \max\{\mu,0\}) > 0$ satisfying $J_1(u) \geq \overline{\rho}$ if $\|u\| = \overline{r}$. ∎

The next two lemmas are crucial since we can relax the condition on q, p, h.

Lemma 3.6. *Assume that* $(w0)$ *and* (H) *hold. If* $\{x \in \overline{\Omega} : \min\{p(x), h(x)\} > q(x)\} \neq \emptyset$, *then for any* $\lambda > 0$ *and* $\mu \in \mathbb{R}$, *there exists* $\phi \in X, \phi \geq 0$ *such that* $J_1(t\phi) < 0$ *for all small* $t > 0$. *In the case* $\{x \in \overline{\Omega} : p(x) > q(x)\} \neq \emptyset$, *the conclusion remains valid for any* $\lambda > 0$ *and* $\mu \geq 0$.

Proof. First, consider the case $q(x_0) < \min\{p(x_0), h(x_0)\}$ for some $x_0 \in \overline{\Omega}$. Let $\lambda > 0$ and $\mu \in \mathbb{R}$ be arbitrary. Let δ_0 be such that $0 < 2\delta_0 < \min\{p(x_0), h(x_0)\} - q(x_0)$. Since $p, q, h \in C(\overline{\Omega})$, there is an open ball B_0 such that $\overline{B_0} \subset \Omega$ and $|q(x) - q(x_0)| < \delta_0, |p(x) - p(x_0)| < \delta_0, |h(x) - h(x_0)| < \delta_0$ for all $x \in B_0$. This yields

$$
q(x) < q(x_0) + \delta_0, p(x) > p(x_0) - \delta_0, h(x) > h(x_0) - \delta_0 \text{ for all } x \in B_0. \tag{3.6}
$$

Let $\phi \in C_0^\infty(\Omega) \setminus \{0\}$ be such that $\mathrm{supp}(\phi) \subset B_0$ and $0 \leq \phi \leq 1$ on B_0. Then for $t \in (0,1)$, we have

$$
J_1(t\phi) \leq \frac{t^{p(x_0)-\delta_0}}{p^-}\int_{B_0} w(x)|\nabla\phi|^{p(x)}dx - \frac{\lambda A t^{q(x_0)+\delta_0}}{q^+} + \frac{|\mu|t^{h(x_0)-\delta_0}}{h^-}\int_{B_0} b(x)dx,
$$

where $A := \int_{B_0} a(x)\phi^{q(x)}dx > 0$. Thus, $J_1(t\phi) < 0$ for sufficiently small $t > 0$ since $q(x_0) + \delta_0 < \min\{p(x_0) - \delta_0, h(x_0) - \delta_0\}$.

For the last case, let $q(x_0) < p(x_0)$ for some $x_0 \in \overline{\Omega}$ and $\lambda > 0, \mu \geq 0$. Once again, since $q, p \in C(\overline{\Omega})$, there exist $\delta_0 > 0$ and an open ball B_0 such that $\overline{B_0} \subset \Omega$ and $q(x) < q(x_0) + \delta_0 < p(x_0) - \delta_0 < p(x)$ for all $x \in B_0$. Let ϕ be as above. Then for $t \in (0,1)$, we have

$$
J_1(t\phi) \leq \frac{t^{p(x_0)-\delta_0}}{p^-}\int_{B_0} w(x)|\nabla\phi|^{p(x)}dx - \frac{\lambda t^{q(x_0)+\delta_0}}{q^+}\int_{B_0} a(x)\phi^{q(x)}dx.
$$

We obtain the same conclusion by repeating the argument above. ∎

Lemma 3.7. *Assume that* $(w0)$ *and* (H) *hold. If* $\{x \in \overline{\Omega} : \max\{p(x), q(x)\} < h(x)\} \neq \emptyset$, *then for any* $\lambda \in \mathbb{R}$ *and* $\mu > 0$, *there exists* $\phi \in X \setminus \{0\}$ *such that* $\phi \geq 0$ *and* $J_1(t\phi) \to -\infty$ *as* $t \to +\infty$.

Proof. Suppose that $\max\{p(x_0), q(x_0)\} < h(x_0)$ for some $x_0 \in \overline{\Omega}$, and take δ_0 such that $0 < 2\delta_0 < h(x_0) - \max\{p(x_0), q(x_0)\}$. Since $p, q, h \in C(\overline{\Omega})$, there is an open ball B_0 such that $\overline{B_0} \subset \Omega$ and

$$q(x) < q(x_0) + \delta_0, p(x) < p(x_0) + \delta_0, h(x) > h(x_0) - \delta_0 \text{ for all } x \in B_0.$$

Let $\lambda \in \mathbb{R}$ and $\mu > 0$ be arbitrary and $\phi \in C_0^\infty(\Omega) \setminus \{0\}$ be such that $\operatorname{supp}(\phi) \subset B_0$ and $0 \leq \phi \leq 1$ on B_0. Then for $t > 1$, we have

$$J_1(t\phi) \leq \frac{t^{p(x_0)+\delta_0}}{p^-} \int_{B_0} w(x)|\nabla \phi|^{p(x)} dx + \frac{|\lambda| t^{q(x_0)+\delta_0}}{q^-} \int_{B_0} a(x) dx - \frac{\mu B t^{h(x_0)-\delta_0}}{h^+},$$

where $B := \int_{B_0} b(x)\phi^{h(x)} dx > 0$. Thus, we arrive at $J_1(t\phi) \to -\infty$ as $t \to +\infty$ since $\max\{q(x_0) + \delta_0, p(x_0) + \delta_0\} < h(x_0) - \delta_0$. ∎

We next show the compactness that is crucial for seeking critical points.

