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ASYMPTOTIC ANALYSIS OF FOURTH ORDER QUASILINEAR DIFFERENTIAL EQUATIONS IN THE FRAMEWORK OF REGULAR VARIATION

Jelena Milošević and Jelena V. Manojlović

Abstract. Under the assumptions that $p(t),\ q(t)$ are regularly varying functions satisfying condition

$$\int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty,$$

existence and asymptotic form of regularly varying intermediate solutions are studied for a fourth-order quasilinear differential equation

$$\left(p(t)|x''(t)|^{\alpha-1}x''(t)\right)'' + q(t)|x(t)|^{\beta-1}x(t) = 0, \quad \alpha > \beta > 0.$$

It is shown that the asymptotic behavior of all such solutions is governed by a unique explicit law.

1. INTRODUCTION

This paper is concerned with positive solutions of fourth-order quasilinear differential equations of the form

$$\text{(E)} \qquad \left(p(t)|x''(t)|^{\alpha-1}x''(t)\right)'' + q(t)|x(t)|^{\beta-1}x(t) = 0, \quad t \ge a > 0,$$

where α and β are positive constants such that $\alpha > \beta$ and $p(t), q(t)$ are positive continuous functions defined on $[a, \infty)$ and $p(t)$ satisfies

$$\text{(1.1)} \qquad \int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty.$$

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The equation (E) is called *sub-half-linear* if $\beta < \alpha$ and *super-half-linear* if $\beta > \alpha$. By a solution of (E) we mean a function $x(t) : [T, \infty) \to \mathbb{R}$, $T \geq a$, such that $x(t)$ and $p(t)|x''(t)|^{\alpha-1}x''(t)$ is twice continuously differentiable on $[T, \infty)$ and satisfies the equation (E) at every point of $[T, \infty)$. A solution $x(t)$ of (E) is said to be *nonoscillatory* if $x(t) \neq 0$ for all large t and *oscillatory* otherwise. In other words, a solution $x(t)$ of (E) is nonoscollatory if $x(t)$ is eventually positive or eventually negative. If $x(t)$ is a solution of (E), then so does $-x(t)$. Therefore, there is no loss of generality in assuming that a nonoscillatory solution of (E) is eventually positive.

Throughout this paper extensive use is made of the symbol $\sim$ to denote the asymptotic equivalence of two positive functions, i.e.,

$$f(t) \sim g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

We also use the symbol $\prec$ to denote the dominance relation between two positive functions in the sense that

$$f(t) \prec g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty.$$

The oscillatory and asymptotic behavior of solutions of the equation (E) has been recently considered by Wu [18] and Naito and Wu [5] under the conditions

$$\int_a^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\, dt = \infty, \tag{1.2}$$

or, more strongly,

$$\int_a^\infty \frac{t}{p(t)^{\frac{1}{\alpha}}}\, dt = \infty \quad \wedge \quad \int_a^\infty \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}} dt = \infty. \tag{1.3}$$

We note that (1.1) implies both of (1.2) and (1.3).

The aim of this paper is to obtain a more detailed information on the asymptotic behavior of positive solutions of equation (E) under the condition (1.1).

The main body of the paper is divided into six sections. In Section 2 we classify the totality of positive solutions of (E) into several types according to their asymptotic behavior at infinity. There a crucial role is played by the four functions

$$\psi_1(t) = 1, \quad \psi_2(t) = t, \quad \psi_3(t) = \int_a^t\!\!\int_a^s \frac{1}{p(r)^{\frac{1}{\alpha}}}\, drds, \quad \psi_4(t) = \int_a^t \int_a^s \left(\frac{r}{p(r)}\right)^{\frac{1}{\alpha}} drds,$$

which are the particular solutions of the unperturbed differential equation

$$(p(t)|x''(t)|^{\alpha-1}x''(t))'' = 0.$$

It is to be noted that the functions define above satisfy the dominance relation

$$\psi_1(t) \prec \psi_2(t) \prec \psi_3(t) \prec \psi_4(t), \quad t \to \infty.$$

Necessary and sufficient integral conditions for the existence of positive solutions $x(t)$ of (E) which are asymptotic to positive constant multiplies by the function $\psi_i(t),\ i = \{1, 2, 3, 4\}$, called *primitive solutions* of (E), have been given in [5, 18]. Our goal is to show that equation (E), except primitive solutions, possesses two more types of positive solutions such that

$$\text{(I)} \quad \psi_1(t) \prec x(t) \prec \psi_2(t) \quad \text{or} \quad \text{(II)} \quad \psi_3(t) \prec x(t) \prec \psi_4(t) \quad \text{as} \quad t \to \infty.$$

