TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 3, pp. 919-926, June 2015
DOI: 10.11650/tjm.19.2015.5138
This paper is available online at http://journal.taiwanmathsoc.org.tw

TWO WEIGHT INEQUALITIES FOR THE BERGMAN PROJECTION WITH DOUBLING MEASURES

Xiang Fang and Zipeng Wang*

Abstract. In this note we show that the problem of characterizing two weight norm inequalities for the Bergman projection under the assumption of doubling measures admits a surprisingly simple solution. Our principal discovery is that Sawyer-type testing can be avoided. This stands in sharp contrast with the current folklore in two weight theory and with the corresponding result for the Hilbert transform.

1. Introduction and Main Results

The problem of characterizing the boundedness of two weight inequality $T : L^2(\mu) \to L^2(\omega)$ for a classical operator T is, in general, a notoriously difficult problem. Since Sawyer's seminal work [13], it has been part of the folklore among experts that one needs an A_2-type condition, plus Sawyer-type testing, to characterize these inequalities. This is amply manifested in the case of the Hilbert transform, due to a series of deep works [6, 7, 8, 11, 15]. Moreover, for concrete situations, the hard-to-verify part is usually Sawyer-testing. After the Hilbert transform, the Bergman projection naturally becomes a focus point along this line of research. The purpose of this note is to exhibit a pleasant surprise for the Bergman projection.

We first introduce a new concept called the reverse doubling property for measures over the unit disk. This property enables us to prove a result which not only solves the problem referred in the title, but also includes Bokelle-Bonami's classical result [3] on the one weight problem for the Bergman projection as a special case.

Definition 1. A measure μ on the unit disk $\mathbb{D} \subset \mathbb{R}^2$ has the reverse doubling property if there is a constant $\delta < 1$ such that

$$\frac{|B_I|_\mu}{|Q_I|_\mu} < \delta$$

Received August 8, 2014, accepted August 18, 2014.
Communicated by Duy-Minh Nhieu.
2010 *Mathematics Subject Classification*: 47G10, 42A50.
Key words and phrases: Reverse doubling property, Weighted norm inequalities, Bergman projection.
*Corresponding author.

for any interval $I = [a, b) \subset \mathbb{T} = \partial\mathbb{D}$. Here $Q_I = \{z \in \mathbb{D} : 1 - |I| < |z| < 1, \frac{z}{|z|} \in I\}$ is the Carleson box associated with I, and $B_I = \{z \in \mathbb{D} : 1 - \frac{|I|}{2} < |z| < 1, \frac{z}{|z|} \in I\}$, where $|I|$ is the normalized arc length so that $|\mathbb{T}| = 1$.

Let $0 < \sigma, \omega \in L^1_{\text{loc}}(\mathbb{D})$ be weights. By Sawyer's duality trick [13], the Bergman projection

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} dA(w)$$

is bounded from $L^2(\mathbb{D}, \sigma)$ to $L^2(\mathbb{D}, \omega)$ if and only if

$$P_{\sigma^{-1}} f(z) = \int_{\mathbb{D}} \frac{f(w)\sigma^{-1}(w)}{(1 - z\bar{w})^2} dA(w)$$

is bounded from $L^2(\mathbb{D}, \sigma^{-1})$ to $L^2(\mathbb{D}, \omega)$.

Our main result in this note is the following Theorem 2. Its proof is surprisingly simple when compared with currently known results in two weight theory.

Theorem 2. *Let σ and ω be two weights. If both σ and ω have the reverse doubling property, then $P_\sigma : L^2(\mathbb{D}, \sigma) \to L^2(\mathbb{D}, \omega)$ is bounded if and only if the joint Berezin condition holds. That is,*

$$\sup_{z \in \mathbb{D}} B(\sigma)(z) B(\omega)(z) < \infty, \tag{1}$$

where the Berezin transform is given by

$$B(\sigma)(z) = \int_{\mathbb{D}} \frac{(1 - |z|^2)^2 \sigma(w)}{|1 - z\bar{w}|^4} dA(w).$$

Corollary 3. [Bekolle-Bonami]. *Let σ be a weight on $\mathbb{D}$. The Bergman projection P is bounded on $L^2(\mathbb{D}, \sigma)$ if and only if σ satisfies the B_2-condition*

$$\sup_{Q_I : I \subset \mathbb{T}} \langle \sigma^{-1} \rangle_{Q_I} \langle \sigma \rangle_{Q_I} < \infty,$$

where $\langle \sigma \rangle_{Q_I} = \dfrac{1}{|Q_I|} \displaystyle\int_{Q_I} \sigma(z) dA(z).$

Corollary 3 follows from Theorem 2 and the following lemma.