Lemma 3.8. *Assume that* $(w0)$ *and* (H) *hold. Then* J_1 *holds* (PS) *condition if one of the following conditions is satisfied;*

(i) $q^+ < p^- \leq p^+ < h^-$ *and* $\lambda \in \mathbb{R}, \mu \geq 0$,

(ii) $q^+ = p^- \leq p^+ < h^-$ *and* $\lambda, \mu \in \mathbb{R}$ *with* $|\lambda| < \lambda_0 := \frac{\frac{1}{p^+} - \frac{1}{h^-}}{(\frac{1}{q^-} - \frac{1}{h^-}) C_a^{q^+}}, \mu \geq 0$, *where* C_a *is as in* (3.1),

(iii) $p^+ < h^-, q^+ \leq h^-$ *and* $\lambda \leq 0, \mu \geq 0$.

Proof. Because of the reflexiveness of X and the (S_+)-property of J_1', to show that J_1 holds (PS) condition, it is enough to show that every (PS) sequence is bounded. Suppose that $q^+, p^+ \leq h^-$ and $\lambda \in \mathbb{R}, \mu \geq 0$. Let $\{u_n\}$ be a (PS) sequence, i.e.,

$$\begin{cases} |J_1(u_n)| < M, \quad \forall n \in \mathbb{N}, \\ J_1'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty, \end{cases} \tag{3.7}$$

for some positive constant M. This implies that, for n large, we have

$$\begin{aligned} M + \|u_n\| &\geq J_1(u_n) - \frac{1}{h^-}\langle J_1'(u_n), u_n\rangle \\ &\geq \int_\Omega w(x)\left(\frac{1}{p(x)} - \frac{1}{h^-}\right)|\nabla u_n|^{p(x)} dx \\ &\quad -\lambda \int_\Omega \left(\frac{1}{q(x)} - \frac{1}{h^-}\right) a(x)|u_n^+|^{q(x)} dx. \end{aligned} \tag{3.8}$$

In case (iii), (3.8) and Proposition 2.2 yield

$$M + \|u_n\| \geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right)\left(\|u_n\|^{p^-} - 1\right).$$

This follows the boundedness of $\{u_n\}$ since $p^- > 1$. In cases (i) and (ii), by taking into account Proposition 2.2 and (3.1), (3.8) implies that

$$\begin{aligned} M + \|u_n\| &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right)\left(\|u_n\|^{p^-} - 1\right) - \left(\frac{1}{q^-} - \frac{1}{h^-}\right)|\lambda|\left(1 + |u_n|^{q^+}_{L^{q(x)}(a,\Omega)}\right) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right)\left(\|u_n\|^{p^-} - 1\right) - \left(\frac{1}{q^-} - \frac{1}{h^-}\right)|\lambda|\left(1 + C_a^{q^+}\|u_n\|^{q^+}\right). \end{aligned}$$

Through this, we easily deduce the boundedness of $\{u_n\}$ in both cases (i) and (ii). ■

Remark 3.9. Since the proofs of Lemmas 3.5-3.8 can be similarly checked for J, all statements for J_1 are still valid for J.

Now, we are ready to give the proofs of Theorems 3.1-3.2.

Proof of Theorem 3.1. For (i), let $p^+ < h^-$ and $\{x \in \overline{\Omega} : q(x) < p(x)\} \neq \emptyset$ and $\mu \in \mathbb{R}$. Then we take $\lambda^* = \lambda^0(\mu)$ with $\lambda^0(\mu)$ as in Lemma 3.5(i). Thus, for a fixed $\lambda \in (0, \lambda^*)$, there exist $r, \rho > 0$ such that $J_1(u) \geq \rho$ if $\|u\| = r$. Let us denote $c := \inf_{u \in \overline{B}_r} J_1(u)$, where $B_r := \{u \in X : \|u\| < r\}$ with a boundary ∂B_r. Then by (3.2) and Lemma 3.6, we deduce $-\infty < c < 0$.