Such solutions will be called *intermediate solutions* of (E). In Section 4 we established sufficient conditions for the existence of such solutions of (E) with continuous coefficients $p(t)$ and $q(t)$. In Section 5 we consider equation (E) with generalized regularly varying $p(t)$ and $q(t)$, while the definition of generalized regularly varying functions and some of their basic properties are summarized in Section 3. After showing that each of two classes of intermediate generalized regularly varying solutions of type (I) and (II) can be divided into three disjoint subclasses according to their asymptotic behavior at infinity, we establish necessary and sufficient conditions for the existence of solutions belonging to each of these three solution subclasses of types (I) and (II). Our discussions include determining the asymptotic behavior of solutions contained in each of the six subclasses explicitly and precisely. In the final Section 6 it is shown that our main results, when specialized to the case where $p(t)$ and $q(t)$ are regularly varying functions in the sense of Karamata, provide thorough information about the existence and asymptotic behavior of regularly varying solutions in the sense of Karamata for equation (E). This information combined with that of the primitive solutions of (E) (cf. Theorems 2.1-2.4) enables us to depict a clear picture of the structure of all regularly varying solutions for equations of the form (E) with regularly varying coefficients.

2. Classification of Positive Solutions

We begin by classification the set of all possible positive solutions of (E) according to their asymptotic behavior as $t \to \infty$. Let $x(t)$ be a positive solution of (E). It is known (see [18]) that $x(t)$ satisfies either

$$x'(t) > 0, \quad x''(t) > 0, \quad (p(t)|x''(t)|^{\alpha-1}x''(t))' > 0 \quad \text{for all large } t, \tag{2.1}$$

or

$$x'(t) > 0, \quad x''(t) < 0, \quad (p(t)|x''(t)|^{\alpha-1}x''(t))' > 0 \quad \text{for all large } t. \tag{2.2}$$

Since (E) implies that $(p(t)|x''(t)|^{\alpha-1}x''(t))'$ is decreasing and positive, there exists a finite limit $\lim_{t\to\infty}(p(t)|x''(t)|^{\alpha-1}x''(t))' = \omega_3 \geq 0$.

Solutions satisfying (2.1). First let $x(t)$ satisfy (2.1) on $[t_0, \infty)$. Since $x'(t)$ is positive and increasing, we see that $x'(t) \geq x'(t_0)$, $t \geq t_0$, which by integration gives $x(t) \to \infty$, $t \to \infty$.

Suppose that $\omega_3 > 0$. Then, since $(p(t)\, x''(t)^\alpha)' \sim \omega_3,\ t \to \infty$, integrating this relation on $[t_0, t]$, we obtain

$$x''(t) \sim \omega_3^{\frac{1}{\alpha}} \left(\frac{t}{p(t)}\right)^{\frac{1}{\alpha}}, \quad t \to \infty,$$

from which, integrating twice on $[t_0, t]$ we find that

$$x(t) \sim \omega_3^{\frac{1}{\alpha}} \int_{t_0}^{t} \int_{t_0}^{s} \left(\frac{r}{p(r)}\right)^{\frac{1}{\alpha}} drds, \quad t \to \infty,$$

i.e., $x(t) \sim \omega_3^{\frac{1}{\alpha}}\, \psi_4(t)$ as $t \to \infty$.

Suppose that $\omega_3 = 0$. Then, since $p(t)x''(t)^\alpha$ is positive and increasing, we have $\lim_{t\to\infty} p(t)x''(t)^\alpha = \omega_2 \in (0, \infty]$. If ω_2 is finite, then integrating the relation $x''(t) \sim (\omega_2/p(t))^{\frac{1}{\alpha}},\ t \to \infty$ twice on $[t_0, t]$, we obtain

$$x(t) \sim \omega_2^{\frac{1}{\alpha}} \int_{t_0}^{t} \int_{t_0}^{s} \frac{1}{p(r)^{\frac{1}{\alpha}}} drds, \quad t \to \infty,$$

i.e., $x(t) \sim \omega_2^{\frac{1}{\alpha}}\, \psi_3(t),\ t \to \infty$. On the other hand, if $\omega_2 = \infty$, we first integrate (E) on $[t, \infty)$ and then on $[t_0, t]$ to obtain

