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0-PRIMITIVE NEAR-RINGS, MINIMAL IDEALS AND SIMPLE NEAR-RINGS

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Abstract. We study the structure of 0-primitive near-rings and are able to answer an open question in the theory of minimal ideals in near-rings to the negative, namely if the heart of a zero symmetric subdirectly irreducible near-ring is subdirectly irreducible again. Also, we will be able to classify when a simple near-ring with an identity and containing a minimal left ideal is a Jacobson radical near-ring. Such near-rings are known to exist but have unusual properties. Along the way we prove results on minimal ideals and left ideals in near-rings which so far were known to hold or have been established in the DCCN case, only.

1. Introduction and Basic Definitions

In what follows, we consider right near-rings, this means the right distributive law holds, but not necessarily the left distributive law. The notation is that of [14].

Primitive near-rings play the same role in the structure theory of near-rings as primitive rings do in ring theory. However, in the case of near-rings there are several types of primitivity which are interesting to consider. The most general type is the primitivity of type 0. A primitive near-ring is necessarily 0-primitive. While satisfactory results describing so called 2-primitive near-rings with identity exist, a 2-primitive near-ring with identity is either a primitive ring or dense in a so called centralizer near-ring, studying the algebraic structure of 0-primitive near-rings, apart from special cases (see for example [9]), has remained widely untouched. We will address this question and can prove results which still do not completely classify 0-primitive near-rings but we will reach a point where well known structure results for primitive near-rings follow as a corollary. It then turns out that our tools for studying 0-primitive near-rings will help to study minimal ideals. Minimal ideals in near-rings have been studied by various

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authors (see Section 3 for references) but a question has remained open so far: It is well known in ring theory that a minimal ideal of a ring either has zero multiplication or is a simple ring. While it is known that this is not the case for minimal ideals in near-rings it remained an open question if a minimal ideal in a zero symmetric near-ring is subdirectly irreducible or not. We will give an example of a zero symmetric and 0-primitive near-ring which contains a minimal ideal which is not subdirectly irreducible. Also, our methods will allow to address the question how to describe simple near-rings. Zero symmetric and simple near-rings containing a minimal N-subgroup and an identity element are known to be 2-primitive and therefore can be classified using centralizer near-rings. We can extend this result to zero symmetric simple near-rings with identity containing a minimal left ideal in so far, as we can give a precise condition in terms of the algebraic structure of the minimal left ideal when the near-ring is 2-primitive or else J_2-radical. Generally, we will be concerned with near-rings which do not satisfy the DCCN, the descending chain condition on N-subgroups contained in N. Thus, well known results on minimal ideals in near-rings with DCCN will follow as a corollary of our considerations and thus should also prove interesting in its own right. We need to give the most fundamental notation in the following, whenever necessary we will introduce further notation.

Let N be a zero symmetric near-ring, this means that $n * 0 = 0$ for all $n \in N$ where $*$ is the near-ring multiplication. Let Γ be an N-group of the near-ring N. An N-ideal I of Γ is a normal subgroup of the group $(\Gamma, +)$ such that $\forall n \in N\ \forall \gamma \in \Gamma\ \forall \delta \in I : n(\gamma + \delta) - n\gamma \in I$. A left ideal L of a near-ring N is an N-ideal of the natural N-group N and in case N is zero symmetric, a left ideal is also an N-group. The left ideal L is an ideal, if $LN \subseteq L$. A non-zero N-group Γ of the near-ring N is of type 0 if there is an element $\gamma \in \Gamma$ such that $N\gamma = \Gamma$, γ is then called a generator of the N-group, and there are no non-trivial N-ideals in Γ. A non-zero N-group Γ is of type 1 if it is of type 0 and N acts strongly monogenic on Γ. N acting strongly monogenic on Γ means that $N\gamma = \Gamma$ or $N\gamma = \{0\}$ for all $\gamma \in \Gamma$.

Let U be a subgroup of the N-group Γ. U is called an N-subgroup of Γ if $NU \subseteq U$. The N-group Γ is called N-group of type 2 if $N\Gamma \neq \{0\}$ and there are no non-trivial N-subgroups in Γ. In case N has an identity element an N-group is of type 1 if and only if it is of type 2 (see [14, Proposition 3.7 and Proposition 3.4]).

Given an N-group Γ and a non-empty subset $S \subseteq \Gamma$ then $(0 : S) = \{n \in N | \forall \gamma \in S : n\gamma = 0\}$ will be called the annihilator of S. Such annihilators always are left ideals of the near-ring N. Γ will be called faithful if $(0 : \Gamma) = \{0\}$. Annihilators of N-groups are always ideals of a near-ring N. In particular, let $v \in \{0, 1, 2\}$. Then, the intersection of the annihilators of all N-groups of type v of a near-ring N is an ideal and called the Jacobson radical of type v, $J_v(N)$. A near-ring is called v-primitive if it acts on a faithful N-group Γ of type v.

The paper is organised as follows: In Section 2 we will see that minimal left ideals which do not have zero multiplication and have a special type of (a mild) chain condition are N-groups of type 0 and give rise to certain non-zero ideals in a near-ring. In Section 3 we will study minimal ideals in a near-ring N, N not necessarily satisfying any chain condition and use the results of Section 2 to prove some decomposition theorems for minimal ideals which generalise well known standard results in near-ring theory. Examples will guarantee the existence of such minimal ideals, also in case the near-ring does not satisfy the DCCI, the descending chain condition on ideals. In Section 5 we use these results and are able to prove a structure theorem for 0-primitive near-rings with DCCL, the descending chain condition on left ideals of N, from which the well known structure results for 1-primitive near-rings with DCCL follow immediately. Then, in Section 6, we use the theory developped in Section 5 to give an example of a zero symmetric and subdirectly irreducible near-ring with heart H which is not subdirectly irreducible. In Section 7, we can classify when a zero symmetric and simple near-ring with identity and a minimal left ideal is 2-primitive and when it is J_2-radical. Examples will illustrate the theory. Finally, we will address some open problems which arise during our discussion.

2. Minimal Left Ideals

We would like to point out that the minimal structures we consider, like minimal left ideals or minimal ideals are always understood to be non-zero. When we have a minimal left ideal L of a zero symmetric near-ring N, then it is not true that this minimal left ideal L has to be also minimal as an N-subgroup. This is a big difference to ring theory, for if the near-ring N happens to be a ring, then left ideals and N-subgroups of the near-ring coincide. Instead of giving a single example we discuss more generally when the situation that a minimal left ideal is not minimal as N-subgroup happens. Let N be a zero symmetric near-ring with DCCN which is 0-primitive but not 1-primitive on the N-group Γ with generator γ. Consider the annihilator $(0 : \gamma)$ and suppose there would be an ideal I of N contained in $(0 : \gamma)$. Then, $\{0\} = I\gamma \supseteq (IN)\gamma = I\Gamma$, which contradicts the faithfulness of Γ. According to [14, Theorem 3.53] a minimal ideal of N is a direct sum of minimal left ideals. Consequently, there must be a minimal left ideal L of N such that $L \not\subseteq (0 : \gamma)$. Thus, by [14, Proposition 3.10], $L \cong_N \Gamma$. The symbol $\cong_N$ means being N-isomorphic as N-subgroups. This notation will be frequently used. If Γ is of type 0 but not of type 1, then Γ must properly contain an N-subgroup and consequently, by N-isomorphism L does, as well. The structure of zero symmetric 0-primitive near-rings with DCCN can be quite completely described and we will do so in Theorem 5.3 and Corollary 5.8. Hence, it is easy to find 0-primitive near-rings which are not 1-primitive. For example, let Γ be a finite group which contains a non-trivial subgroup S. Then the near-ring $N := \{f \in M_0(\Gamma) | f(S) \subseteq S\}$ is finite and 0-primitive on Γ but not 1-primitive on Γ. Here, $M_0(\Gamma)$ is the near-ring of zero preserving functions mapping from Γ to Γ where near-ring addition is the pointwise addition of functions and multiplication is function composition. Clearly, N is faithful on Γ and every element in $\Gamma \setminus S$ is a generator of the N-group Γ. Let $s \in S \setminus \{0\}$. Then $\{0\} \neq Ns \subseteq S$. Thus, N does not act strongly monogenic on Γ and hence, N cannot act 1-primitively on Γ. But N acts 0-primitively on Γ. Let $s \in S \setminus \{0\}$ and $\gamma \in \Gamma \setminus S$. Let $f \in M_0(\Gamma)$ such that $f(\gamma + s) = \delta \notin S$ and $f(\gamma_1) = 0$ for every $\gamma_1 \in \Gamma \setminus \{\gamma + s\}$. Then, $f \in N$ and $f(\gamma + s) - f(\gamma) \notin S$. So, S is not an N-ideal. We will meet other examples of 0-primitive near-rings which are not 1-primitive near-rings, not necessarily of the type just presented, in several places in this paper.

Note that it is also very easy to construct minimal left ideals which are not N-groups of type 0. Take $(N, +)$ to be a simple group and define the zero multiplication on N. Then, obviously N is a near-ring where N is minimal as a left ideal but certainly not an N-group of type 0. In case a minimal left ideal does not have zero multiplication the situation is much different as we will see in Lemma 2.2.

Our focus will be on minimal left ideals L which do not properly contain N-subgroups which are N-isomorphic to L. The next proposition shows that such a situation naturally occurs when studying near-rings with descending chain condition on near-ring subgroups. To fix a notation we will keep throughout the paper, the symbol $\supset$ means a proper subset. Also, let M be a subnear-ring of a near-ring N. We say that M satisfies the DCC on N-subgroups contained in M if there is no properly decreasing chain of N-subgroups of N contained in M. Similary, we say that M satisfies the DCC on left ideals contained in M if there is no properly decreasing chain of left ideals of N contained in M.

Proposition 2.1. *Let N be a zero symmetric near-ring. Let L be a minimal left ideal such that L satisfies the DCC on N-subgroups contained in L. Suppose $M \subseteq L$ is an N-subgroup such that $M \neq L$. Then, L and M cannot be N-isomorphic.*

Proof. Suppose to the contrary that M and L are N-isomorphic. Then, due to N-isomorphism also M contains a proper subgroup M_1 which is N-isomorphic to M. Then also M_1 does, and so on. From that we get an infinite decreasing chain of N-isomorphic N-subgroups $L \supset M \supset M_1 \supset \ldots$. ■

In particular, Proposition 2.1 applies to zero symmetric near-rings with DCCN. The condition that a minimal left ideal L of a zero symmetric near-ring N does not properly contain N-subgroups which are N-isomorphic to L is much weaker than the usual DCCN. It will be used frequently in this paper and plays a key role in proving some of our main results. For example, it will be precisely this condition which makes a simple near-ring containing a minimal left ideal and an identity element a 2-primitive near-ring, see Section 7.

Natural examples of zero symmetric near-rings N containing a minimal left ideal L where the minimal left ideal does have an infinite decreasing chain of N-subgroups but does not have proper N-subgroups which are N-isomorphic to L exist, see the class of examples in Section 4.

We continue with a key lemma of this paper, which shows us where to get N-groups of type 0 once we have given a zero symmetric near-ring.

Lemma 2.2. *Let N be a zero symmetric near-ring and let L be a minimal left ideal such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then L contains a multiplicative right identity e when considered as subnear-ring of N. Furthermore, L is an N-group of type 0.*

Proof. We first show that L has a generator. Since we have $L^2 \neq \{0\}$, there is an element $l \in L$ such that $Ll \neq \{0\}$. By minimality of L as a left ideal, this implies $L \cap (0 : l) = \{0\}$. Consequently, the map $\psi_l : L \to Ll, j \mapsto jl$ is injective. Certainly, ψ_l is a surjective N-homomorphism and thus L and Ll are N-isomorphic. By assumption this implies $L = Ll$. So, we see that L has the generator l.

What is more, L contains an idempotent e which is a right identity in L. To see this, let $e \in L$ such that $el = l$. Such an e exists since $Ll = L$. Thus, $e^2 l = el$ and consequently, $(e^2 - e)l = 0$. So, $e^2 - e \in L \cap (0 : l) = \{0\}$ and we see that $e = e^2$. Let $j \in L$. Then, $je = je^2$, so $(j - je)e = 0$. Hence, $j - je \in L \cap (0 : e)$. Since $e \in Le$ by idempotence of e we see that $Le \neq \{0\}$ and so, by minimality of L we have that $L \cap (0 : e) = \{0\}$. Hence, $j = je$ and e is a multiplicative right identity in L.