Putting $0 < \epsilon < \inf_{u \in \partial B_r} J_1(u) - c$, by the Ekeland variational principle, we find $u_\epsilon \in \overline{B}_r$ such that

$$\begin{cases} J_1(u_\epsilon) \leq c + \epsilon, \\ J_1(u_\epsilon) < J_1(u) + \epsilon\|u - u_\epsilon\|, \ \forall u \in \overline{B}_r, u \neq u_\epsilon. \end{cases} \tag{3.9}$$

This implies that $u_\epsilon \in B_r$ since $J_1(u_\epsilon) \leq c + \epsilon < \inf_{u \in \partial B_r} J_1(u)$. From these facts, we have that u_ϵ is a local minimum of the funtional $\widetilde{J}(u) = J_1(u) + \epsilon\|u - u_\epsilon\|$. Therefore, for $v \in B_1$ and sufficiently small $t > 0$, we have

$$0 \leq \frac{\widetilde{J}(u_\epsilon + tv) - \widetilde{J}(u_\epsilon)}{t} = \frac{J_1(u_\epsilon + tv) - J_1(u_\epsilon)}{t} + \epsilon\|v\|.$$

Therefore, letting $t \to 0^+$, we obtain

$$\langle J_1'(u_\epsilon), v\rangle + \epsilon\|v\| \geq 0.$$

Replacing v by $-v$ in the argument above, we get

$$-\langle J_1'(u_\epsilon), v\rangle + \epsilon\|v\| \geq 0.$$

Hence, we have

$$|\langle J_1'(u_\epsilon), v\rangle| \leq \epsilon\|v\|, \forall v \in \overline{B}_1.$$

Thus, we infer

$$\|J_1'(u_\epsilon)\|_{X^*} \leq \epsilon. \tag{3.10}$$

From (3.9) and (3.10), we deduce a sequence $\{u_n\} \subset B_r$ such that

$$\begin{cases} J_1(u_n) \to c \ \text{ as } \ n \to \infty, \\ J_1'(u_n) \to 0 \ \text{ in } X^* \text{ as } n \to \infty. \end{cases} \tag{3.11}$$

Thus, $\{u_n\}$ is a bounded (PS) sequence in the reflexive Banach space X. Due to the $(S_+)-$property of J_1', $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to \underline{u}$ in X as $n_k \to \infty$. It follows from this and (3.11) that $J_1(\underline{u}) = c$ and $J_1'(\underline{u}) = 0$. Hence, $\underline{u}$ is a nontrivial nonnegative solution to (1.1) with $J(\underline{u}) = J_1(\underline{u}) < 0$.

For (ii), let $q^+ < p^- \leq p^+ < h^-$ and μ be a positive real number and take $\lambda^{**} = \lambda^0(\mu)$, where $\lambda^0(\mu)$ is as in Lemma 3.5(i). Then, for a fixed $\lambda < \lambda^{**}$, there exist $r, \rho > 0$ such that $J_1(u) \geq \rho$ if $\|u\| = r$; by Lemma 3.7, there is $e \in X$ such that $\|e\| > r$ and $J_1(e) < 0$. Moreover, by Lemma 3.8(i), J_1 satisfies (PS) condition. The Mountain Pass Theorem implies that J_1 has a critical point $\overline{u}$ satisfying $J_1(\overline{u}) \geq \rho$. Hence, (1.1) has a nontrivial nonnegative solution $\overline{u}$ with $J(\overline{u}) > 0$.

The proof of (iii) is similar to that of (ii). Here we take $\lambda^{***} = \min\{\lambda_0, \lambda^0(\mu)\}$, where $\lambda^0(\mu)$ is as in Lemma 3.5(i) and λ_0 is as in Lemma 3.8(ii). Then, for a fixed $\lambda \in (-\lambda^{***}, \lambda^{***})$, we verify that J_1 satisfies all conditions of the Mountain Pass Theorem using Lemma 3.5(i), Lemma 3.7, and Lemma 3.8(ii). ∎

Proof of Theorem 3.2. The proofs of (i) and (ii) are similar to that of Theorem 3.1 (i). Let $q^+ < p^-$ and λ be a positive real number. If $\mu^* = \mu^{**} = \mu^0(\lambda)$, where $\mu^0(\lambda) > 0$ is as in Lemma 3.5(ii), then all arguments in the proof of Theorem 3.1(i) are still workable for any $\mu \in [0, \mu^*)$ in case (i) and for any $\mu < \mu^{**}$ in case (ii).

To show (iii) and (iv), we apply the Mountain Pass Theorem for J_1. In case (iii), for a given $\lambda \in \mathbb{R}$, take $\mu^{***} = \mu^0(\lambda)$ with $\mu^0(\lambda)$ in Lemma 3.5(ii), and let $\mu \in (0, \mu^{***})$. Then, by Lemma 3.5(ii), there exist $\overline{r}, \overline{\rho} > 0$ such that $J_1(u) \geq \overline{\rho}$ if $\|u\| = \overline{r}$; by Lemma 3.7, there is $\overline{e} \in X$ such that $\|\overline{e}\| > \overline{r}$ and $J_1(\overline{e}) < 0$. Lemma 3.8 (i) guarantees that J_1 satisfies the (PS) condition. So J_1 has a critical point $\overline{u}$ satisfying $J_1(\overline{u}) \geq \rho$. Hence, (1.1) has a nontrivial nonnegative solution $\overline{u}$ with $J(\overline{u}) > 0$. In case (iv), for any $\lambda \leq 0, \mu > 0$, Lemma 3.5(i), Lemma 3.7, and Lemma 3.8(iii) deduce that J_1 satisfies all conditions of the Mountain Pass Theorem. ∎

4. Existence of Infinitely Many Solutions

The existence of infinitely many solutions for (1.1) was studied in [16] when $w \equiv 1$ and $q^+ < p^- \leq p^+ < h^-$. In this section, we obtain the same or similar results as in [16] when degeneracy and various situations regarding the order of p, q, h and the range of the parameters are considered.