Lemma 4. *If σ is a B_2-weight, then both σ and σ^{-1} have the reverse doubling property.*

Corollary 5. *If both σ and ω are doubling measures, then $P_\sigma : L^2(\mathbb{D},\sigma) \to L^2(\mathbb{D},\omega)$ is bounded if and only if the joint Berezin condition (1) holds.*

In [11, 15] Nazarov, Treil and Volberg developed a deep program and solved the corresponding problem for the Hilbert transform $\mathcal{H}$ using an A_2 condition, corresponding to (1) above, and the Sawyer-type testing condition; see Theorem 15.1 in [15]. As of today, it is probably part of the folklore that one needs these two types of conditions to characterize two weight inequalities. Hence, it is somehow surprising to see that Sawyer-testing is not needed in Corollary 5. In other words, the hard-to-verify part is indeed unnecessary.

Later, the two weight problem for $\mathcal{H}$ was solved in its full generality by Lacey, Sawyer, Shen, Uriarte-Tuero and Hytonen in [7, 8] and [6]. Also in [2], Aleman, Pott and Reguera proved a special case of two weight inequalities for P, but their answer still depends on Sawyer-testing.

Corollary 5 follows from Theorem 2 and the following lemma.

Lemma 6. *If σ is a doubling measure on $\mathbb{D}$, then σ has the reverse doubling property.*

2. Proofs of Lemma 4 and Lemma 6

Proof of Lemma 4. Let $Q_I = B_I \cup T_I$ be a Carleson box induced by an interval $I \subset \mathbb{T}$, where

$$T_I = \{z \in \mathbb{D} : 1 - |I| < |z| \le 1 - \frac{|I|}{2}, \frac{z}{|z|} \in I\}.$$

Assume that $\sup_{Q_I : I \subset \mathbb{T}} \langle \sigma^{-1} \rangle_{Q_I} \langle \sigma \rangle_{Q_I} = c_1 < \infty$. Then

$$\begin{aligned}\frac{|T_I|}{|Q_I|} &\le \frac{[\int_{T_I} \sigma(z)dA(z)]^{\frac12}[\int_{Q_I} \sigma^{-1}(z)dA(z)]^{\frac12}}{|Q_I|}\\ &= \frac{[\int_{T_I} \sigma(z)dA(z)]^{\frac12}}{[\int_{Q_I} \sigma(z)dA(z)]^{\frac12}} \frac{[\int_{Q_I} \sigma(z)dA(z)]^{\frac12}[\int_{Q_I} \sigma^{-1}(z)dA(z)]^{\frac12}}{|Q_I|}.\end{aligned}$$

Then

$$\frac{\int_{T_I} \sigma(z)dA(z)}{\int_{Q_I} \sigma(z)dA(z)} \ge \frac{1}{9c_1},$$

and

$$\frac{|B_I|_\sigma}{|Q_I|_\sigma} \leq 1 - \frac{1}{9c_1} < 1. \qquad \blacksquare$$

Proof of Lemma 6. Let $I \subset \mathbb{T}$ be an interval. Since σ is doubling, there is a constant $c_2 > 1$ such that

$$|Q_I|_\sigma \leq c_2 |T_I|_\sigma.$$

Then

$$\frac{|B_I|_\sigma}{|Q_I|_\sigma} < 1 - \frac{1}{c_2} < 1. \qquad \blacksquare$$

3. Proof of Sufficiency in Theorem 2

Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Consider the following well known dyadic grids on $\mathbb{T}$,

$$\mathcal{D}^0 = \{[\frac{2\pi m}{2^j}, \frac{2\pi(m+1)}{2^j}) : m \in \mathbb{Z}_+, j \in \mathbb{Z}_+, 0 \leq m < 2^j\}$$

and

$$\mathcal{D}^{\frac{1}{3}} = \{[\frac{2\pi m}{2^j} + \frac{2\pi}{3}, \frac{2\pi(m+1)}{2^j} + \frac{2\pi}{3}) : m \in \mathbb{Z}_+, j \in \mathbb{Z}_+, 0 \leq m < 2^j\}.$$

For each $\beta \in \{0, \frac{1}{3}\}$, let $\mathcal{Q}^\beta$ denote the collection of Carleson boxes Q_I with $I \in \mathcal{D}^\beta$ and we call $\mathcal{Q}^\beta$ a Carleson box system over $\mathbb{D}$.