Consequently, we have a Peirce decomposition of N as $N = (0 : e) \dot{+} Ne = (0 : e) \dot{+} L$. Suppose that $I \subseteq L$ is an N-ideal contained in L. Consequently, $(I, +)$ is a normal subgroup of $(L, +)$. Let $n \in N$. Then $n = n_1 + n_2$ with $n_1 \in (0 : e)$ and $n_2 \in L$. Let $i \in I$. Then, $n + i - n = n_1 + n_2 + i - n_2 - n_1$. Since I is normal in L, $n_2 + i - n_2 \in I$, so $n + i - n = n_1 + i_1 - n_1$ for some element $i_1 \in I \subseteq L$. Since $(N, +)$ is the direct sum of the normal subgroups $(L, +)$ and $((0 : e), +)$ of $(N, +)$ we have that $n_1 + i_1 = i_1 + n_1$. Consequently,

$$n + i - n = n_1 + i_1 - n_1 = i_1 \in I$$

and this proves that $(I, +)$ is normal in $(N, +)$. Let $n, m \in N$. So, there is an element $l \in L$ and $a \in (0 : e)$ such that $m = a + l$. Let $i \in I$. Then,

$$n(m + i) - nm = n((a + l) + i) - n(a + l).$$

By [14, Proposition 2.29] the sum $N = (0 : e) \dot{+} L$ is distributive, so

$$n((a + l) + i) = n(a + (l + i))$$

$$n(a + (l + i)) = na + n(l + i)$$

and

$$n(a + l) = na + nl.$$

Consequently,

$$n(m+i) - nm = na + n(l+i) - nl - na.$$

By assumption, $n(l+i) - nl \in I$ and since $(I,+)$ is a normal subgroup of $(N,+)$ we have that

$$na + n(l+i) - nl - na \in I.$$

This shows that $I \subseteq L$ is a left ideal of N. By minimality of L as a left ideal we either have $I = \{0\}$ or $I = L$. Thus, L contains no non-trivial N-ideals and this proves that L is an N-group of type 0. ■

The author does not know an example of a minimal left ideal without zero multiplication which is not an N-group of type 0. Proposition 2.1 and Lemma 2.2 show that one has to look at near-rings without DCCN to be in a position to find such examples at all.

The next result tells us something about N-subgroups which are contained in a minimal left ideal of a near-ring N. A minimal left ideal of a near-ring N does not have to be minimal as an N-subgroup of the near-ring, as we have already seen.

Lemma 2.3. *Let N be a zero symmetric near-ring and L a minimal left ideal. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Let $M \subseteq L$ be an N-subgroup of N, $M \neq L$. Then $LM = \{0\}$.*

Proof. Let $m \in M$. Then $(0:m) \cap L$ is a left ideal contained in L and so, by minimality of L either $(0:m) \cap L = L$ which gives $Lm = \{0\}$ or $(0:m) \cap L = \{0\}$. Suppose that $Lm \neq \{0\}$ and hence, $(0:m) \cap L = \{0\}$. Then, the map $\psi_m : L \to Lm, l \mapsto lm$ is injective. ψ_m clearly is a surjective N-homomorphism and so we have that L and Lm are N-isomorphic. Since $Lm \subseteq L$ it follows from our assumption that $L = Lm$. Consequently, $L = Lm \subseteq M$, which contradicts the fact that M is properly contained in L. Thus, for all $m \in M$, $Lm = \{0\}$. ■

The next lemma guarantees the existence of certain non-zero ideals provided we have non-nilpotent minimal left ideals which do not have zero multiplication in a zero symmetric near-ring N. Before, we introduce a notation we will keep throughout the paper.

Definition 2.4. Let N be a zero symmetric near-ring and L be a left ideal. Then, $\theta_0^L := \{l \in L | Nl \neq L\}$ and $\theta_1^L := \{l \in L | Nl = L\}$.

Note that the zero 0 of N is always contained in θ_0^L. For certain left ideals L, θ_1^L may be the empty set. However, we will meet only situations where the left ideals do have generators as an N-group.

Lemma 2.5. *Let N be a zero symmetric near-ring and let L be a minimal left ideal such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then, $L \subseteq (0 : \theta_0^L)$ and $(0 : \theta_0^L)$ is a non-zero ideal of N.*

Proof. Let $l \in \theta_0^L$. If $Nl = \{0\}$ then certainly $Ll = \{0\}$ as well. Suppose that $Nl \neq \{0\}$. Thus, Nl is a non-trivial N-subgroup properly contained in L. By Lemma 2.3, we get $LNl = \{0\}$. By Lemma 2.2, L contains a multiplicative right identity e and therefore, $LNl = \{0\}$ implies $Lel = Ll = \{0\}$. So we have shown that for $l \in \theta_0^L$, $Ll = \{0\}$.

As an annihilator, $(0 : \theta_0^L)$ is a left ideal of N. Let $l \in \theta_0^L$ and $n \in N$. Then $N(nl) \subseteq Nl \neq L$, so $nl \in \theta_0^L$. Let $a \in (0 : \theta_0^L)$, $n \in N$ and $l \in \theta_0^L$. Then $(an)l = a(nl) = 0$ because $nl \in \theta_0^L$. So, we have shown that $(0 : \theta_0^L)$ is an ideal of N, containing L and thus being non-zero. ■

If we have a minimal left ideal L with $L^2 \neq \{0\}$ in a zero symmetric near-ring N where L properly contains N-subgroups which are N-isomorphic to L, then the result of Lemma 2.5 may not be longer true as the Example 7.4 in the Section 7 shows.

3. Structure of Minimal Ideals

Minimal ideals in near-rings have been the subject of several papers. Unlike the situation in rings where a minimal ideal of a ring either has zero multiplication or is a simple ring (see [1] for references) the situation in near-rings is much more complicated. When the near-ring has DCCN, then S. Scott proved the following theorem (see [14, Theorem 3.54 and Corollary 3.55]):

Theorem 3.1. (S. Scott, [15]). *Let N be a zero symmetric near-ring with DCCN and I a minimal ideal. Then I is isomorphic to a finite direct sum of minimal left ideals of the near-ring N, all of the summands being N-isomorphic. I contains a minimal left ideal L such that $L^2 \neq \{0\}$ precisely when $I^2 \neq \{0\}$.*

Note that the result of Theorem 3.1 also applies to minimal ideals all of whose minimal left ideals have zero multiplication. On the other hand, the DCCN of the near-ring N is needed. In [6] minimal ideals of a zero symmetric near-ring N which satisfy the descending chain condition on N-subgroups contained in I are described using a method which the author calls "Polin near-rings". In [1] a minimal ideal I of a near-ring N where $N/(0 : I)$ satisfies the DCCL is shown to be a 2-primitive near-ring or else $J_2(I) = I$. We do not restrict ourselves to near-rings satisfying the DCCN or (more generally) to near-ring ideals satisfying the descending chain condition on N-subgroups contained in the ideal. Also, we have to be able to deal with situations where $J_2(I) = I$, I a minimal ideal of a zero symmetric near-ring. In Proposition 3.4

and Theorem 3.7 we partially generalize the requirements in Theorem 3.1. We can re-prove the result of Theorem 3.1 in a more general setting as long as the minimal ideal contains a non-nilpotent minimal left ideal. From that point of view, the results of this section should be interesting in their own right. We start with an easy to prove, yet fundamental lemma.

Lemma 3.2. *Let N be a zero symmetric near-ring and let I be an ideal of N. Let Γ be an N-group of type 0 such that $I \cap (0 : \Gamma) = \{0\}$. Let $\gamma \in \Gamma$ such that $N\gamma = \Gamma$. Then, $I\gamma = \Gamma$.*

Proof. Let $\gamma \in \Gamma$ such that $N\gamma = \Gamma$. Suppose that $I\gamma = \{0\}$. Then $I\Gamma = IN\gamma \subseteq I\gamma = \{0\}$ contradicting the fact that $I \cap (0 : \Gamma) = \{0\}$. Since $\Gamma = N\gamma$ we have that $I\gamma$ is an N-ideal in Γ. By assumption, Γ is an N-group of type 0 implying that $I\gamma = \Gamma$. ∎

The notation of the next definition will be kept throughout the remainder of the paper.

Definition 3.3. Let N be a zero symmetric near-ring and Γ an N-group. Let $\theta_1 := \{\gamma \in \Gamma | N\gamma = \Gamma\}$ and $\theta_0 := \{\gamma \in \Gamma | N\gamma \neq \Gamma\}$.

Note that if Γ is an N-group of type 0 in Definition 3.3, then θ_1 is not empty. We are now in a position to study certain types of ideals in a zero symmetric near-ring which turn out to be minimal ideals. To recall a notation, the intersection of all 0-modular left ideals of a near-ring N is called $J_{\frac{1}{2}}(N)$ and is the greatest quasi-regular left ideal in a near-ring N (see [14, Theorem 5.37]). Also, an ideal I of a zero symmetric near-ring is also an N-group of the near-ring N. When we consider I as an N-group, then we will use the notation ${}_N I$.

Proposition 3.4. *Let N be a zero symmetric near-ring. Let I be an non-zero ideal of N and Γ be an N-group of type 0 such that $I \cap (0 : \Gamma) = \{0\}$. Suppose that $I \subseteq (0 : \theta_0)$ and there is a finite number n of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0 : \gamma_i) \cap I) = \{0\}$. Then ${}_N I$ is a finite direct sum of minimal left ideals of the near-ring N, all of them being N-isomorphic to Γ and being N-groups of type 0. Moreover, I is a minimal ideal. I contains a right identity element 1_r of I such that $J_{\frac{1}{2}}(N) \subseteq (0 : 1_r)$ and thus, $N = (0 : 1_r) \dot{+} I$. So, I is a direct summand as a left ideal of N.*

Proof. Since Γ is an N-group of type 0, θ_1 is not empty and $\Gamma = \theta_0 \cup \theta_1$. Since $I \subseteq (0 : \theta_0)$ and $I \cap (0 : \Gamma) = \{0\}$ we must have $(0 : \theta_1) \cap I = \{0\}$.

Suppose there exists a generator $\gamma \in \theta_1$ such that $(0 : \gamma) \cap I = \{0\}$. Then, $\psi : I \to \Gamma, i \mapsto i\gamma$ is injective. By Lemma 3.2, ψ is also surjective and so we see that I is N-isomorphic to Γ. Since Γ is of type 0, we see that I is minimal as a left ideal. By N-isomorphism, I is an N-group of type 0. Since ψ is an N-isomorphism, there is an element $e \in I$ such that $e\gamma = \gamma$. Let $i \in I$. Then, $(ie - i)\gamma = 0$. So, $ie - i \in I \cap (0 : \gamma) = \{0\}$. Thus, we see that e is a right identity element for I. So, $I = Ne \cong_N N/(0 : e)$. Since Γ is an N-group of type 0, $(0 : e)$ is a 0-modular left ideal of N by [14, Proposition 3.23] and so, $J_{\frac{1}{2}}(N) \subseteq (0 : e)$. Since e is a right identity of I we have that $N = (0 : e)\dot{+}I$. The proof would be complete in this case. So we now assume that for any $\gamma \in \theta_1$ we have $(0 : \gamma) \cap I \neq \{0\}$.

By assumption, there is a finite number $n \geq 2$ of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0 : \gamma_i) \cap I) = \{0\}$. Hence, there is a natural number $s \geq 2$ and a set $\{\sigma_1, \ldots, \sigma_s\} \subseteq \theta_1$ minimal with respect to $\cap_{i=1}^{s}((0 : \sigma_i) \cap I) = \{0\}$. For $i \in \{1, \ldots, s\}$ let

$$L_i := \cap_{j=1, j\neq i}^{s}((0 : \sigma_j) \cap I).$$

Since $s \geq 2$, L_i exists for $i \in \{1, \ldots s\}$ and L_i is non-zero by choice of $\{\sigma_1, \ldots, \sigma_s\}$.

For all $\gamma \in \theta_1$ we have $I\gamma = \Gamma$ by Lemma 3.2. So, $h : I \to \Gamma, i \mapsto i\gamma$ is an N-epimorphism with kernel $(0 : \gamma) \cap I$. Thus, $I/I \cap (0 : \gamma) \cong_N \Gamma$. Since Γ is an N-group of type 0, we have that $I/I \cap (0 : \gamma)$ is an N-group of type 0. Therefore, for all $\gamma \in \theta_1$, $(0 : \gamma) \cap I$ is a maximal N-ideal of the N-group I (see [14, Theorem 1.30]). Thus, the N-group I contains a finite set of maximal N-ideals $(0 : \sigma_i) \cap I$, $i \in \{1, \ldots, s\}$ with zero intersection and thus, by [14, Theorem 2.50] the N-group I is the direct sum $I = \sum_{i=1}^{s} L_i$, where $L_i := \cap_{j=1, j\neq i}^{s}((0 : \sigma_j) \cap I)$. We now show that for all $i \in \{1, \ldots, s\}$, $L_i \cong_N \Gamma$.