Recall that, if X is a separable reflexive Banach space, then it is well-known that there exist $\{e_n\}_{n=1}^\infty \subset X$ and $\{f_n\}_{n=1}^\infty \subset X^*$ such that $X = \overline{\text{span}\{e_n\}_{n=1}^\infty}$, $X^* = \overline{\text{span}\{f_n\}_{n=1}^\infty}$ and

$$\langle f_i, e_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the duality product of X^* and X (see [35, Section 17]). Denote

$$X_n = \text{span}\{e_n\}, \ Y_n = \oplus_{k=1}^n X_k, \ Z_n = \overline{\oplus_{k=n}^\infty X_k}.$$

For a separable reflexive Banach space X and X_n, Y_n, Z_n as above, let us recall fundamental theorems to obtain a sequence of critical values.

Proposition 4.1. [33, Fountain Theorem] *Assume that $J \in C^1(X, \mathbb{R})$ is even and that, for each $n = 1, 2, \ldots$, there exist $\rho_n > \gamma_n > 0$ such that*

$(H1)$ $b_n = \inf_{\{u \in Z_n \colon \|u\| = \gamma_n\}} J(u) \to +\infty$ *as* $n \to \infty$;

$(H2)$ $a_n = \max_{\{u \in Y_n \colon \|u\| = \rho_n\}} J(u) \leq 0$;

$(H3)$ *J satisfies $(PS)_c$-condition for every $c > 0$.*

Then J has a sequence of critical values tending to $+\infty$.

Proposition 4.2. [33, Dual Fountain Theorem] *Assume that $J \in C^1(X, \mathbb{R})$ is even. If there exists $n_0 (\in \mathbb{N}) > 0$ such that, for each $n \geq n_0$, there exist $\rho_n > \gamma_n > 0$ such that*

$(D1)$ $\inf_{\{u \in Z_n \colon \|u\| = \rho_n\}} J(u) \geq 0$;

$(D2)$ $b_n = \max_{\{u \in Y_n \colon \|u\| = \gamma_n\}} J(u) < 0$;

$(D3)$ $d_n = \inf_{\{u \in Z_n \colon \|u\| \leq \rho_n\}} J(u) \to 0$ *as* $n \to \infty$;

$(D4)$ *J satisfies $(PS)_c^*$-condition for every $c \in [d_{n_0}, 0)$, i.e., for any $c \in [d_{n_0}, 0)$, if $\{u_{n_j}\}$ is a sequence in X such that $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \to c$, $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$, then $\{u_{n_j}\}$ has a subsequence converging to a critical point of J.*

Then J has a sequence of negative critical values tending to 0.

Now, we state the main result of this section.

Theorem 4.3. *Assume that* $(w0)$ *and* (H) *hold.*

(i) *If* $q^+ < p^- \le p^+ < h^-$, *then for any* $\lambda \in \mathbb{R}, \mu > 0$, (1.1) *has a sequence of solutions* $\{\pm u_n\}$ *such that* $J(\pm u_n) \to \infty$ *as* $n \to \infty$.

(ii) *If* $q^+ = p^- \le p^+ < h^-$, *then for any* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \lambda_0$ *and* $\mu > 0$, *the conclusion above remains valid (here,* λ_0 *is as in Lemma 3.8*(ii).

(iii) *If* $q^+ < p^- \le p^+ < h^-$, *then for any* $\lambda > 0, \mu \in \mathbb{R}$, (1.1) *has a sequence of solutions* $\{\pm v_n\}$ *such that* $J(\pm v_n) < 0, \forall n$, *and* $J(\pm v_n) \to 0$ *as* $n \to \infty$.

(iv) *If* $q^+ < \min\{p^-, h^-\}$ *and* $p^+ < h^+$, *then for any* $\lambda > 0, \mu \le 0$, *the conclusion in* (iii) *remains valid.*

Proof. Note that $J \in C^1(X, \mathbb{R})$, and J is even. Write $J(u) = \int_\Omega \frac{w(x)}{p(x)} |\nabla u|^{p(x)} dx - \Psi(u)$, where $\Psi : X \to \mathbb{R}$ is defined by $\Psi(u) = \lambda \int_\Omega \frac{a(x)}{q(x)} |u|^{q(x)} dx + \mu \int_\Omega \frac{b(x)}{h(x)} |u|^{h(x)} dx$. By Proposition 2.9, Ψ is of $C^1(X, \mathbb{R})$, and Ψ is sequentially weakly continuous.