$$x''(t) = \frac{1}{p(t)^{\frac{1}{\alpha}}} \left(c_2 + \int_{t_0}^{t} \int_{s}^{\infty} q(r)x(r)^\beta\, drds\right)^{\frac{1}{\alpha}}, \quad t \geq t_0, \tag{2.3}$$

where $c_2 = p(t_0)x''(t_0)^\alpha > 0$. Integrating the above twice on $[t_0, t]$ then yields

$$\begin{aligned} x(t) &= c_0 + c_1(t - t_0) \\ &\quad + \int_{t_0}^{t} \int_{t_0}^{s} \frac{1}{p(r)^{\frac{1}{\alpha}}} \left(c_2 + \int_{t_0}^{r} \int_{u}^{\infty} q(v)x(v)^\beta\, dvdu\right)^{\frac{1}{\alpha}} drds, \quad t \geq t_0, \end{aligned} \tag{2.4}$$

where $c_1 = x'(t_0) > 0$ and $c_0 = x(t_0) > 0$. Since $\int_{t_0}^{t} \int_{s}^{\infty} q(r)x(r)^\beta drds = O(t)$ as $t \to \infty$, the condition (1.3) implies from (2.3) that $\lim_{t\to\infty} x'(t) = \infty$. Using L' Hospital's rule, we easily see from (2.4) that $\lim_{t\to\infty} x(t)/\psi_3(t) = \infty$ and $\lim_{t\to\infty} x(t)/\psi_4(t) = 0$, or equivalently $\psi_3(t) \prec x(t) \prec \psi_4(t)$ as $t \to \infty$.

It follows from above observation that there are three types of possible asymptotic behavior for positive solutions $x(t)$ of (E) satisfying (2.1)

$$x(t) \sim k_1\psi_3(t), \quad \text{or} \quad \psi_3(t) \prec x(t) \prec \psi_4(t), \quad \text{or} \quad x(t) \sim k_2\psi_4(t), \quad \text{as} \quad t \to \infty,$$

where k_1 and k_2 are some positive constants.

Solutions satisfying (2.2). Let $x(t)$ satisfy (2.2) on $[t_0, \infty)$. It is necessary that $\omega_3 = 0$, so that we have

$$-\Big(p(t)(-x''(t))^{\alpha}\Big)' = \int_t^{\infty} q(s)x(s)^{\beta} ds, \quad t \geq t_0. \tag{2.5}$$

Moreover, since $p(t)(-x''(t))^{\alpha}$ and $x'(t)$ are positive and decreasing , there exist finite limits $\lim_{t\to\infty} p(t)(-x''(t))^{\alpha} = \omega_2 \geq 0$ and $\lim_{t\to\infty} x'(t) = \omega_1 \geq 0$. In fact, it must be $\omega_2 = 0$, because otherwise, integration of the relation $x''(t) \sim (-\omega_2/p(t))^{\frac{1}{\alpha}}$, $t \to \infty$ leads to $x'(t) \sim -\omega_2^{\frac{1}{\alpha}} \int_{t_0}^{t} ds/p(s)^{\frac{1}{\alpha}}$, $t \to \infty$. Thus, we conclude with the help of (1.1) that $\lim_{t\to\infty} x'(t) = -\infty$, an impossibility. Using this fact and integrating (2.5) twice on $[t, \infty)$, we obtain

$$x'(t) = \omega_1 + \int_t^{\infty} \left(\frac{1}{p(s)} \int_s^{\infty} (r-s)q(r)x(r)^{\beta}\, dr\right)^{\frac{1}{\alpha}} ds, \quad t \geq t_0, \tag{2.6}$$

which, integrated on $[t_0, t]$, gives

$$\begin{aligned} x(t) &= c_0 + \omega_1(t - t_0) \\ &\quad + \int_{t_0}^{t} \int_s^{\infty} \left(\frac{1}{p(r)} \int_r^{\infty} (u-r)q(u)x(u)^{\beta}\, du\right)^{\frac{1}{\alpha}} dr ds, \quad t \geq t_0, \end{aligned} \tag{2.7}$$

where $c_0 = x(t_0) > 0$. It follows that if $\omega_1 > 0$, then $x(t) \sim \omega_1\, \psi_2(t)$, $t \to \infty$ and that if $\omega_1 = 0$, there are two possibilities: either $x(t)$ tends to a finite limit or $x(t)$ grows to infinity as $t \to \infty$. In the latter case it is clear that $\psi_1(t) \prec x(t) \prec \psi_2(t)$ as $t \to \infty$.