Let $i \in \{1, \ldots, s\}$. By definition of L_i, $L_i\sigma_j = \{0\}$ if $i \neq j$ and $L_i \cap (0 : \sigma_i) = \cap_{i=1}^{s}((0 : \sigma_i) \cap I) = \{0\}$. As $\sigma_i \in \theta_1$, we have $N\sigma_i = \Gamma$. Consequently, $L_i\sigma_i$ is an N-ideal of Γ which is an N-group of type 0. Thus we either have $L_i\sigma_i = \Gamma$ or $L_i\sigma_i = \{0\}$. Now, $L_i\sigma_i = \{0\}$ would contradict $L_i \cap (0 : \sigma_i) = \{0\}$. Thus we have $L_i\sigma_i = \Gamma$. So we see that for every $i \in \{1, \ldots, s\}$, $h_i : L_i \to \Gamma, l_i \mapsto l_i\sigma_i$ is an N-isomorphism. Since Γ is an N-group of type 0 we now also have that L_i is an N-group of type 0, in particular, L_i is a minimal left ideal.

Next we prove that I is a minimal ideal. Suppose there is an ideal J with J being properly contained in I. We already know that $I = \sum_{i=1}^{s} L_i$ for some natural number s, and L_i, $i \in \{1, \ldots, s\}$ being minimal left ideals of N and being N-isomorphic to Γ. Thus, there must be a $j \in \{1, \ldots, s\}$ such that $L_j \not\subseteq J$. By minimality of L_j as a left ideal we get $L_j \cap J = \{0\}$ and hence, $JL_j \in L_j \cap J = \{0\}$. Consequently, $J \subseteq I \cap (0 : L_j)$. Since L_j is N-isomorphic to Γ, it follows that $J \subseteq I \cap (0 : \Gamma) = \{0\}$. Hence, $J = \{0\}$ and I is minimal as an ideal.

We now prove that I contains a right identity element. Since for all $i \in \{1, \ldots, s\}$, h_i is an N-isomorphism, for all $i \in \{1, \ldots, s\}$ there is an element $e_i \in L_i$ such that $e_i\sigma_i = \sigma_i$. Let $i \in \{1, \ldots, s\}$ and $l_i \in L_i$. Then, $(l_ie_i - l_i)\sigma_i = 0$. So, $l_ie_i - l_i \in L_i \cap (0 : \sigma_i) = \{0\}$. Thus, we see that e_i is a right identity element for L_i. Let $j \in \{1, \ldots, s\}$ and $j \neq i$ and $e_j \in L_j$ such that $e_j\sigma_j = \sigma_j$. Since $L_i\sigma_j = \{0\}$, we have $0 = e_i\sigma_j = e_ie_j\sigma_j$. Thus, $e_ie_j \in L_j \cap (0 : \sigma_j) = \{0\}$.

Thus, the idempotents e_i, $i \in \{1, \ldots, s\}$ are orthogonal idempotents and right identities of the left ideals L_i. Let $i \in I$. Since $I = \sum_{i=1}^{s} L_i$, for all $i \in \{1, \ldots, s\}$ there exists $l_i \in L_i$ such that $i = l_1 + \ldots + l_s$. For $i, j \in \{1, \ldots, s\}$ we have $l_ie_j = l_i$ if $i = j$ and otherwise $l_ie_j = 0$. By [14, Proposition 2.29], the direct sum $I = \sum_{i=1}^{s} L_i$ is distributive, so we see that $(l_1+\ldots+l_s)(e_1+\ldots+e_s) = (l_1+\ldots+l_s)$. Consequently, $1_r := e_1 + \ldots + e_s$ is a right identity in I. Let $i \in \{1, \ldots, s\}$. So, we have that $L_i = L_ie_i$. From the N-homomorphism $\psi_i : N \to L_i, n \mapsto ne_i$ we see that $L_i \cong_N N/(0 : e_i)$. Since L_i is an N-group of type 0, we see that $(0 : e_i)$ is a 0-modular left ideal of N by [14, Proposition 3.23] and therefore, $J_{\frac{1}{2}}(N) \subseteq (0 : e_i)$.

By idempotence of 1_r, we have a decomposition of N as $N = (0 : 1_r) + N1_r = (0 : 1_r) + I$. Let $a \in \cap_{i=1}^{s}(0 : e_i)$. 1_r is the distributive sum of the idempotents e_i, $i \in \{1, \ldots, s\}$. Consequently, $a1_r = ae_1 + \ldots + ae_s = 0$ and we see that $\cap_{i=1}^{s}(0 : e_i) \subseteq (0 : 1_r)$.

Let $a \in (0 : 1_r)$. Again we use that 1_r is the distributive sum of the idempotents e_i, $i \in \{1, \ldots, s\}$. So, $0 = a1_r = ae_1+\ldots+ae_s$. Since I is the direct sum $I = \sum_{i=1}^{s} L_i$, it follows that $ae_i = 0$, for all $i \in \{1, \ldots, s\}$. Consequently, $(0 : 1_r) \subseteq \cap_{i=1}^{s}(0 : e_i)$.

We now have that $\cap_{i=1}^{s}(0 : e_i) = (0 : 1_r)$ and $J_{\frac{1}{2}}(N) \subseteq (0 : 1_r)$. ∎

We keep the notation of Proposition 3.4 for a discussion. Clearly, the assumption that there is a finite number n of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0 : \gamma_i) \cap I) = \{0\}$ is a kind of finiteness condition of the near-ring. As we will see, it may be achieved by assuming a DCC condition on left ideals of N contained in I. Apart from that, N has to satisfy no finiteness condition at all. We will see in Section 4 that such ideals do indeed exist even in near-rings which do not satisfy the DCCI.

Lemma 3.5. *Let N be a zero symmetric near-ring. Let I be a non-zero ideal of N and Γ be an N-group of type 0 such that $I \cap (0 : \Gamma) = \{0\}$. Suppose that I does not contain an infinite strictly decreasing chain of left ideals of N. Suppose that $I \subseteq (0 : \theta_0)$. Then there is a finite number n of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0 : \gamma_i) \cap I) = \{0\}$.*

Proof. Suppose there exists a generator $\gamma \in \theta_1$ such that $(0 : \gamma) \cap I = \{0\}$. Then we are done. So we now assume that for every $\gamma \in \theta_1$ we have $(0 : \gamma) \cap I \neq \{0\}$. For the proof we introduce a notation. We let $(0 : \gamma) \cap I := (0 : \gamma)_I$ for $\gamma \in \theta_1$.

Note that $(0 : \theta_1) \cap I = \{0\}$. This follows from the fact that $I \subseteq (0 : \theta_0)$, $\Gamma = \theta_0 \cup \theta_1$ and $I \cap (0 : \Gamma) = \{0\}$. Let $\gamma_1 \in \theta_1$. Thus, $(0 : \gamma_1)_I \neq \{0\}$ and therefore, there is an element $\gamma_2 \in \theta_1$ such that $(0 : \gamma_1)_I\gamma_2 \neq \{0\}$. Consequently, $(0 : \gamma_1)_I \not\subseteq (0 : \gamma_2)_I$ and it follows that

$$(0 : \gamma_1)_I \cap (0 : \gamma_2)_I \subset (0 : \gamma_1)_I.$$

In the same manner, if $(0:\gamma_1)_I \cap (0:\gamma_2)_I \neq \{0\}$ we obtain an element $\gamma_3 \in \theta_1$ such that $((0:\gamma_1)_I \cap (0:\gamma_2)_I)\gamma_3 \neq \{0\}$, so

$$(0:\gamma_1)_I \cap (0:\gamma_2)_I \cap (0:\gamma_3)_I \subset (0:\gamma_1)_I \cap (0:\gamma_2)_I$$

and

$$(0:\gamma_1)_I \cap (0:\gamma_2)_I \subset (0:\gamma_1)_I.$$

Thus we obtain a strictly decreasing chain of left ideals of N which are contained in I. By assumption, this chain must eventually stop after finitely many steps. Thus, there is a finite number $n \geq 2$ of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0:\gamma_i) \cap I) = \{0\}$. ■

We can get a kind of converse to Proposition 3.4. We will need an additional lemma to do this.

Lemma 3.6. *Let N be a zero symmetric near-ring. Let I be a minimal ideal of N containing a minimal left ideal L of N such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then, $I\theta_0^L = \{0\}$ and for all $l \in \theta_1^L$, $Il = L$. In particular, I acts strongly monogenic on L.*

Proof. By Lemma 2.5 we have that $L \subseteq (0:\theta_0^L)$ and $(0:\theta_0^L)$ is an ideal of N. So, $L \subseteq I \cap (0:\theta_0^L)$. Minimality of I as an ideal now implies that $I \subseteq (0:\theta_0^L)$.

Let $l \in \theta_1^L$. Suppose that $Il = \{0\}$. Then $IL = INl \subseteq Il = \{0\}$ contradicting the fact that $L^2 \neq \{0\}$. Since $L = Nl$ we have that Il is an N-ideal in L. From Lemma 2.2 we have that L is an N-group of type 0 implying that $Il = L$. ■

Lemma 3.6 shows that a minimal ideal I of a near-ring N containing a minimal left ideal L with $L^2 \neq \{0\}$ acts faithfully (due to the minimality of I) and strongly monogenic on L. Following from [16, Theorem 2.3 and Lemma 2.7] we have that $I/J_1(I)$ is a 1-primitive near-ring and $IJ_1(I) = \{0\}$. From [16, Lemma 2.8] we have that $J_1(I)$ is the unique maximal ideal of I in case I has DCC on I-subgroups of I. From these results we obtain more knowledge on the structure of minimal ideals of near-rings N not necessarily satisfying the DCCN but we do not follow this line of discussion here.

The results of Lemma 3.6 and Proposition 3.4 now allow us to prove a decomposition result for certain minimal ideals in zero symmetric near-rings.

Theorem 3.7. *Let N be a zero symmetric near-ring. Let I be a minimal ideal of N containing a minimal left ideal L of N such that $L^2 \neq \{0\}$. Suppose that I does not contain an infinite strictly decreasing chain of left ideals of N and suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then ${}_N I$ is a finite direct sum of minimal left ideals of the near-ring N, all of them being N-isomorphic to L and being N-groups of type 0. Furthermore, I contains a* *right identity element* 1_r *of* I *such that* $J_{\frac{1}{2}}(N) \subseteq (0 : 1_r)$ *and* I *is a direct summand as a left ideal of* N*, so* $N = (0 : 1_r)\dot{+}I$.

Proof. Let L be the minimal left ideal contained in I such that $L^2 \neq \{0\}$ and such that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. By minimality of I as an ideal we must have $I \cap (0 : L) = \{0\}$. From Lemma 3.6 we get that $I\theta_0^L = \{0\}$. So, Lemma 3.5 shows that I fulfills all the assumptions of Proposition 3.4 and the result follows. ■

Let I be a minimal ideal of a zero symmetric near-ring N. If we assume that I has the DCC on N-subgroups of N being contained in I, then everything gets much nicer. In particular, the following Corollary 3.8 applies to finite minimal ideals which may be contained in a near-ring N which satisfies no chain condition at all (in such a case Theorem 3.1 cannot be applied).

Corollary 3.8. *Let* N *be a zero symmetric near-ring and* I *a minimal ideal satisfying the DCC on* N*-subgroups of* N *being contained in* I*. Let* L *be a minimal left ideal of* N *being contained in* I *such that* $L^2 \neq \{0\}$*. Then* I *has all the properties as described in Theorem 3.7.*

Proof. Let L be a minimal left ideal of N such that $L \subseteq I$ and $L^2 \neq \{0\}$. Since I satisfies the DCC on N-subgroups of N being contained in I, L satisfies the DCC on N-subgroups of N contained in L and I also satisfies the DCC on left ideals of N being contained in I. Proposition 2.1 shows that L cannot contain N-subgroups properly contained in L and being N-isomorphic to L. So, Theorem 3.7 applies. ■

We want to point out that ideals which are the direct sum of minimal left ideals which are not necessarily N-isomorphic N-groups of type 0 have been studied in [2] for zero symmetric near-rings with identity element. As shown in [2], such ideals can be used to study isomorphism types of N-groups of type 0 of zero symmetric near-rings with identity.

4. An Example

We will demonstrate by a class of examples of zero symmetric near-rings N that ideals as discussed in Proposition 3.4 exist, even in near-rings without DCCI. Moreover, these examples contain a minimal ideal I to which Proposition 3.4 applies but I does not satisfy the DCC for N-groups contained in I. Also, these examples demonstrate that there exist zero symmetric near-rings N containing a minimal left ideal L without zero multiplication where the minimal left ideal does contain an infinite decreasing chain of N-subgroups but does not contain proper N-subgroups which are N-isomorphic to L. The construction of these near-rings is not tricky at all, to prove the desired properties requires some effort.