To prove cases (i) and (ii), we only need to verify that J satisfies $(H1) - (H3)$ in Fountain Theorem. Let $q^+ \le p^- \le p^+ < h^-$ and $\lambda \in \mathbb{R}, \mu > 0$. $(H1)$ can be shown using the same argument as in [16, Proof of Theorem 4.1 (5)]. For case (i), $(H3)$ is deduced from Lemma 3.8(i); for case (ii), $(H3)$ is deduced from Lemma 3.8(ii) with Remark 3.9. To show $(H2)$, taking $u \in X$ with $\|u\| > 1$ and using Proposition 2.2, we have

$$J(u) \le \frac{1}{p^-} \|u\|^{p^+} + \frac{|\lambda|}{q^-} \left(|u|^{q^+}_{L^{q(x)}(a,\Omega)} + 1 \right) - \frac{\mu}{h^+} \left(|u|^{h^-}_{L^{h(x)}(b,\Omega)} - 1 \right). \tag{4.1}$$

Since, on the finite dimensional space Y_n, norms $\|\cdot\|, |\cdot|_{L^{q(x)}(a,\Omega)}$ and $|\cdot|_{L^{h(x)}(b,\Omega)}$ are equivalent and $q^+ \le p^+ < h^-$, (4.1) implies that $J(u) \le 0$ for all $u \in Y_n$ with sufficiently large $\|u\|$. This follows $(H2)$.

To show cases (iii) and (iv), we verify that J satisfies $(D1) - (D4)$ in Dual Fountain Theorem. $(D1)$ and $(D3)$ can be shown as in [16, Proof of Theorem 4.1 (6)], in which $\rho_n = 1, \forall n \in \mathbb{N}$. For $(D2)$, taking r such that $0 < r < \min\left\{ \frac{1}{C_a}, \frac{1}{C_b} \right\}$, where C_a, C_b are as in (3.1), we get that, for $\|u\| = r$,

$$J(u) \le \frac{1}{p^-} \|u\|^{p^-} - \frac{\lambda}{q^+} |u|^{q^+}_{L^{q(x)}(a,\Omega)} + \frac{|\mu|}{h^-} |u|^{h^-}_{L^{h(x)}(b,\Omega)}.$$

Since $q^+ < \min\{p^-, h^-\}$ in both cases (iii) and (iv) and norms $\|\cdot\|, |\cdot|_{L^{q(x)}(a,\Omega)}$ and $|\cdot|_{L^{h(x)}(b,\Omega)}$ are equivalent on the finite dimensional space Y_n, we can choose $\gamma_n \in (0, \rho_n)$ such that $J(u) \le 0$ for all $u \in Y_n$ with $\|u\| = \gamma_n$. This implies $(D2)$. Finally, to verify $(D4)$, we show that J satisfies $(PS)^*_c$-condition for every $c \in \mathbb{R}$. Let

$\{u_{n_j}\} \subset X$ be such that $u_{n_j} \in Y_{n_j}, J(u_{n_j}) \to c, (J|_{Y_{n_j}})'(u_{n_j}) \to 0$ as $n_j \to \infty$. So, for large n, we have that, for $p^+ < h^-$ and $\mu \geq 0$,

$$\begin{aligned} c+1+\|u_{n_j}\| &\geq J(u_{n_j}) - \frac{1}{h^-}\langle (J|_{Y_{n_j}})'(u_{n_j}), u_{n_j}\rangle \\ &= J(u_{n_j}) - \frac{1}{h^-}\langle J'(u_{n_j}), u_{n_j}\rangle \\ &\geq \left(\frac{1}{p^+}-\frac{1}{h^-}\right)(\|u_{n_j}\|^{p^-}-1) - \left(\frac{1}{q^-}-\frac{1}{h^-}\right)\lambda\left(1+C_a^{q^+}\|u_{n_j}\|^{q^+}\right), \end{aligned}$$

and for $p^+ < h^+$ and $\mu \leq 0$,

$$\begin{aligned} c+1+\|u_{n_j}\| &\geq J(u_{n_j}) - \frac{1}{h^+}\langle (J|_{Y_{n_j}})'(u_{n_j}), u_{n_j}\rangle \\ &= J(u_{n_j}) - \frac{1}{h^+}\langle J'(u_{n_j}), u_{n_j}\rangle \\ &\geq \left(\frac{1}{p^+}-\frac{1}{h^+}\right)(\|u_{n_j}\|^{p^-}-1) - \left(\frac{1}{q^-}-\frac{1}{h^+}\right)\lambda\left(1+C_a^{q^+}\|u_{n_j}\|^{q^+}\right). \end{aligned}$$

This implies the boundedness of $\{u_{n_j}\}$ in both cases (iii) and (iv). Since the remaining process to verify $(D4)$ is the same as in [16, Proof of Theorem 4.1 (6)], we omit it. ■

Remark 4.4. Theorems 3.1-3.2 and Theorem 4.3 remain valid if $(w0)$ and (H) are replaced by $(w1)$ and (H^*), respectively.

5. Caffarelli-Kohn-Nirenberg Type Problems

In this section, we discuss some Caffarelli-Kohn-Nirenberg type problems. The following is similar to the result in [16] for the case of no degeneracy, but the proof should be different since $\delta(x)$ may be zero.