Lemma 7. *Let $\beta \in \{0, \frac{1}{3}\}$. If a weight σ has the reverse doubling property, then there is a constant c_3 such that for any $K \in \mathcal{D}^\beta$,*

$$\sum_{Q_I \in \mathcal{Q}^\beta : Q_I \subset Q_K} |Q_I|_\sigma \leq c_3 |Q_K|_\sigma.$$

Proof. For simplicity, we fix $\beta = 0$. Also fix any $K \subset \mathcal{D}^0$. Define $C^{(0)}(Q_K) = Q_K$, and

$$C^{(1)}(Q_K) = \{Q_I : I \in \mathcal{D}^0 \text{ is a son of } K\}.$$

Observe that $C^{(1)}(Q_K)$ has two members: $Q_{I_{11}}$ and $Q_{I_{12}}$, and $B_K = Q_{I_{11}} \cup Q_{I_{12}}$. By induction, for any $j \geq 2$, we define

$$C^{(j)}(Q_K) = \{Q_I \in C^{(1)}(\tilde{Q}_I) : \tilde{Q}_I \in C^{(j-1)}(Q_K)\}.$$

Then

$$\sum_{Q_I \in \mathcal{Q}^0 : Q_I \subset Q_K} |Q_I|_\sigma = \sum_{j=0}^{\infty} \sum_{Q_I \in C^{(j)}(Q_K)} |Q_I|_\sigma.$$

By the reverse doubling property, it is not hard to observe that for any $j \geq 1$,

$$\frac{\sum_{Q_I \in C^{(j)}(Q_K)} |Q_I|_\sigma}{|Q_K|_\sigma} < \delta^j,$$

where δ is from Definition 1. It follows that

$$\sum_{Q_I \in \mathcal{Q}^0 : Q_I \subset Q_K} |Q_I|_\sigma \leq \frac{1}{1-\delta} |Q_K|_\sigma. \qquad \blacksquare$$

Now, we need the next lemma which illustrates the relationship between an arbitrary interval $J \subset \mathbb{T}$ and intervals in $\mathcal{D}^\beta, \beta \in \{0, \frac{1}{3}\}$.

Lemma 8. [9]. *Let $J \subset \mathbb{T}$ be an interval. Then there exists an interval $L \in \mathcal{D}^0 \cup \mathcal{D}^{\frac{1}{3}}$ such that $J \subset L$ and $|L| \leq 6|J|$.*

Lemma 9. *There is a positive constant c_3 such that for any $z, w \in \mathbb{D}$, there exists a Carleson box Q_I such that $z, w \in Q_I$ and*

$$\frac{1}{c_4} |Q_I|^{\frac{1}{2}} \leq |1 - z\bar{w}| \leq c_4 |Q_I|^{\frac{1}{2}}.$$

For a proof, one can consult [2, 5]. By Lemma 8 and Lemma 9, there is a constant c_5 such that for any $f \geq 0$ and $z \in \mathbb{D}$,

$$P_\sigma^+ f(z) \leq c_5 [T_\sigma^0 f(z) + T_\sigma^{\frac{1}{3}} f(z)],$$

where

$$P_\sigma^+ f(z) = \int_{\mathbb{D}} \frac{f(w)\sigma(w)}{|1 - z\bar{w}|^2} dA(w)$$

and for $\beta \in \{0, \frac{1}{3}\}$,

$$T_\sigma^\beta f(z) = \sum_{I : I \in \mathcal{D}^\beta, z \in Q_I} \frac{1}{|Q_I|} \int_{Q_I} f(w)\sigma(w) dA(w) \chi_{Q_I}(z).$$

Next we need a result on Carleson embedding.

Lemma 10. *Let σ be a measure on $\mathbb{D}$. Let $\mathcal{Q}^\beta$, $\beta \in \{0, \frac{1}{3}\}$, be a Carleson system over $\mathbb{D}$. If there is a constant c_6 such that for any $K \in \mathcal{D}^\beta$*

$$\sum_{Q_I \in \mathcal{Q}^\beta : Q_I \subset Q_K} |Q_I|_\sigma \leq c_6 |Q_K|_\sigma,$$

then there is a constant c_7 such that

$$\sum_{Q_I \in \mathcal{Q}^\beta} |Q_I|_\sigma \cdot \left[\frac{1}{|Q_I|_\sigma} \int_{Q_I} f(z)\sigma(z) dA(z)\right]^2 \leq c_7 \int_{\mathbb{D}} |f(z)|^2 \sigma(z) dA(z), \quad f \in L^2(\mathbb{D}, \sigma).$$

Remark. For the proof, one can easily adopt known arguments on Carleson embeddings, say, Theorem 3.1 in [10] or Theorem 5.8 in [14]. So we skip the details.