Let G be a group with more than one element and S be a group which contains an infinite descending chain of subgroups S_i, $i \in \mathbb{N}$ with $S_1 := S$ and $S_i \supset S_{i+1}$. Let $G^* := G \setminus \{0\}$ and for all $i \in \mathbb{N}$, $S_i^* := S_i \setminus \{0\}$. Let

$$N := \{f \in M_0(G \times S) | \text{for all } g \in G \text{ and } i \in \mathbb{N} : f(g \times S_i^*) \subseteq \{0\} \times S_{i+1}\}.$$

One easily checks that N is a zero symmetric near-ring w.r.t pointwise addition of functions and function composition which is acting faithfully on the N-group $G \times S$.

Suppose that I is a non-trivial N-ideal of $G \times S$. Thus, for all $f \in N$, for all $(a, b) \in I$ and all $(g, s) \in G \times S$ we must have

$$f((g, s) + (a, b)) - f((g, s)) \in I.$$

Let $(a, b) \in I$ and suppose that $a \neq 0$. Then,

$$f((-a, 0) + (a, b)) - f((-a, 0)) = f((0, b)) - f((-a, 0)) \in I.$$

Let $z \in (G \times S) \setminus I$. Define $f : G \times S \to G \times S$ such that

$$f((-a, 0)) = -z$$

and

$$f((g, s)) = (0, 0) \text{ for} (g, s) \in (G \times S) \setminus \{(-a, 0)\},$$

in particular $f((0, b)) = (0, 0)$. Then, $f \in N$ and

$$f((0, b)) - f((-a, 0)) = z \notin I.$$

Thus, the only non-trivial N-ideals of $G \times S$ must be contained in $\{0\} \times S$. Let I be an N-ideal contained in $\{0\} \times S$ and $(0, b) \in I$, $b \neq 0$. Let $x \in G^*$. Then, for all $f \in N$,

$$f((x, -b) + (0, b)) - f((x, -b)) = f((x, 0)) - f((x, -b)) \in I.$$

Define $f : G \times S \to G \times S$ such that

$$f((x, 0)) := (c, 0), c \neq 0$$

and

$$f((g, s)) = (0, 0) \text{ for} (g, s) \in (G \times S) \setminus \{(x, 0)\},$$

in particular $f((x, -b)) = (0, 0)$. Then, $f \in N$ and

$$f((x, 0)) - f((x, -b)) = (c, 0) \notin I.$$

So, there is no non-trivial N-ideal in $G \times S$.

Consequently, $G \times S$ is an N-group of type 0 with set

$$\theta_1 := \{(g,0)|g \in G^*\}$$

and

$$\theta_0 := \{(g,s)|g \in N \text{ and } s \in S^*\} \cup \{(0,0)\},$$

where θ_1 is the set of generators and θ_0 is the set of non-generators of Γ. Thus, N is a 0-primitive near-ring.

We now show that N does not satisfy the DCCI. Note that each group $\{0\} \times S_i$, $i \in \mathbb{N}$ is an N-subgroup of $G \times S$ and thus the annihilator $(0 : \{0\} \times S_i)$, $i \in \mathbb{N}$ is an ideal of N. Since $S_i \supset S_{i+1}$ we have that $(0 : \{0\} \times S_i) \subseteq (0 : \{0\} \times S_{i+1})$ for $i \in \mathbb{N}$. Let $s_i \in S_i \setminus S_{i+1}$. Let $f : G \times S \to G \times S$ such that

$$f((0,s_i)) = (0, s_{i+1}) \text{ with } s_{i+1} \in S^*_{i+1}$$

and

$$f((g,s)) = (0,0) \text{ for } (g,s) \in (G \times S) \setminus \{(0,s_i)\}.$$

Then, $f \in N$ and $f \in (0 : \{0\} \times S_{i+1})$ but $f \notin (0 : \{0\} \times S_i)$.

So, $(0 : \{0\} \times S_i)$, $i \in \mathbb{N}$ is an infinite properly descending chain of ideals of the near-ring N. So N does not satisfy the DCCI, nor the DCCL nor the DCCN.

Clearly, $(0 : \theta_0)$ is a non-zero left ideal of the near-ring. Let $\delta \in \theta_0$ and $n \in N$. Then $N(n\delta) \subseteq N\delta \neq \Gamma$ and so we see that $N\theta_0 \subseteq \theta_0$. Hence, $(0 : \theta_0)$ is an ideal. By faithfulness of N on $G \times S$ we have that $(0 : \theta_1) \cap (0 : \theta_0) = \{0\}$. From now on we assume that G is a finite group. Since $(0 : \theta_1) = \cap_{\gamma \in \theta_1}(0 : \gamma)$ and $\theta_1 = \{(g,0)|g \in G^*\}$, there is a finite number n of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^n((0 : \gamma_i) \cap (0 : \theta_0)) = \{0\}$. So, Proposition 3.4 applies to $(0 : \theta_0)$ and we see that $(0 : \theta_0)$ is a minimal ideal. In fact, $(0 : \theta_0)$ is the unique minimal ideal as we will see in Proposition 5.1.

We now demonstrate that for certain choices of G and S we do not get proper N-subgroups of $G \times S$ which are N-isomorphic to $G \times S$.

Let U be an N-subgroup of $G \times S$ which is N-isomorphic to $G \times S$. Since $G \times S$ is an N-group of type 0, also U is of type 0 and must have a generator $(a,b) \in U$. Suppose that U is properly contained in $G \times S$. Suppose that $b = 0$ and thus, $a \neq 0$. Then, $U = N((a,0)) = G \times S$, since $\theta_1 = \{(g,0)|g \in G^*\}$. This contradicts the fact that U is a proper N-subgroup. So, $b \neq 0$. Then, by definition of N,

$$U = N((a,b)) \subseteq \{0\} \times S_2 \subseteq \{0\} \times S.$$

Hence, a proper N-subgroup of $G \times S$ which is N-isomorphic to $G \times S$ must be contained in $\{0\} \times S$. G was assumed to be a finite group, so if S is a group without elements of finite order (and having an infinite descending chain of subgroups S_i,

$i \in \mathbb{N}$ with $S_1 := S$ and $S_i \supset S_{i+1}$, $\mathbb{Z}$ for example), then we cannot have that $\{0\} \times S$ contains a subgroup isomorphic to $G \times S$. Note that the condition that an isomorphic copy of $G \times S$ is contained in $\{0\} \times S$ may fail for other reasons also, depending on the groups S and G.

Now we demonstrate that $(0 : \theta_0)$ is a minimal ideal which does not satisfy the DCC on N-subgroups contained in $(0 : \theta_0)$. Also we give an example of a minimal left ideal L without zero multiplication which does not properly contain an N-subgroup N-isomorphic to itself but does not satisfy the DCC on N-subgroups contained in L. From Proposition 3.4 we know that $(0 : \theta_0)$ is the finite direct sum of minimal left ideals of N which are all N-isomorphic to the N-group $G \times S$, so $(0 : \theta_0) = \sum_{i=1}^{s} L_i$ for some $s \in \mathbb{N}$ and L_i, $i \in \{1, \ldots, s\}$ minimal left ideals of N. Let L_1 be a minimal left ideal N-isomorphic to $G \times S$ from this decomposition of $(0 : \theta_0)$. By N-isomorphism to $G \times S$, L_1 cannot contain proper N-subgroups isomorphic to L_1. Since $(0 : G \times S) = \{0\}$ we must have $L_1^2 \neq \{0\}$. Clearly, any group $\{0\} \times S_i$, $i \in \mathbb{N}$ is an N-subgroup of $G \times S$. So by N-isomorphism, also L_1 does contain an infinite properly decreasing chain of N-subgroups and so does $(0 : \theta_0)$.

5. 0-Primitive Near-rings

The results of Section 3 will now be applied to study 0-primitive near-rings. First we note that 0-primitive near-rings containing a minimal ideal are subdirectly irreducible near-rings.

Proposition 5.1. *Let N be a zero symmetric near-ring containing a minimal ideal H. Suppose that N is 0-primitive on the N-group Γ. Then N is a subdirectly irreducible near-ring with heart H.*

Proof. Suppose there is a non-zero ideal I such that $H \not\subseteq I$. By minimality of H, this implies $H \cap I = \{0\}$ and therefore, $HI = \{0\}$. Let $\gamma \in \Gamma$ be a generator of the N-group Γ. Since $(0 : \Gamma) = \{0\}$, $I \cap (0 : \Gamma) = \{0\}$ and $H \cap (0 : \Gamma) = \{0\}$, so Lemma 3.2 shows that $I\gamma = \Gamma = H\gamma$. Consequently, there is $i \in I$ such that $\gamma = i\gamma$. Thus, $H\gamma = Hi\gamma = \{0\}$, a contradiction. Thus, N is subdirectly irreducible. ■

Note that the class of near-rings discussed in Section 4 are examples of zero symmetric near-rings which are 0-primitive and subdirectly irreducible but do not have the DCCI.

Proposition 3.4 now allows us to describe the algebraic structure of a 0-primitive near-ring. We restrict to the situation of non-rings only, since primitive near-rings which are rings are primitive rings in the usual sense (see [14, Proposition 4.8]). When dealing with non-rings, the following Proposition is of great use.

Proposition 5.2. ([14, Proposition 3.4]). *Let N be a zero symmetric near-ring and Γ be a faithful N-group with generator γ. Let L_1, L_2 be two left ideals of N such that $L_1 + (0 : \gamma) = L_2 + (0 : \gamma) = N$, but $L_1 \cap L_2 \subseteq (0 : \gamma)$. Then N is a ring.*

Also, similar to Proposition 3.4, we need some finiteness condition. As we will see, this finiteness condition may be achieved by assuming that the near-ring has the DCCL, for example. Remember that $\theta_0 := \{\gamma \in \Gamma | N\gamma \neq \Gamma\}$ for an N-group Γ (see Definition 3.3).

Theorem 5.3. *Let N be a zero symmetric near-ring which is 0-primitive on Γ. We assume that N is not a ring. Suppose that $I := (0 : \theta_0) \neq \{0\}$ and there is a finite number n of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0 : \gamma_i) \cap I) = \{0\}$. Then, $I = (0 : \theta_0)$ is the unique minimal ideal of N and has all the properties described in Proposition 3.4. In particular, $(0 : \theta_0)$ contains a right identity element 1_r and is a direct summand as a left ideal of N and $N = (0 : \theta_1) \dot{+} (0 : \theta_0)$ with $J_{\frac{1}{2}}(N) \subseteq (0 : \theta_1)$. If there is another N-group Γ_1 on which N acts 0-primitively also, then $\Gamma \cong_N \Gamma_1$.*

Proof. As an annihilator, $I := (0 : \theta_0)$ is a left ideal of N. Since $N\theta_0 \subseteq \theta_0$ one easily sees that I is an ideal. By assumption I is not zero and by faithfulness of Γ, $I \cap (0 : \Gamma) = \{0\}$. Hence, Proposition 3.4 applies to the ideal $I = (0 : \theta_0)$. In particular, $(0 : \theta_0)$ is a minimal ideal and by Proposition 5.1 we have that $I = (0 : \theta_0)$ is the unique minimal ideal. It remains to show that $N = (0 : \theta_1) \dot{+} (0 : \theta_0)$. To this end, we use that Proposition 3.4 guarantess the existence of a right identity 1_r in $(0 : \theta_0)$ and show that $(0 : \theta_1) = (0 : 1_r)$.

The right identity 1_r of $(0 : \theta_0)$ gives rise to a decomposition of N as $N = (0 : 1_r) \dot{+} N1_r = (0 : 1_r) \dot{+} (0 : \theta_0)$. Let $a \in (0 : 1_r)$ and suppose there is a generator $\gamma \in \theta_1$ such that $a \notin (0 : \gamma)$. Thus, $(0 : 1_r) \not\subseteq (0 : \gamma)$. Since $(0 : \gamma)$ is a maximal left ideal by [14, Proposition 3.4], we get $(0 : 1_r) + (0 : \gamma) = N$. By Lemma 3.2, $(0 : \theta_0)\gamma = \Gamma$, so $(0 : \theta_0) \not\subseteq (0 : \gamma)$. Hence, $(0 : \theta_0) + (0 : \gamma) = N$. Since $(0 : 1_r) \cap (0 : \theta_0) = \{0\}$, Proposition 5.2 now tells us that N is a ring, contradicting the assumptions. Therefore $a \in (0 : \gamma)$ for all $\gamma \in \theta_1$.