Theorem 5.1. *Assume that $p \in C_+(\overline{\Omega})$ and that $(w0)$ holds. Assume also that $r, \delta \in C(\overline{\Omega})$ such that $0 \leq \delta(x) < N$ for all $x \in \overline{\Omega}$, and*

(5.1) $$1 \leq r(x) < \frac{N-\delta(x)}{N} p_s^*(x),\ \forall x \in \overline{\Omega}.$$

Then we have the continuous compact imbedding

$$W^{1,p(x)}(w,\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(|x|^{-\delta(x)},\Omega).$$

Moreover, if $(w0)$ and $p_s^*(x)$ are replaced by $(w1)$ and $p^*(x)$, respectively, the imbedding remains valid.

Proof. By the continuity of r, δ, p, s, and the compactness of $\overline{\Omega}$, there exists $\epsilon > 0$ such that $\delta(x) < N - 2\epsilon$ and $r(x) < \frac{N-2\epsilon-\delta(x)}{N-\epsilon} p_s^*(x)$ for all $x \in \overline{\Omega}$. Let $\gamma(x) = \frac{N-\epsilon}{\delta(x)+\epsilon}$

and $d(x) = |x|^{-\delta(x)-\epsilon}$, then d, γ satisfy the hypotheses of Proposition 2.6. Thus, by Proposition 2.6, we obtain

$$W^{1,p(x)}(w,\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(|x|^{-\delta(x)-\epsilon},\Omega). \tag{5.2}$$

The boundedness of Ω implies that $L^{r(x)}(|x|^{-\delta(x)-\epsilon},\Omega) \hookrightarrow L^{r(x)}(|x|^{-\delta(x)},\Omega)$. Hence, this and (5.2) imply that

$$W^{1,p(x)}(w,\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(|x|^{-\delta(x)},\Omega).$$

Suppose that $(w1)$ holds and $1 \le r(x) < \frac{N-\delta(x)}{N}p^*(x),\ \forall x \in \overline{\Omega}$. Then, by the continuity of r, δ, p, and the compactness of $\overline{\Omega}$, there exists a constant $s_0 > \max\{\frac{N}{p^-}, \frac{1}{p^- -1}\}$ such that

$$1 \le r(x) < \frac{N-\delta(x)}{N}p^*_{s_0}(x),\ \forall x \in \overline{\Omega}.$$

This implies that $(w0)$ and (5.1) hold for $s(x) \equiv s_0$, and this completes the proof. ■

Remark 5.2. Theorem 2.1 and Corollary 2.1 in [16] are the special cases of Corollary 2.7 and Theorem 5.1, respectively, when $w(x) \equiv 1$ on $\overline{\Omega}$.

Some interesting consequences follow from Theorem 5.1.

Corollary 5.3. *Assume that $p \in C_+(\overline{\Omega})$ and $\theta, r, \delta \in C(\overline{\Omega})$ such that $0 \le \theta(x) < \frac{N}{s_0}$ for some constant $s_0 \ge \max\{\frac{N}{p^-}, \frac{1}{p^- -1}\}$ and $0 \le \delta(x) < N$ for all $x \in \overline{\Omega}$ and*

$$1 \le r(x) < \frac{N-\delta(x)}{N}p^*_{s_0}(x),\ \forall x \in \overline{\Omega}. \tag{5.3}$$

Then, we have the continuous compact imbedding

$$W^{1,p(x)}(|x|^{\theta(x)},\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(|x|^{-\delta(x)},\Omega).$$

Consequently, we obtain a Caffarelli-Kohn-Nirenberg type inequality;

$$|u|_{L^{r(x)}(|x|^{-\delta(x)},\Omega)} \le C|\nabla u|_{L^{p(x)}(|x|^{\theta(x)},\Omega)}, \forall u \in W_0^{1,p(x)}(|x|^{\theta(x)},\Omega).$$

In particular, when $r(x) \equiv r$ (constant) and $p(x) \equiv p$ (constant), we have

$$\left(\int_\Omega |x|^{-\delta(x)}|u|^r dx\right)^{1/r} \le C\left(\int_\Omega |x|^{\theta(x)}|\nabla u|^p dx\right)^{1/p}, \forall u \in W_0^{1,p}(|x|^{\theta(x)},\Omega).$$

Proof. By the hypothesis, we have $\theta^+ < \frac{N}{s_0}$. Let s_1 be such that $s_0 < s_1$ with $\theta^+ s_1 < N$. This and (5.3) imply that

$$1 \le r(x) < \frac{N-\delta(x)}{N}p^*_{s_1}(x),\ \forall x \in \overline{\Omega}.$$

It is clear that $w(x) = |x|^{\theta(x)} \in L^1_{loc}(\Omega)$. Take $R > 1$ such that $\overline{\Omega} \subset B(0, R)$. Then, $\int_{1\le|x|\le R} w(x)^{-s_1} dx < \infty$. Note that $w(x)^{-s_1} \le |x|^{-\theta^+ s_1}$ for $|x| \le 1$ and $\int_{|x|\le 1} |x|^{-\theta^+ s_1} dx < \infty$ since $\theta^+ s_1 < N$, we infer that $\int_{|x|\le 1} w(x)^{-s_1} dx < \infty$. These facts imply that $w^{-s_1} \in L^1(B(0, R))$; hence, $w^{-s_1} \in L^1(\Omega)$. So $w(x) = |x|^{\theta(x)}$ satisfies $(w0)$ for $s(x) \equiv s_1$; hence, we obtain the desired conclusion in view of Theorem 5.1 . ∎

The next is for case $(w1)$.