Now we are ready to wrap up the proof of sufficiency in Theorem 2. Observe that it is sufficient to show that $T_\sigma^\beta : L^2(\mathbb{D},\sigma) \to L^2(\mathbb{D},\omega)$ is bounded for $\beta \in \{0, \frac{1}{3}\}$. Let Q_I be a Carleson box and any $z_0 \in Q_I$ such that $1 - |z_0| = \frac{|I|}{2}$. There is a constant $c_8 > 0$ independent of z_0 and Q_I such that

$$B(\sigma)(z_0) \geq c_8 \frac{1}{|Q_I|} \int_{Q_I} \sigma(z) dA(z). \tag{2}$$

Since $\sup_{z\in\mathbb{D}} B(\sigma)(z)B(\omega)(z) < \infty$, it follows that

$$c_9 \doteq \sup_{Q_I : I \subset \mathbb{T}} \frac{|Q_I|_\sigma^{\frac{1}{2}} |Q_I|_\omega^{\frac{1}{2}}}{|Q_I|} < \infty.$$

Let $f \in L^2(\mathbb{D},\sigma)$ and $g \in L^2(\mathbb{D},\omega)$. Then

$$\begin{aligned} |\langle T_\sigma^\beta f, g\rangle_{L^2(\mathbb{D},\omega)}| &= |\sum_{Q_I : Q_I \in \mathcal{Q}^\beta} \frac{1}{|Q_I|} \int_{Q_I} f(z)\sigma(z)dA(z) \int_{Q_I} g(z)\omega(z)dA(z)| \\ &\leq c_9 \sum_{Q_I : Q_I \in \mathcal{Q}^\beta} \frac{1}{|Q_I|_\sigma^{\frac{1}{2}}} \int_{Q_I} |f(z)|\sigma(z)dA(z) \frac{1}{|Q_I|_\omega^{\frac{1}{2}}} \int_{Q_I} |g(z)|\omega(z)dA(z) \\ &\leq c_9 [\sum_{Q_I : Q_I \in \mathcal{Q}^\beta} |Q_I|_\sigma (\frac{1}{|Q_I|_\sigma} \int_{Q_I} |f(z)|\sigma(z)dA(z))^2]^{\frac{1}{2}} \\ &\quad \cdot [\sum_{Q_I : Q_I \in \mathcal{Q}^\beta} |Q_I|_\omega (\frac{1}{|Q_I|_\omega} \int_{Q_I} |g(z)|\omega(z)dA(z))^2]^{\frac{1}{2}}. \end{aligned}$$

Since σ and ω have the reverse doubling property, by Lemma 10 and Lemma 7, there is a constant c_{10} such that for $\beta \in \{0, \frac{1}{3}\}$,

$$|\langle T_\sigma^\beta f, g\rangle_{L^2(\mathbb{D},\omega)}| \leq c_{10} \|f\|_{L^2(\mathbb{D},\sigma)} \|g\|_{L^2(\mathbb{D},\omega)}.$$

Hence $P_\sigma^+ : L^2(\mathbb{D},\sigma) \to L^2(\mathbb{D},\omega)$ is bounded, and so is P_σ.

4. Proof of Necessity

We show that the joint Berezin condition is always necessary for two weight norm inequalities for the Bergman projection P.