Now let $a \in (0 : \theta_1)$. We know that $(0 : \theta_0) = \sum_{i=1}^{s} L_i$, $s \in \mathbb{N}$ and for $i \in \{1, \ldots, s\}$, L_i is a minimal left ideal of N and each of these minimal left ideals is N-isomorphic to Γ. Consequently, we may write $1_r = \sum_{i=1}^{s} e_i$, with $e_i \in L_i$. Each of the e_i, $i \in \{1, \ldots, s\}$ is a right identity of the left ideal L_i, by [14, Theorem 3.43]. Suppose there is a $j \in \{1, \ldots, s\}$ such that $ae_j \neq 0$, so $(0 : \theta_1) \not\subseteq (0 : e_j)$. We know that $N/(0 : e_j) \cong_N Ne_j = L_j \cong_N \Gamma$. Thus, $(0 : e_j)$ is a maximal left ideal in N and therefore we have $L_j + (0 : e_j) = N$ as well as $(0 : \theta_1) + (0 : e_j) = N$. By minimality of L_j we either have $L_j \cap (0 : \theta_1) = L_j$ or $L_j \cap (0 : \theta_1) = \{0\}$. But $L_j \cap (0 : \theta_1) = L_j$ implies $L_j\theta_1 = \{0\}$ and since we have that $L_j\theta_0 = \{0\}$, this contradicts the faithfulness of Γ. So, $L_j \cap (0 : \theta_1) = \{0\}$ holds and again, by Proposition 5.2 we get that N is a ring, contradicting the assumptions. Therefore, $ae_j = 0$ for all $j \in \{1, \ldots, s\}$. Since $1_r = e_1 + \ldots + e_s$ is a distributive sum by [14, Proposition 2.29], $a \in (0 : 1_r)$ and the equality $(0 : 1_r) = (0 : \theta_1)$ is proved.

We now show that two N-groups of type 0 are N-isomorphic. Let Γ_1 be an N-group of type 0 on which N acts 0-primitively also. Let θ_0^1 be the set of non-generators of Γ_1 and θ_1^1 be the set of generators of Γ_1. Let $\gamma_1 \in \theta_1^1$. By Lemma 3.2 we have that $(0 : \theta_0)\gamma_1 = \Gamma_1$. We know from Proposition 3.4 that $(0 : \theta_0) = \sum_{i=1}^{s} L_i$, $s \in \mathbb{N}$ and for $i \in \{1, \ldots, s\}$, L_i is a minimal left ideal of N and each of these minimal left ideals is N-isomorphic to Γ. Consequently, there must be a minimal left ideal L_1 of N contained in $(0 : \theta_0)$ such that $L_1 \cong_N \Gamma$ and $L_1 \not\subseteq (0 : \gamma_1)$. Thus, by [14, Proposition 3.10] $L_1 \cong_N \Gamma_1$. This shows that $\Gamma_1 \cong_N \Gamma$. ∎

In particular, Theorem 5.3 applies to 0-primitive near-rings with DCCL, provided that $(0 : \theta_0) \neq \{0\}$. This follows from the next corollary.

Corollary 5.4. *Let N be a zero symmetric near-ring which is 0-primitive on Γ. We assume that N is not a ring. Suppose that $I := (0 : \theta_0) \neq \{0\}$. If I satisfies the descending chain condition on left ideals of N contained in I, then Theorem 5.3 applies.*

Proof. Due to faithfulness of Γ we must have $I \cap (0 : \Gamma) = \{0\}$ and so Lemma 3.5 applies. It shows that there is a finite number n of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0 : \gamma_i) \cap I) = \{0\}$, so we can apply Theorem 5.3. ∎

We keep the notation of Theorem 5.3 for a discussion and introduce a notation.

Definition 5.5. Let N be a zero symmetric near-ring which is 0-primitive on Γ and let L be a left ideal of N. On θ_1 we define an equivalence relation $\sim_L$ by $\gamma_1 \sim_L \gamma_2$ iff $(0 : \gamma_1) \cap L = (0 : \gamma_2) \cap L$.

The finiteness condition of Theorem 5.3 can also be achieved by assuming that $I = (0 : \theta_0) \neq \{0\}$ has only finitely many equivalence classes w.r.t $\sim_I$ on θ_1.

Corollary 5.6. *Let N be a zero symmetric near-ring which is 0-primitive on Γ. We assume that N is not a ring. Suppose that $I := (0 : \theta_0) \neq \{0\}$ and there are only finitely many different equivalence classes w.r.t $\sim_I$. Then, there is a finite number n of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0 : \gamma_i) \cap I) = \{0\}$ and Theorem 5.3 applies.*

Proof. By faithfulness of Γ we have that $I \cap (0 : \Gamma) = \{0\}$. The fact that $I\theta_0 = \{0\}$ implies that $\{0\} = I \cap (0 : \theta_1) = I \cap (\cap_{\gamma \in \theta_1}(0 : \gamma))$. In case there are only finitely many different equivalence classes w.r.t $\sim_I$ it follows that there is a finite number n of elements $\{\gamma_1, \ldots, \gamma_n\} \subseteq \theta_1$ such that $\cap_{i=1}^{n}((0 : \gamma_i) \cap I) = \{0\}$. ∎

Also, we have re-proven a well known and important result in the structure theory of near-rings. We will point this out by formulating a corollary.

Corollary 5.7. ([14, Theorem 4.46]). *Let N be a zero symmetric 1-primitive near-ring with DCCL. Assume that N is not a ring. Then N is simple, contains a right identity element and is the finite direct sum of N-isomorphic minimal left ideals which are N-groups of type 1. Any two N-groups of type 1 are N-isomorphic.*

Proof. Let Γ be the N-group on which N acts 1-primitively. Thus, N acts strongly monogenic on Γ and we have $(0 : \theta_0) = N$ and $(0 : \theta_1) = \{0\}$. Corollary 5.4 allows us to apply Theorem 5.3 which gives the result. ■

So, when we have a zero symmetric and 0-primitive near-ring N with DCCL we can describe its heart, provided that $(0 : \theta_0) \neq \{0\}$. Given a zero symmetric 0-primitive near-ring we unfortunately cannot be assured that $(0 : \theta_0) \neq \{0\}$, as Proposition 7.8 and Example 7.4 in Section 7 shows. On the other hand, the example in Section 4 shows that there exist zero symmetric 0-primitive near-rings even without DCCI such that $(0 : \theta_0) \neq \{0\}$. When using the DCCN, we can be assured that $(0 : \theta_0) \neq \{0\}$ in a zero symmetric and 0-primitive near-ring. This is the content of the next corollary.

Corollary 5.8. *Let N be a 0-primitive near-ring with DCCN which acts 0-primitively on the N-group Γ. Then, $(0 : \theta_0) \neq \{0\}$ and hence, Theorem 5.3 applies.*

Proof. We have to prove that $(0 : \theta_0) \neq \{0\}$. Let $\gamma \in \theta_1$. Consider the annihilator $(0 : \gamma)$ and suppose there is an ideal I of N contained in $(0 : \gamma)$. Then $\{0\} = I\gamma$, contradicting Lemma 3.2. According to Theorem 3.1 a minimal ideal of N is a direct sum of minimal left ideals. Consequently, there must be a minimal left ideal L of N such that $L \not\subseteq (0 : \gamma)$. Thus, by [14, Proposition 3.10] $L \cong_N \Gamma$. Let $\delta \in \theta_0$ and suppose that $L\delta \neq \{0\}$. Thus, by minimality of L we must have $L \cap (0 : \delta) = \{0\}$ and consequently, $\psi : L \to L\delta, l \mapsto l\delta$ is an N-isomorphism. Since $\delta \in \theta_0$, $L\delta \neq \Gamma$. Thus, Γ contains a proper N-subgroup which is N-isomorphic to L and since $L \cong_N \Gamma$, $L\delta \cong_N \Gamma$. By N-isomorphism of L and Γ, L must now also have a proper N-subgroup which is N-isomorphic to L. This violates the statement of Proposition 2.1. Thus, $L \subseteq (0 : \theta_0)$. Corollary 5.4 now shows that we can apply Theorem 5.3. ■

If we have a decomposition of a zero symmetric and 0-primitive near-ring $N = (0 : \theta_1)\dot{+}(0 : \theta_0)$ as in Theorem 5.3, then cleary the minimal ideal $I := (0 : \theta_0)$ acts faithfully and strongly monogenic on Γ. We will see in the example of the sub-section 5 that I can act 1-primitively on Γ, the example in Section 6 shows that there exist zero symmetric 0-primitive near-rings where this is not the case.

A partial converse of Proposition 5.1 is the next theorem. Remember that for a left ideal L in a near-ring N we have introduced the notation $\theta_0^L := \{l \in L | Nl \neq L\}$ (see Definition 2.4).

Theorem 5.9. *Let N be a zero symmetric and subdirectly irreducible near-ring which is not a ring with heart H. Suppose that H satisfies the decending chain condition on left ideals of N contained in H and suppose H contains a minimal left ideal L of N such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then, N is 0-primitive on L, $H = (0 : \theta_0^L)$ and H has a structure as described in Theorem 3.7.*

Proof. Lemma 2.2 gives that L is an N-group of type 0. Suppose that $\{0\} \neq (0 : L)$. Then, $H \subseteq (0 : L)$ contradicting the fact that $L^2 \neq \{0\}$, so N is 0-primitive on L. Lemma 2.5 shows that $(0 : \theta_0^L)$ is a non-zero ideal containing L. Corollary 5.4 now allows us to apply Theorem 5.3 which gives the result. ∎

In the presence of the DCCN, Proposition 5.1 and Theorem 5.9 can be combined nicely. This is not a new observation and was pointed out for example in [5] but should be added at this point for completeness.

Theorem 5.10. *Let N be a zero symmetric near-ring with DCCN. Then the following are equivalent:*

(1) *N is 0-primitive.*

(2) *N is subdirectly irreducible with non-nilpotent heart.*

Proof. $(1) \Rightarrow (2)$ follows from Proposition 5.1.

$(2) \Rightarrow (1)$: Let H be the non-nilpotent heart. [14, Corollary 3.55] now shows that a minimal left ideal contained in H is non-nilpotent. Let $L \subseteq H$ be a minimal left ideal. Since $L^2 \neq \{0\}$ we must have $H \not\subseteq (0 : L)$. Thus, $(0 : L) = \{0\}$. Lemma 2.1 shows that we can apply Lemma 2.2 to prove that L is an N-group of type 0. ∎

Theorem 5.3 shows that we can describe the algebraic structure of a 0-primitive near-ring quite well if we have a kind of finiteness condition and if we know that $(0 : \theta_0) \neq \{0\}$. Proposition 5.1 shows that a zero symmetric 0-primitive near-ring will have a unique minimal ideal H, provided that there exist minimal ideals. We now show that we can prove some interesting results concerning this minimal ideal H without any further assumptions. In fact, we now prove a kind of "density" result for the heart H of a zero symmetric and 0-primitive near-ring. The result is similar to the well known [14, Theorem 4.30] which states that a zero symmetric and 0-primitive near-ring allows interpolation on generators of Γ which have different annihilators. We now will see that, although H does not have to be 0-primitive again (see [14, Remark 4.50] or the example in Section 6), H also allows such a kind of interpolation. In our case we will explicitely assume that the 0-primitive near-ring is not 2-primitive. Ideals in 2-primitive near-rings are known to be 2-primitive again, see [14, Theorem 4.49], so we do not loose any information. Note that the example in Section 6 will show that the heart H of a zero symmetric and 0-primitive near-ring need not be 0-primitive on Γ again, so [14, Theorem 4.30] does not directly carry over. We need a lemma first:

Lemma 5.11. *Let N be a zero symmetric and 0-primitive near-ring N which acts 0-primitively on Γ. Let I be an ideal of N. Assume that I is not a ring. Let $m \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_m \in \theta_1$ such that for $i, j \in \{1, \ldots, m\}, i \neq j$, $\gamma_i \not\sim_I \gamma_j$. Let $\gamma \in \theta_1$. Then, $\cap_{i=1}^m (0 : \gamma_i) \cap I \subseteq (0 : \gamma) \cap I$ implies that there is a $j \in \{1, \ldots, m\}$ such that $(0 : \gamma_j) \cap I = (0 : \gamma) \cap I$.*

Proof. Let $\gamma \in \theta_1$. By Lemma 3.2 we have $I\gamma = \Gamma$. So, $h : I \to \Gamma, i \mapsto i\gamma$ is an N-epimorphism with kernel $(0 : \gamma) \cap I$. For the proof we introduce a notation. Let $\gamma \in \theta_1$. Then, $(0 : \gamma) \cap I := (0 : \gamma)_I$. Thus, $I/(0 : \gamma)_I \cong_N \Gamma$. Since Γ is an N-group of type 0, we have that $I/(0 : \gamma)_I$ is an N-group of type 0. Therefore, $(0 : \gamma)_I$ is a left ideal of I and a maximal N-ideal of the N-group I.