Corollary 5.4. *Assume that* $0 \in \partial\Omega$, $p \in C_+(\overline{\Omega})$, *and* $\theta, r, \delta \in C(\overline{\Omega})$ *such that* $0 \le \theta(x), 0 \le \delta(x) < N$ *for all* $x \in \overline{\Omega}$ *and*

$$1 \le r(x) < \frac{N - \delta(x)}{N} p^*(x),\ \forall x \in \overline{\Omega}.$$

Then we have the continuous compact imbedding

$$W^{1,p(x)}(|x|^{-\theta(x)}, \Omega) \hookrightarrow\hookrightarrow L^{r(x)}(|x|^{-\delta(x)}, \Omega).$$

Consequently, we obtain a Caffarelli-Kohn-Nirenberg type inequality;

$$|u|_{L^{r(x)}(|x|^{-\delta(x)},\Omega)} \le C|\nabla u|_{L^{p(x)}(|x|^{-\theta(x)},\Omega)}, \forall u \in W_0^{1,p(x)}(|x|^{-\theta(x)}, \Omega).$$

In particular, when $r(x) \equiv r$ *(constant) and* $p(x) \equiv p$ *(constant), we have*

$$\left(\int_\Omega |x|^{-\delta(x)}|u|^r dx\right)^{1/r} \le C \left(\int_\Omega |x|^{-\theta(x)}|\nabla u|^p dx\right)^{1/p}, \forall u \in W_0^{1,p}(|x|^{-\theta(x)}, \Omega).$$

Lastly, we summarize our results regarding Caffarelli-Kohn-Nirenberg type questions as below.

Remark 5.5. (i) It is worth comparing our results with the original Caffarelli-Kohn-Nirenberg inequality [6];
For $N \ge 2, p \in (1, N)$, there exists a positive constant $C_{a,b}$ such that

$$\left(\int_\Omega |x|^{-br}|u|^r dx\right)^{1/r} \le C_{a,b} \left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{1/p}, \forall u \in C_0^1(\Omega), \tag{5.4}$$

where $-\infty < a < \frac{N-p}{p}, a \le b \le a + 1, r = \frac{Np}{N-p-ap+bp}$, and Ω is an arbitrary open domain in $\mathbb{R}^N$.

In fact, Corollary 5.3 and Corollary 5.4 can be interpreted when all variable exponents are constants and $N \ge 2, p \in (1, N)$ as follows;
"There exists a positive constant C such that

$$\left(\int_\Omega |x|^{-br}|u|^r dx\right)^{1/r} \le C \left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{1/p}, \forall u \in W_0^{1,p}(|x|^{-ap}, \Omega),$$

where $-\frac{\alpha}{p} < a \leq 0 \leq b < 1 + \frac{N(p-1)-\alpha}{p}, 1 \leq r < \frac{Np}{N-p+\alpha+bp}$ for some $0 < \alpha \leq \min\{p, N(p-1)\}$."
and
"Suppose that $0 \in \partial\Omega$. Then, there exists a positive constant C such that

$$\left(\int_\Omega |x|^{-br}|u|^r dx\right)^{1/r} \leq C\left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{1/p}, \forall u \in W_0^{1,p}(|x|^{-ap}, \Omega),$$

where $0 \leq a, 0 \leq b < 1 + \frac{N(p-1)}{p}, 1 \leq r < \frac{Np}{N-p+bp}$."

Note that the condition for a, b in [6] is $a \leq 0 \leq b \leq a+1, r = \frac{Np}{N-p-ap+bp}$ when $a \leq 0 \leq b$ and is $0 \leq a \leq \frac{N-p}{p}, a \leq b \leq a+1, r = \frac{Np}{N-p-ap+bp}$ when $0 \leq a, b$.

(ii) In view of Corollary 5.3 and Corollary 5.4, we studied Caffarelli-Kohn-Nirenberg type problem (1.5) under each condition:

(I) $0 \in \overline{\Omega}$, $p \in C_+(\overline{\Omega})$, and $\theta, \xi, \delta, q, h \in C(\overline{\Omega})$ such that $0 \leq \theta(x) < \frac{N}{s_0}$, $0 \leq \xi(x), \delta(x) < N$ and

$$1 < q(x) < \frac{N-\xi(x)}{N}p^*_{s_0}(x),\ 1 < h(x) < \frac{N-\delta(x)}{N}p^*_{s_0}(x),\ \forall x \in \overline{\Omega},$$

for some constant $s_0 \geq \max\{\frac{N}{p^-}, \frac{1}{p^- -1}\}$,

(II) $0 \in \partial\Omega$, $p \in C_+(\overline{\Omega})$, and $\theta, \xi, \delta, q, h \in C(\overline{\Omega})$ such that $\theta(x) \leq 0 \leq \xi(x), \delta(x) < N$ and

$$1 < q(x) < \frac{N-\xi(x)}{N}p^*(x),\ 1 < h(x) < \frac{N-\delta(x)}{N}p^*(x),\ \forall x \in \overline{\Omega}.$$

In other words, Theorems 3.1-3.2 and 4.3 for (1.5) in either condition (I) or (II) remain valid when applying either Corollary 5.3 or Corollary 5.4, respectively.