Proposition 11. *Let σ and ω be two weights on $\mathbb{D}$. If the Bergman projection*

$$P : L^2(\mathbb{D},\sigma) \to L^2(\mathbb{D},\omega)$$

is bounded, then

$$\sup_{z\in\mathbb{D}} B(\sigma^{-1})(z)B(\omega)(z) < \infty.$$

Proof. For any $z_0 \in \mathbb{D}$, let

$$k_{z_0}(w) = \frac{1-|z_0|^2}{(1-\overline{z_0}w)^2}.$$

We define a rank one operator

$$T_{z_0}f(z) = \int_{\mathbb{D}} f(w)\overline{k_{z_0}}(w)dA(w)k_{z_0}(z)$$

on $L^2(\mathbb{D},\sigma)$. Then

$$\|T_{z_0}\|^2_{L^2(\mathbb{D},\sigma)\to L^2(\mathbb{D},\omega)} = B(\sigma^{-1})(z_0)B(\omega)(z_0)$$

As in (2.7) of [1], or by the projection formula in [4], there is a sequence $\{c_n\}_{n=0}^{\infty}$ satisying $\sum_{n=1}^{\infty}|c_n| < \infty$ such that for any $f \in L^2(\mathbb{D},\sigma)$ and $z \in \mathbb{D}$,

$$T_{z_0}(f)(z) = P(f)(z) - \sum_{n=0}^{\infty} c_n\varphi_{z_0}^n P(\overline{\varphi_{z_0}^n}f)(z),$$

where

$$\varphi_{z_0}(w) = \frac{z_0 - w}{1-\bar{z}_0 w}.$$

Since $|\varphi_{z_0}(w)| < 1$, if $P : L^2(\mathbb{D},\sigma) \to L^2(\mathbb{D},\omega)$ is bounded, then

$$\|T_{z_0}\|_{L^2(\mathbb{D},\sigma)\to L^2(\mathbb{D},\omega)} \leq (1+\sum_{n=0}^{\infty}|c_n|)\|P\|_{L^2(\mathbb{D},\sigma)\to L^2(\mathbb{D},\omega)} < \infty.$$

The proof of Proposition 11 is complete now. ■

Acknowledgment

X. Fang is supported by NSC of Taiwan (102-2115-M-008-016-MY2). Z. Wang is supported by NSFC (11371096).

References

1. A. Aleman and O. Constantin, The Bergman projection on vector-valued L^2-spaces with operator-valued weights, *J. Funct. Anal.*, **262(5)** (2012), 2359-2378.

2. A. Aleman, S. Pott and M. Reguera, *Sarason Conjecture on the Bergman Space*, preprint, 2013.

3. D. Bekolle and A. Bonami, Inegalites a poids pour le noyau de Bergman, (French), *C. R. Acad. Sci. Paris A-B*, **286(18)** (1978), 775-778.

4. X. Fang, Canonical operator models over Reinhardt domains, *Taiwanese J. Math.*, **11(1)** (2007), 75-94.

5. X. Fang, K. Guo and Z. Wang, *Composition Operators on the Bergman Space via Quasiconformal Mappings*, preprint, 2014.

6. T. Hytonen, *The Two-weight Inequality for the Hilbert Transform with General Measures*, preprint, 2014.

7. M. Lacey, Two weight inequality for the Hilbert transform: a real variable characterization, II, *Duke Math. J.*, to appear.

8. M. Lacey, E. Sawyer, C. Shen and I. Uriarte-Tuero, Two weight inequality for the Hilbert transform: a real variable characterization, I, *Duke Math. J.*, to appear.

9. T. Mei, BMO is the intersection of two translates of dyadic BMO, *C. R. Acad. Sci. Paris, Ser. I.*, **336** (2003), 1003-1006.

10. F. Nazarov, S. Treil and A. Volberg, The Tb-theorem on non-homogeneous spaces, *Acta Math.*, **190(2)** (2003), 151-239.

11. F. Nazarov, S. Treil and A. Volberg, *Two Weight $T1$ Theorem for the Hilbert Transform: The Case of Doubling Measures*, preprint, 2005.

12. S. Pott and M. Reguera, Sharp Bekolle estimates for the Bergman projection, *J. Funct. Anal.*, **265(12)** (2013), 3233-3244.

13. E. Sawyer, A characterization of a two-weight norm inequality for maximal operators, *Studia Math.*, **75(1)** (1982), 1-11.

14. X. Tolsa, *Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderon-Zygmund Theory*, Progress in Mathematics, **307**, Birkhauser/Springer, 2014.

15. A. Volberg, *Calderon-Zygmund Capacities and Operators on Nonhomogeneous Spaces*, CBMS Vol. **100**, American Mathematical Society, Providence, RI, 2003.

Xiang Fang
Department of Mathematics
National Central University
Chung-Li 32001
Taiwan
E-mail: xfang@math.ncu.edu.tw

Zipeng Wang
School of Mathematics Science
Fudan University
Shanghai 200433
P. R. China
E-mail: zipengwang11@fudan.edu.cn