We prove the statement by induction on m. Let $m = 1$, $\gamma \in \theta_1$ and $\gamma_1 \in \theta_1$. Both, $(0 : \gamma_1)_I$ and $(0 : \gamma)_I$ are maximal N-ideals of the N-group I and therefore, $(0 : \gamma_1)_I \subseteq (0 : \gamma)_I$ implies $(0 : \gamma_1)_I = (0 : \gamma)_I$.

Now assume that $\cap_{i=1}^{m+1}(0 : \gamma_i)_I \subseteq (0 : \gamma)_I$. If $(0 : \gamma_{m+1})_I \subseteq (0 : \gamma)_I$, then we are done since maximality of these N-ideals of I imply equality. By induction hypothesis we are done if $\cap_{i=1}^{m}(0 : \gamma_i)_I \subseteq (0 : \gamma)_I$, also. So assume that $\cap_{i=1}^{m}(0 : \gamma_i)_I \not\subseteq (0 : \gamma)_I$ and $(0 : \gamma_{m+1})_I \not\subseteq (0 : \gamma)_I$. Maximality of $(0 : \gamma)_I$ as an N-ideal now implies that

$$I = (0 : \gamma_{m+1})_I + (0 : \gamma)_I$$

as well as

$$I = \cap_{i=1}^{m}(0 : \gamma_i)_I + (0 : \gamma)_I.$$

Γ is a faithful I-group with generator γ and I is assumed not to be a ring. So, Proposition 5.2 shows that

$$\cap_{i=1}^{m+1}(0 : \gamma_i)_I = \cap_{i=1}^{m}(0 : \gamma_i)_I \cap (0 : \gamma_{m+1})_I \not\subseteq (0 : \gamma)_I.$$

But this violates the induction assumption. So the proof is complete. ∎

Theorem 5.12. *Let N be a zero symmetric near-ring which is 0-primitive and not 2-primitive on Γ. Let H be a minimal ideal of N. Let $m \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_m \in \theta_1$ such that for $i, j \in \{1, \ldots, m\}, i \neq j$, $\gamma_i \not\sim_H \gamma_j$. Let $\delta_1, \ldots, \delta_m \in \Gamma$. Then H is not a ring and there exists an element $h \in H$ such that $h\gamma_i = \delta_i$ for all $i \in \{1, \ldots, m\}$.*

Proof. By Proposition 5.1, N is subdirectly irreducible with heart H. If $J_2(N) = \{0\}$, then N is a subdirect product of 2-primitive near-rings and following from subdirect irreducibility, N is 2-primitive, contradicting our assumptions. Thus, $H \subseteq J_2(N)$. By [14, Theorem 5.21] $J_2(H) = J_2(N) \cap H = H$. First assume that H is a ring. Then, by [14, Proposition 5.3] $J_0(H) = H$. Thus, by [14, Theorem 5.37] H is quasiregular in H. This means by definition of quasiregularity, [14, Definition 3.36], that for all $z \in H$, z is contained in the left ideal L_z^H of H which is generated by the set $\{h - hz | h \in H\}$. We proceed to show that z is a quasiregular element of the near-ring N. Thus we have to show that z is contained in the left ideal L_z which is generated by the set $\{n - nz | n \in N\}$.

Let L be the left ideal of N which is generated by the set $\{h - hz | h \in H\}$. Since $\{h - hz | h \in H\} \subseteq \{n - nz | n \in N\}$ we have that $L \subseteq L_z$. On the other hand, $L \cap H$ is a left ideal of H containing the set $\{h - hz | h \in H\}$, since $\{h - hz | h \in H\} \subseteq H$.

Now, L_z^H is the intersection of all left ideals of H containing $\{h - hz | h \in H\}$ and hence, $L_z^H \subseteq L \cap H$. So we see that $L_z^H \subseteq L \subseteq L_z$. So, $z \in L_z$ and we see that z is quasiregular in N. Thus, H is a quasiregular ideal of N and by [14, Theorem 5.37] $H \subseteq J_0(N)$. This contradicts 0-primitivity of N. Therefore we can be assured that H is not a ring.

Note that the statement of the theorem is trivial if $m = 1$, so let $m \geq 2$. For $i \in \{1, \ldots, m\}$ we define

$$L_i := \cap_{j=1, j\neq i}^m (0 : \gamma_j) \cap H.$$

Lemma 5.11 shows that $L_i \not\subseteq (0 : \gamma_i) \cap H$. Since L_i is a left ideal of N we get that $L_i\gamma_i$ is a non-zero N-ideal of Γ and therefore, by 0-primitivity of N, $L_i\gamma_i = \Gamma$. So, for all $i \in \{1, \ldots, m\}$ there is an element $l_i \in L_i$ such that $l_i\gamma_i = \delta_i$. By definition of L_i we have $L_i\gamma_j = \{0\}$ if $j \neq i$, $j \in \{1, \ldots, m\}$. Now let $k = (l_1 + \ldots + l_m)$. Then, $k\gamma_i = \delta_i$ for all $i \in \{1, \ldots, m\}$. Since for $i \in \{1, \ldots, m\}$, $L_i \subseteq H$ we have that $k \in H$ and the proof is complete. ∎

5.1. An example. Theorem 5.3 may be used to construct special examples of 0-primitive near-rings. We will use this in the next Section 6 to prove that the non-nilpotent heart of a zero symmetric subdirectly irreducible near-ring does not have to be subdirectly irreducible again. Before, we will construct a class of finite 0-primitive near-rings which are not 1-primitive (hence not simple) which have the property that they are J_s-radical and do not contain a right identity element (this is in contrast to the situation of finite zero symmetric 1-primitive near-rings which are simple and contain a right identity element, see [14, Theorem 4.46]). Also, they contain a minimal ideal which is a 1-primitive near-ring (in contrast to the example in the next Section 6). The J_s-radical of a near-ring N is a Jacobson type radical which is the smallest ideal containing $J_{\frac{1}{2}}(N)$ if N has DCCL (see [13, Corollary 8.23]). It is mentioned in [13, p. 146] that examples of a near-ring N with $J_0(N) \neq J_s(N)$ are not very common and usually quite complicated to construct. So we take the opportunity to present an easy to construct example of a zero symmetric and 0-primitive near-ring N where we will have $J_s(N) = N$. Also, this example serves as an example how to use Theorem 5.3 to study the structure of a 0-primitive near-ring.

Let $(\Gamma, +)$ be a group containing two non-trivial subgroups S_1 and S_2, $S_1 \subset S_2$, which form a chain, thus $\{0\} \subset S_1 \subset S_2 \subset \Gamma$. Let

$$N := \{f : \Gamma \to \Gamma | f(0) = 0 \text{ and } f(S_2) \subseteq S_1 \text{ and } f(S_1) = \{0\}\}.$$

It is straightforward to see that N is a zero symmetric near-ring which acts faithfully but not strongly monogenic on Γ. By definition of functions in N we see that the elements in $\Gamma \setminus S_2$ can be mapped arbitrarily, so we have $\theta_1 = \Gamma \setminus S_2$ and $\theta_0 = S_2$. Suppose there exists a non-trivial N-ideal I in Γ. Since N is zero symmetric we must have that I is an N-subgroup also, so $I \subseteq \theta_0$. Let $i \in I \setminus \{0\}$. Thus, $i \in \theta_0 = S_2$. Let $\gamma_1 \in \Gamma \setminus S_2$. Then, since $(\theta_0, +)$ is a group, we have that $\gamma_1 + i \notin \theta_0$. Consequently, we can define a function $f : \Gamma \to \Gamma$ such that $f(\gamma_1 + i) = \gamma_1$ and $f(\delta) = 0$, $\delta \in \Gamma \setminus \{\gamma_1\}$. Then, $f \in N$ and $f(\gamma_1 + i) - f(\gamma_1) = \gamma_1 \notin \theta_0$. Hence, I is not an N-ideal. This shows that N acts 0-primitively but not 1-primitively on Γ. Clearly (and also from Corollary 5.8), $(0 : \theta_0) \neq \{0\}$. Consequently, Theorem 5.3 shows that $N = (0 : \theta_1) \dot{+} (0 : \theta_0)$ and $(0 : \theta_1) \neq \{0\}$.

We now show that $(0 : \theta_1)^2 = \{0\}$. Let $f_1, f_2 \in (0 : \theta_1)$. For $\gamma \in \theta_1$, $f_1(f_2(\gamma)) = 0$ since the functions in N are zero preserving. Let $\delta \in \theta_0 = S_2$. Then, $f_1(f_2(\delta)) = f_1(\delta_1)$ with $\delta_1 \in S_1$ because $f_2(S_2) \subseteq S_1$. But $f_1(\delta_1) = 0$ since $f_1(S_1) = \{0\}$. This shows that $(0 : \theta_1)$ is nilpotent. Thus, by [14, Theorem 5.37] $(0 : \theta_1) \subseteq J_{\frac{1}{2}}(N)$. Theorem 5.3 shows that $J_{\frac{1}{2}}(N) \subseteq (0 : \theta_1)$, so $J_{\frac{1}{2}}(N) = (0 : \theta_1)$. Also from Theorem 5.3 we have that $N = J_{\frac{1}{2}}(N) \dot{+} (0 : \theta_0)$, $(0 : \theta_0)$ being the heart of N. So, the smallest ideal containing $J_{\frac{1}{2}}(N)$ now must equal N, so $J_s(N) = N$.

Note that $(N/(0 : \theta_0))^2 = \{0\}$. So the factor near-ring of a 0-primitive near-ring by $(0 : \theta_0)$ does not have to be 0-primitive again, not even 0-semisimple. In our case it is 0-radical, so $J_0(N/(0 : \theta_0)) = N/(0 : \theta_0)$ by [14, Theorem 5.37].

Also, the example shows that contrary to the situation in 1-primitive near-rings (see [14, Theorem 4.46]) a finite 0-primitive near-ring does not necessarily have a multiplicative right identity. Suppose $g \in N$ would be a multiplicative right identity. Then, by [14, Theorem 3.43], $g = e_1 + e_2$ with $e_1 \in (0 : \theta_1)$ and $e_2 \in (0 : \theta_0)$ and $e_1^2 = e_1$. But we just proved that $(0 : \theta_1)^2 = \{0\}$, so $e_1^2 = 0$. Thus, $g = e_2$ and so, $N = (0 : \theta_0)$ which is not true, since N is not acting 1-primitively on Γ. So, N has no right identity.

Clearly, $H := (0 : \theta_0)$ acts faithfully and strongly monogenic on Γ and any element in θ_1 generates Γ as an H-group (note that this must be the case by Lemma 3.2). Suppose that U is a non-trivial H-ideal and thus an H-group in Γ. Then, $U \subseteq \theta_0$. Let $u \in U \setminus \{0\}$. Let $\gamma_1 \in \Gamma \setminus S_2$. Then, since $(\theta_0, +)$ is a group, we have that $\gamma_1 + u \notin \theta_0$. Consequently, similar as before, we can define a function $f : \Gamma \to \Gamma$ such that $f(\gamma_1 + u) = \gamma_1$ and $f(\delta) = 0$, $\delta \in \Gamma \setminus \{\gamma_1\}$. Then, $f \in H$ and $f(\gamma_1 + u) - f(\gamma_1) = \gamma_1 \notin \theta_0$. Hence, U is not an H-ideal. This shows that $(0 : \theta_0)$ acts 1-primitively on Γ and is a 1-primitive near-ring and hence, $(0 : \theta_0)$ is a simple near-ring. That this is not always the case will be shown in the example of the next section. What is more, Proposition 5.1 shows that N is subdirectly irreducible. Since N is not 1-primitive we have $J_1(N) \neq \{0\}$. Thus, since $(0 : \theta_0)$ is the heart of N we have $(0 : \theta_0) \subseteq J_1(N)$.

6. A Minimal Ideal Which Is Not Nilpotent and Not Subdirectly Irreducible

The result of Theorem 5.3 now allows us to adress an open question in the theory of minimal ideals of near-rings. In [1] the question was raised if a non-nilpotent minimal ideal of a zero symmetric near-ring is a subdirectly irreducible near-ring. This is true and was proved in [7] for example when the near-ring is distributively generated (see also [4] concerning the topic of ideals in distributively generated near-rings). The general case remained open in [1] and this problem was also adressed in [3]. For more details and references concerning this topic consult [1] or [3].

We will now give an example that in general a minimal ideal which is not nilpotent does not have to be subdirectly irreducible. In fact, the example shows that the non-nilpotent heart of a zero symmetric subdirectly irreducible near-ring need not be subdirectly irreducible again. To fix the notation, let $(\Gamma, +)$ be a group. Then $M_0(\Gamma)$ is the near-ring of zero preserving functions mapping from Γ to Γ where near-ring addition is the pointwise addition of functions and multiplication is function composition.