Acknowledgment

The second author was supported by a National Research Foundation of Korea grant funded by the Korean government (MEST) (NRF-2012R1A1A2000739).

References

1. B. Abdellaoui, E. Colorado and I. Peral, Some critical quasilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions: relation with Sobolev and Hardy-Sobolev optimal constants, *J. Math. Anal. Appl.*, **332** (2007), 1165-1188.

2. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349-381.

3. A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.*, **122(2)** (1994), 519-543.

4. T. Bartsch, S. Peng and Z. Zhang, Existence and non-existence of solutions to elliptic equations related to the Caffarelli-Kohn-Nirenberg inequalities, *Calc. Var. Partial Differential Equations*, **30(1)** (2007), 113-136.

5. H. Brezis and L. Nirenberg, H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris Sér. I Math.*, **317(5)** (1993), 465-472.

6. L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, *Compositio Math.*, **53** (1984), 259-275.

7. Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.*, **66(4)** (2006), 1383-1406.

8. R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 1, Physical Origins and Classical Methods, Springer-Verlag, Berlin, 1985.

9. D. de Figueiredo, J.-P. Gossez and P. Ubilla, Local "superlinearity" and "sublinearity" for the p-Laplacian, *J. Funct. Anal.*, **257** (2009), 721-752.

10. L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics 2017, Springer-Verlag, Heidelberg, 2011.

11. P. Drábek, A. Kufner and F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities. de Gruyter Series in Nonlinear Analysis and Applications* 5, Walter de Gruyter and Co., Berlin, 1997.

12. I. Ekeland and N. Ghoussoub, Selected new aspects of the calculus of variations in the large, *Bull. Amer. Math. Soc.*, **39** (2002), 207-265.

13. X. Fan, J. S. Shen and D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **262** (2001), 749-760.

14. X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.

15. X. Fan and X. Han, Existence and multiplicity of solutions for p(x)-Laplacian equations in R^N, *Nonlinear Anal.*, **59** (2004), 173-188.

16. X. Fan, Solutions for p(x)-Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.*, **312** (2005), 464-477.

17. X. L. Fan, On the sub-supersolution method for p(x)-Laplacian equations, *J. Math. Anal. Appl.*, **330** (2007), 665-682.

18. J. García, I. Peral and J. Manfredi, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, *Commun. Contemp. Math.*, **2(3)** (2000), 385-404.

19. N. Ghoussoub and C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents, *Trans. Amer. Math. Soc.*, **12** (2000), 5703-5743.

20. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Heidelberg, 2001.

21. Z. Guo and Z. Zhang, $W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations, *J. Math. Anal. Appl.*, **286** (2003), 32-50.

22. K. Ho and I. Sim, Existence and some properties of solutions for degenerate elliptic equations with exponent variable, *Nonlinear Anal.*, **98** (2014), 146-164.

23. K. Ho and I. Sim, *Existence results for degenerate $p(x)$-Laplace equations with Leray-Lions type operators*, submitted.

24. Y. H. Kim and I. Sim, Existence of solutions and positivity of the infimum eigenvalue for degenerate elliptic equations with variable exponents, *Discrete Contin. Dyn. Syst. Supplement*, (2013), 695-707.

25. Y. H. Kim, L. Wang and C. Zhang, Global bifurcation of a class of degenerate elliptic equations with variable exponents, *J. Math. Anal. Appl.*, **371** (2010), 624-637.

26. O. Kovăcik and J. Răkosnik, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.*, **41(116)** (1991), 592-618.

27. V. K. Le, On a sub-supersolutionmethod for variational inequalities with Leray-Lions operators in variable exponent spaces, *Nonlinear Anal.*, **71** (2009), 3305-3321.

28. M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. R. Soc. A*, **462** (2006), 2625-2641.

29. M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.*, **135(9)** (2007), 2929-2937 (electronic).

30. M. Mihăilescu, V. Rădulescu and D. Stancu-Dumitru, A Caffarelli-Kohn-Nirenberg-type inequality with variable exponent and applications to PDEs, *Complex Var. Elliptic Equ.*, **56(7-9)** (2011), 659-669.

31. V. Murthy and G. Stampacchia, Boundary value problems for some degenerate-elliptic operators, *Ann. Mat. Pura Appl.*, **80** (1968), 1-122

32. M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, 2000.

33. M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996.

34. V. V. Zhikov, On some variational problems, *Russ. J. Math. Phys.*, **5** (1997), 105-116.

35. J. F. Zhao, *Structure Theory of Banach Spaces*, Wuhan University Press, Wuhan, 1991 (in Chinese).

Ky Ho and Inbo Sim
Department of Mathematics
University of Ulsan
Ulsan 680-749
Republic of Korea
E-mail: hnky81@gmail.com
ibsim@ulsan.ac.kr