Let $(\Gamma, +) := (\mathbb{Z}_8 \times \mathbb{Z}_8, +)$. Let $\theta_0 := \{0, 2, 4, 6\} \times \{0, 2, 4, 6\}$. $(\theta_0, +)$ is a subgroup of $(\Gamma, +)$. Let $S_1 := \{0, 4\} \times \{0\}$ and $S_2 := \{0\} \times \{0, 4\}$. Both $(S_1, +)$ and $(S_2, +)$ are subgroups of $(\theta_0, +)$. Let

$$N := \{f \in M_0(\Gamma) | f(\theta_0) \subseteq \theta_0 \text{ and } \forall (x, y) \in \Gamma \setminus \theta_0 :$$
$$f((x, y)) = f((x, y) + (4, 0)) = f((x, y) + (0, 4))\}.$$

It is easy to see that N is a zero symmetric subnear-ring of $M_0(\Gamma)$ acting faithfully on Γ. Note that whenever $(x, y) \in \Gamma \setminus \theta_0$, then also $\{(x, y)+(4, 0), (x, y)+(0, 4)\} \subseteq \Gamma \setminus \theta_0$. Both elements $(4, 0)$ and $(0, 4)$ are of additive order 2. So, for an element $(x, y) \in \Gamma \setminus \theta_0$ and $f \in N$, $f(x, y) = f(4 + x, y) = f(x, y + 4) = f(x + 4, y + 4)$. From that we see that always 4 elements in $\Gamma \setminus \theta_0$ are mapped to the same value by a function in N. Each element in $\Gamma \setminus \theta_0$ is a generator of Γ as an N-group. So, a proper N-ideal of Γ must be contained in θ_0. Since elements from θ_0 can be mapped arbitrarily into θ_0 again, we see that the only subgroup of $(\theta_0, +)$ which is invariant under the near-ring action is $(\theta_0, +)$ itself. So, the only possible N-ideal of Γ is $(\theta_0, +)$ (it certainly is an N-subgroup). Let $f \in N$ and suppose that θ_0 is an N-ideal, indeed. Then, since $(2, 0) \in \theta_0$, we must have $f((1, 0) + (2, 0)) - f((1, 0)) = f((3, 0)) - f((1, 0)) \in \theta_0$. Both, $(3, 0)$ and $(1, 0)$ are elements in $\Gamma \setminus \theta_0$. Since $f((1, 0)) = f((5, 0)) = f((1, 4)) = f((5, 4))$ and $f((3, 0)) = f((7, 0)) = f((3, 4)) = f((7, 4))$ we see that $(3, 0)$ and $(1, 0)$ can be independently mapped arbitrarily into Γ. Let $\delta_1 \in \theta_0$ and $\delta_2 \notin \theta_0$. Let $f : \Gamma \to \Gamma$ such that

$$f((3, 0)) = f((7, 0)) = f((3, 4)) = f((7, 4)) = \delta_1$$

and

$$f((1, 0)) = f((5, 0)) = f((1, 4)) = f((5, 4)) = \delta_2$$

and

$$f(\gamma) = 0 \text{ for } \gamma \in \Gamma \setminus \{(3, 0), (7, 0), (3, 4), (7, 4), (1, 0), (5, 0), (1, 4), (5, 4)\}.$$

Then, $f \in N$ and $f((3, 0)) - f((1, 0)) \notin \theta_0$. So, θ_0 is not an N-ideal. Hence, N acts 0-primitively on Γ. Clearly, $(0 : \theta_0) \neq \{0\}$, this also follows from Corollary 5.8 by the way. So, it follows from Theorem 5.3 that $I := (0 : \theta_0)$ is a minimal and non-nilpotent (it contains a right identity element) ideal of N, in fact I is the heart of the subdirectly irreducible near-ring N by Proposition 5.1. Let

$$J_1 := \{f \in I | \forall \gamma \in \Gamma : f(\gamma) \in S_1\}$$

and

$$J_2 := \{f \in I | \forall \gamma \in \Gamma : f(\gamma) \in S_2\}.$$

Obviously, J_1 and J_2 are non-zero subsets of N. Since S_1 and S_2 are groups and the sum of two elements of I is again in I, J_1 and J_2 are closed w.r.t. pointwise addition of functions. Let $j \in J_1$, $f \in N$. Then, since I is an ideal, $jf \in I$. Let $\gamma \in \Gamma$ and $f(\gamma) = \gamma_1$. Then, $j(f(\gamma)) = j(\gamma_1) \in S_1$ and so we see that J_1 is closed under composition of functions and is even a right ideal of N and so of I (that $(J_1, +)$ is a normal subgroup of $(N, +)$ clearly follows from the fact that $(\Gamma, +)$ is abelian). Similary, this holds for J_2. So we have that J_1 and J_2 are non-zero right ideals of N and therefore of I. We now prove that they are also left ideals of I. Thus we have to show that for all $i_1, i_2 \in I$ and for all $j \in J_1$, $i_1(i_2 + j) - i_1 i_2 \in J_1$ and similary for J_2. Let $i_1, i_2 \in I$ and $j \in J_1$. First note that since I is an ideal, $i_1(i_2 + j) - i_1 i_2 \in I$. Let $\gamma \in \Gamma$. If $j(\gamma) = (0, 0)$, then clearly $(i_1(i_2 + j) - i_1 i_2)(\gamma) = (0, 0) \in S_1$. So suppose that $j(\gamma) \neq (0, 0)$. Since $j(\gamma) \in S_1$ this means that $j(\gamma) = (4, 0)$. First suppose that $i_2(\gamma) \in \theta_0$, so also $i_2(\gamma) + j(\gamma) = i_2(\gamma) + (4, 0) \in \theta_0$, by definition of θ_0. Since $i_1 \in I = (0 : \theta_0)$ we have that

$$(i_1(i_2 + j) - i_1 i_2)(\gamma) = i_1(i_2(\gamma) + j(\gamma)) - i_1 i_2(\gamma)$$

and

$$i_1(i_2(\gamma) + j(\gamma)) - i_1 i_2(\gamma) = (0, 0) - (0, 0) \in S_1.$$

Now suppose that $i_2(\gamma) \notin \theta_0$. Consequently, by definition of functions in N, $i_1(i_2(\gamma) + (4, 0)) = i_1(i_2(\gamma))$. Therefore,

$$(i_1(i_2 + j) - i_1 i_2)(\gamma) = i_1(i_2(\gamma) + j(\gamma)) - i_1 i_2(\gamma)$$

and

$$i_1(i_2(\gamma) + j(\gamma)) - i_1 i_2(\gamma) = i_1(i_2(\gamma) + (4, 0)) - i_1(i_2(\gamma)) = (0, 0) \in S_1.$$

Thus, J_1 is a left ideal and therefore and ideal of I.

Now we have to show that J_2 is a left ideal of I. Thus we have to show that for all $i_1, i_2 \in I$ and for all $j \in J_2$, $i_1(i_2 + j) - i_1 i_2 \in J_2$. Again since I is an ideal, $i_1(i_2 + j) - i_1 i_2 \in I$. Let $i_1, i_2 \in I$ and $j \in J_2$. Let $\gamma \in \Gamma$. If $j(\gamma) = (0, 0)$, then clearly $(i_1(i_2 + j) - i_1 i_2)(\gamma) = (0, 0) \in S_2$. So suppose that $j(\gamma) \neq (0, 0)$. Since $j(\gamma) \in S_2$ this means that $j(\gamma) = (0, 4)$. First suppose that $i_2(\gamma) \in \theta_0$, so also $i_2(\gamma) + j(\gamma) = i_2(\gamma) + (0, 4) \in \theta_0$. Since $i_1 \in I = (0 : \theta_0)$ we have that

$$(i_1(i_2+j)-i_1i_2)(\gamma)=i_1(i_2(\gamma)+j(\gamma))-i_1i_2(\gamma)$$

and

$$i_1(i_2(\gamma)+j(\gamma))-i_1i_2(\gamma)=(0,0)-(0,0)\in S_2.$$

Now suppose that $i_2(\gamma)\notin\theta_0$. Consequently, by definition of functions in N, $i_1(i_2(\gamma)+(0,4))=i_1(i_2(\gamma))$. Therefore,

$$(i_1(i_2+j)-i_1i_2)(\gamma)=i_1(i_2(\gamma)+j(\gamma))-i_1i_2(\gamma)$$

and

$$i_1(i_2(\gamma)+j(\gamma))-i_1i_2(\gamma)=i_1(i_2(\gamma)+(0,4))-i_1(i_2(\gamma))=(0,0)\in S_2.$$

Thus, J_2 is a left ideal and therefore and ideal of I.

Since $S_1\cap S_2=\{(0,0)\}$, $J_1\cap J_2=\{0\}$. Therefore, I contains two non-zero ideals with zero intersection and I cannot be subdirectly irreducible. So, there is no unique minimal ideal in I. Note that N is 0-primitive on Γ but not 1-primitive, so $J_1(N)\neq\{0\}$ as a consequence of Proposition 5.1. Thus, the minimal ideal I is contained in $J_1(N)\subseteq J_2(N)$ and by [14, Theorem 5.21] $J_2(I)=I$. Note also that the ideal I of this example is also an example of a minimal ideal which does not have zero multiplication but is not simple as a near-ring. Consequently, I is not a 1-primitive near-ring, by [14, Theorem 4.46]. Since I acts stronlgy monogenic on Γ this shows that I is not acting 0-primitively on Γ. The first example of a non-nilpotent minimal ideal I in a zero symmetric near-ring such that I is not a simple near-ring was given by K. Kaarli, see [8, Proposition 3.5]. In Kaarli's example I contains a unique minimal ideal S such that $S^2=\{0\}$, so I is subdirectly irreducible.

We summarize the results in this section as a proposition:

Proposition 6.1. *There exists a finite zero symmetric subdirectly irreducible near-ring N with $(N,+)$ abelian and heart H such that H contains a right identity element and is not nilpotent and H is not a subdirectly irreducible near-ring.*

7. Simple Near-rings

For a ring R it is well kown that if R contains a minimal left ideal and is simple, then R is a primitive ring or $R^2=\{0\}$. If one considers a zero symmetric near-ring N which is simple, then $N^2=\{0\}$ or N is 1-primitive if one assumes that the near-ring satisfies the DCCN, as we will see in Corollary 7.6. Since in a near-ring which is a ring, the DCCN equals the DCCL, Corollary 7.6 re-proves the ring theory result. This observation was already made in [6] and will be re-proven in this section. First note the following easy to establish proposition:

Proposition 7.1. *Let N be a zero symmetric simple near-ring which contains a minimal N-subgroup M of N such that $M^2\neq\{0\}$. Then N acts 2-primitively on M.*

Proof. Due to the simplicity of N, $(0 : M) = \{0\}$ or $(0 : M) = N$. In case $NM = \{0\}$, this would imply $M^2 = \{0\}$, a contradiction. Due to minimality of M as an N-subgroup it follows that M is an N-subgroup of type 2 and so, N acts 2-primitively on M. ∎

Proposition 7.1 is easy to prove and well known (see [14, Corollary 4.47]) but it requires a minimal condition on N-subgroups. In this section we address the question what can be said if we require the existence of a minimal left ideal only.

The result concerning simple rings with a minimal left ideal and also Proposition 7.1 can be extended accordingly in the following way:

Theorem 7.2. *Let N be a zero symmetric near-ring which is a simple near-ring. Let L be a minimal left ideal such that $L^2 \neq \{0\}$. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then N is a 1-primitive near-ring, acting 1-primitively on L.*

Proof. By Lemma 2.2, L is an N-group of type 0. We proceed to show that L is an N-group of type 1 in our situation. Thus, we have to show that N acts strongly monogenic on L. Lemma 2.5 implies that $(0 : \theta_0^L)$ is a non-zero ideal of N and by simplicity of N this implies that $(0 : \theta_0^L) = N$. Consequently, for $l \in \theta_0^L$ we have $Nl = \{0\}$, so N acts strongly monogenic on L and L is an N-group of type 1. Suppose $(0 : L) \neq \{0\}$. Then, $(0 : L)$ is a non-zero ideal of N and by simplicity of N, $(0 : L) = N$. This contradicts $L^2 \neq \{0\}$. So, N acts faithfully on L, L being an N-group of type 1. This shows that N is 1-primitive on L. ∎

In case we deal with simple near-rings with identity we can rule out the case of having minimal left ideals with zero multiplication and we get the following:

Theorem 7.3. *Let N be a zero symmetric simple near-ring with identity containing a minimal left ideal L. Then one of the following holds:*

(1) *N is a 2-primitive near-ring, acting 2-primitively on L.*

(2) *Every minimal left ideal J of N contains an N-subgroup M, $M \neq J$, such that M is N-isomorphic to J and $J_2(N) = N$.*

Proof. We assume that there is a minimal left ideal L which does not contain N-subgroups properly contained in L and being N-isomorphic to L. Suppose $L^2 = \{0\}$. Thus, $\{0\} \neq L \subseteq (0 : L)$. By simplicity of N this implies $N = (0 : L)$, so $NL = \{0\}$. But N contains an identity element, so $L \subseteq NL = \{0\}$ which is a contradiction. Consequently, $L^2 \neq \{0\}$ and Theorem 7.2 shows that N is a 1-primitive near-ring acting on L. Since N has an identity element, N is also 2-primitive by [14, Proposition 3.7].

Assume that every minimal left ideal J contains an N-subgroup M, $M \neq J$, such that M is N-isomorphic to J. By simplicity of N we can only have $J_2(N) = \{0\}$ or

$J_2(N) = N$. Suppose that $J_2(N) = \{0\}$. Thus, by [14, Theorem 5.2] the intersection of the 2-primitive ideals of N is zero. Since N is assumed to be simple, the zero ideal must be a 2-primitive ideal and consequently, N must be a 2-primitive near-ring acting 2-primitively on the N-group Γ, say. Thus, Γ is an N-group of type 2. Let J be a minimal left ideal in N. Since Γ is faithful, $J\Gamma \neq \{0\}$, so there is an element $\gamma \in \Gamma$ such that $J\gamma \neq \{0\}$. Since Γ is a strongly monogenic N-group this implies that $N\gamma = \Gamma$. Consequently, $J\gamma$ is an N-ideal in Γ and by 2-primitivity, $\Gamma = J\gamma$. Thus, $\psi : J \to \Gamma, j \mapsto j\gamma$ is an N-epimorphism with kernel $J \cap (0 : \gamma) = \{0\}$ due to minimality of J. Thus, ψ is an N-isomorphism. This is a contradiction to Γ being an N-group of type 2 because J contains a proper N-subgroup M. ∎

Note that the identity in Theorem 7.3 was only needed to guarantee that nilpotent left ideals do not exist in the near-ring. So one could appropriately generalize the statement of the theorem. The situation in item (2) of Theorem 7.3 may occur. There exist zero symmetric simple J_2-radical near-rings with identity and containing minimal left ideals.

Example 7.4. Let D be a principal ideal domain which is not a field. Let D^2 be the natural right D-module of the ring D. Then,

$$M_D(D^2) := \{f : D^2 \to D^2 | \forall g \in D^2 \forall d \in D : f(gd) = f(g)d\}$$

is a zero symmetric near-ring with identity.

It is shown in [11, Theorem 2.12] that $M_D(D^2)$ is a simple near-ring. In [12, Theorem 2.2] it is shown that $J_2(M_D(D^2)) = M_D(D^2)$. In [11, Theorem 3.5] it is shown that $M_D(D^2)$ contains minimal left ideals. Thus, $M_D(D^2)$ is a near-ring whose structure is as described in item (2) of Theorem 7.3.

Consequently, we formulate as a proposition:

Proposition 7.5. *There exists a zero symmetric simple near-ring N with identity and containing a minimal left ideal such that $J_2(N) = N$ and every minimal left ideal L contains an N-subgroup M, $M \neq L$, such that M is N-isomorphic to L.*

As a corollary, Theorem 7.2 triggers a result concerning simple near-rings with DCCN which also was proved in [6].

Corollary 7.6. *Let N be a zero symmetric near-ring with $DCCN$ which is a simple near-ring. Then one of the following holds:*

(1) $N^2 = \{0\}$.

(2) *N is a 1-primitive near-ring, acting 1-primitively on some minimal left ideal L.*

Proof. Suppose that $N \neq \{0\}$. The DCCN guarantees the existence of a minimal left ideal L. Due to simplicity of the near-ring N, N is minimal as an ideal. Suppose first that $L^2 = \{0\}$. Then, L is a nilpotent left ideal. Since $L \cap N \neq \{0\}$, [14, Corollary 3.55] implies that N is nilpotent. Thus, there is a natural number $k > 1$ such that $N^k = \{0\}$ and $N^{k-1} \neq \{0\}$. Consequently, $N^{k-1} \subseteq (0 : N)$. Thus, $(0 : N)$ is a non-zero ideal of N. By simplicity of N this implies $N = (0 : N)$. So, we have proven $N^2 = \{0\}$ in case the minimal left ideal L is such that $L^2 = \{0\}$.

Thus, we now assume that $L^2 \neq \{0\}$. Proposition 2.1 shows that we can apply Theorem 7.2 giving the result. ■

In Proposition 7.1 we have seen that given a zero symmetric simple near-ring N containing a minimal N-subgroup M of N such that $M^2 \neq \{0\}$, then N acts 2-primitively on M. Corollary 7.6 claims that given a zero symmetric near-ring N with $DCCN$ which is a simple near-ring and $N^2 \neq \{0\}$ then N acts 1-primitively on a minimal left ideal L. We now present an example of a finite zero symmetric near-ring N which does not have zero multiplication and which is not acting 1-primitively on some minimal N-subgroup of N but is acting 1-primitively only on a minimal left ideal. While simple near-rings with identity which are J_2-radical are hard to construct (the only class of that type known to the author is the class of near-rings discussed in Example 7.4) it is easy to construct simple J_2-radical near-rings which contain a right identity element, as the example also shows.

Example 7.7. Let $(N, +) := (\mathbb{Z}_9, +)$, the cyclic group of order 9 and define the multiplication $*$ as follows:

$*$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	8	1	0	0	0
2	0	0	0	0	7	2	0	0	0
3	0	0	0	0	6	3	0	0	0
4	0	0	0	0	5	4	0	0	0
5	0	0	0	0	4	5	0	0	0
6	0	0	0	0	3	6	0	0	0
7	0	0	0	0	2	7	0	0	0
8	0	0	0	0	1	8	0	0	0

$(N, +, *)$ is a so called planar near-ring (see [14, Theorem 8.96 and Example 1.4] for the details of constructing planar near-rings using fixedpointfree automorphism groups; here we use the group $(\{id, -id\}, \circ)$ which acts without fixedpoints on $(\mathbb{Z}_9, +)$). $U := (\{0, 3, 6\}, +)$ is the only subgroup of $(N, +)$ and in fact it is an N-subgroup of N. Since $1 * (1 + 3) - 1 * 1 = 1 * 4 - 1 * 1 = 8 \notin U$, U is not a left ideal of N. From that we see that N itself is an N-group of type 1 and N acts 1-primitively on N but not on U which is the minimal N-subgroup contained in N. Clearly, N is a simple near-ring and since $NU = \{0\}$ we have that $J_2(N) = N$ by [14, Corollary 5.45].

In any case zero symmetric simple near-rings which contain a right identity element are 0-primitive near-rings. This parallels the result in ring theory which says that any simple ring with a right identity is primitive. We add one more aspect concerning the Jacobson radical of type 1.

Proposition 7.8. *Let N be a zero symmetric simple near-ring which contains a multiplicative right identity. Then N is a 0-primitive near-ring. Let Γ be the N-group of type 0 the near-ring N acts on 0-primitively. Let $\theta_0 := \{\delta \in \Gamma | N\delta \neq \Gamma\}$. If $J_1(N) = N$, then $(0 : \theta_0) = \{0\}$.*

Proof. Due to the right identity element of N, the zero ideal is a modular ideal (see [14, Remark 3.21]). By simplicity of N, the zero ideal is a maximal ideal and so a maximal modular ideal. Hence, by [14, Theorem 4.36] the zero ideal is a 0-primitive ideal which proves that N is a 0-primitive near-ring.

Suppose that $(0 : \theta_0) \neq \{0\}$. Since $N\theta_0 \subseteq \theta_0$ we have that $(0 : \theta_0)$ is an ideal of N and by simplicity, $N = (0 : \theta_0)$. Hence for any $\delta \in \theta_0$, $N\delta = 0$ and so N acts 1-primitively on Γ. Thus, $J_1(N) = \{0\} \neq N$. ∎

In case a near-ring N has an identity element, $J_2(N) = J_1(N)$. So we see from Proposition 7.8 that a near-ring N as described in Example 7.4 is 0-primitive and serves as an example of a zero symmetric 0-primitive near-ring with $(0 : \theta_0) = \{0\}$ (see Theorem 5.3 and the subsequent discussion).

8. Open Questions

In Section 6 we could answer an open question in the theory of minimal ideals of near-rings. However, new questions arise. In particular it would be interesting to know if given a zero symmetric near-ring N with DCCL which is subdirectly irreducible with heart H and $H^2 \neq \{0\}$, then does H contain a minimal left ideal L of N such that $L^2 \neq \{0\}$? Such a situation occured when studying 0-primitive near-rings in Section 5. In case N has the DCCN, this is true by Theorem 3.1.

Also of interest is the situation in Theorem 5.3. Given a zero symmetric near-ring N which is 0-primitive on Γ and $\theta_0 := \{\gamma \in \Gamma | N\gamma \neq \Gamma\}$ when is $(0 : \theta_0) \neq \{0\}$? From Corollary 5.8 we know that this must be true if N satisfies the DCCN. The example in Section 4 shows that this can be true even if N does not satisfy the DCCI. On the other hand, Example 7.4 shows that this does not have to be true always. The near-ring N of Example 7.4 has the property that every left ideal of N contains a minimal left ideal, by [11, Lemma 3.3] and [11, Theorem 3.5]. But from the results in [11] we see that in general N does not have the DCCL and an example of a near-ring of the type required in Example 7.4 and having the DCCL is not known to the author. So, would the DCCL suffice that $(0 : \theta_0) \neq \{0\}$? In the same setting we may ask if near-rings of the form as described in item (2) of Theorem 7.3 do really exist if we assume the DCCL (instead of the existence of minimal left ideals only). Note that this situation is very different from ring theory, for if we have a simple ring R with identity which contains a minimal left ideal, then R satisfies the DCCL, see [10, Theorem 3.10].

Also interesting would be an example of a minimal left ideal without zero multiplication in a zero symmetric near-ring N which is not an N-group of type 0 (see Lemma 2.2).

References

1. G. Birkenmeier and H. Heatherly, Minimal ideals in near-rings, *Comm. Algebra*, **20(2)** (1992), 457-468.
2. J. F. T. Hartney and A. M. Matlala, Structure theorems for the socle-ideal of a near-ring, *Comm. Algebra*, **36(3)** (2008), 1140-1152.
3. H. Heatherly, Localized distributivity conditions, *Near-rings and near-fields* (Fredericton, NB, 1993), 13-29, Math. Appl., 336, Kluwer Acad. Publ., Dordrecht, 1995.
4. H. Heatherly and G. Mason, Ideals in distributively generated nearrings, *Publ. Math. Debrecen*, **69(1-2)** (2006), 121-135.
5. K. Kaarli, On non-zerosymmetric near-rings with minimum condition, *Nearrings, nearfields and K-loops*, (Hamburg, 1995), 21-33, Math. Appl., 426, Kluwer Acad. Publ., Dordrecht, 1997.
6. K. Kaarli, Minimal ideals in near-rings (Russian), *Tartu Riikl. l. Toimetised Vih.*, **366** (1975), 105-142.
7. K. Kaarli, On minimal ideals of distributively generated near-rings, *Contributions to general algebra*, 7 (Vienna, 1990), 201-204, Hölder-Pichler-Tempsky, Vienna, 1991.
8. K. Kaarli, On Jacobson type radicals of near-rings, *Acta Math. Hungar.*, **50(1-2)** (1987), 71-78.
9. W. F. Ke and J. H. Meyer, Matrix near-rings and 0-primitivity, *Monatsh. Math.*, **165(3-4)** (2012), 353-363.
10. T. Y. Lam, *A first course in noncommutative rings*, Second edition, Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 2001. xx+385 pp. ISBN: 0-387-95183-0.
11. C. J. Maxson and A. P. J. van der Walt, Centralizer near-rings over free ring modules, *J. Austral. Math. Soc. Ser. A*, **50(2)** (1991), 279-296.
12. P. Fuchs, C. J. Maxson, A. P. J. van der Walt and K. Kaarli, Centralizer near-rings determined by PID-modules II, *Period. Math. Hungar.*, **26(2)** (1993), 111-114.
13. J. D. P. Meldrum, *Near-rings and their links with groups*, Research Notes in Mathematics, 134. Pitman (Advanced Publishing Program), Boston, MA, 1985. x+275 pp. ISBN: 0-273-08701-0.

14. G. Pilz, *Near-rings*, The theory and its applications, Second edition. North-Holland Mathematics Studies, 23. North-Holland Publishing Co., Amsterdam, 1983. xv+470 pp. ISBN: 0-7204-0566-1.

15. S. D. Scott, Minimal ideals of near-rings with minimal condition, *J. London Math. Soc.*, **8(2)** (1974), 8-12.

16. G. Wendt, Left ideals in 1-primitive near-rings, *Math. Pannon.*, **16(1)** (2005), 145-151.

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