Thus it follows that the asymptotic behavior of positive solutions $x(t)$ of (E) satisfying (2.2) falls into one of the following three cases:

$$x(t) \sim k_1\psi_1(t), \quad \text{or} \quad \psi_1(t) \prec x(t) \prec \psi_2(t), \quad \text{or} \quad x(t) \sim k_2\psi_2(t), \quad \text{as} \quad t \to \infty,$$

where k_1 and k_2 are some positive constants.

As regards the primitive solutions of equation (E), the existence of four types of such solutions has been completely characterized for both sublinear and superlinear case of (E) with continuous coefficients $p(t)$ and $q(t)$ as the following theorems proven in [5] and [18] show.

Theorem 2.1. *Let* $p(t), q(t) \in C[a,\infty)$. *Equation (E) has a positive solution* $x(t)$ *satisfying* $x(t) \sim k_1\psi_1(t),\ t \to \infty$ *if and only if*

$$\int_a^\infty t\left(\frac{1}{p(t)}\int_t^\infty (s-t)\,q(s)\,ds\right)^{\frac{1}{\alpha}} dt < \infty. \tag{2.8}$$

Theorem 2.2. *Let* $p(t), q(t) \in C[a,\infty)$. *Equation (E) has a positive solution* $x(t)$ *satisfying* $x(t) \sim k_2\psi_2(t),\ t \to \infty$ *if and only if*

$$\int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty (s-t)\,s^\beta\,q(s)\,ds\right)^{\frac{1}{\alpha}} dt < \infty. \tag{2.9}$$

Theorem 2.3. *Let* $p(t), q(t) \in C[a,\infty)$. *Equation (E) has a positive solution* $x(t)$ *satisfying* $x(t) \sim k_3\psi_3(t),\ t \to \infty$ *if and only if*

$$\int_a^\infty t\,q(t)\,\psi_3(t)^\beta\,dt < \infty. \tag{2.10}$$

Theorem 2.4. *Let* $p(t), q(t) \in C[a,\infty)$. *Equation (E) has a positive solution* $x(t)$ *satisfying* $x(t) \sim k_4\psi_4(t),\ t \to \infty$ *if and only if*

$$\int_a^\infty q(t)\,\psi_4(t)^\beta\,dt < \infty. \tag{2.11}$$

Thus we are led to the study of intermediate solutions of equation (E). Our task is to solve two problems: **(i)** establish necessary and sufficient conditions for (E) to possess intermediate solutions of types (I) and (II); **(ii)** determine their asymptotic behavior (or order of growth) at infinity precisely. Since the second problem is very difficult for equation (E) with general continuous coefficients $p(t)$ and $q(t)$, we will make an attempt to solve the problem in the framework of regular variation (in the sense of Karamata), that is, we limit ourselves to the case where $p(t)$ and $q(t)$ are regularly varying functions and focus our attention on regularly varying solutions of (E). The recent development of asymptotic analysis of differential equations by means of regularly varying functions, which was initiated by the monograph of Marić [6], has shown that there exists a variety of nonlinear differential equations for which the problem mentioned above can be solved completely. The reader is referred to the papers [4, 7, 8, 11, 13, 14] for second order equation and to [9, 10, 12] for fourth order equations which are the special cases of (E) with $\alpha = 1$ or $p(t) \equiv 1$. The present work can be considered as a continuation of the previous papers [9, 10, 11], but has features different from them in the sense that the *generalized regularly varying functions* (or generalized Karamata functions) introduced in [3] are crucially utilized in order to make clear the dependence of intermediate solutions on the coefficient $p(t)$. See also [15, 16] for related results regarding fourth order equations.

3. Basic Properties of Regularly Varying Functions

We recall that the set of regularly varying functions of index $\rho \in \mathbb{R}$ is introduced by the following definition.

Definition 3.1. A measurable function $f : (a, \infty) \to (0, \infty)$ for some $a > 0$ is said to be regularly varying at infinity of index $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all } \lambda > 0.$$

The totality of all regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. In the special case when $\rho = 0$, we use the notation SV instead of $\mathrm{RV}(0)$ and refer to members of SV as *slowly varying functions*. Any function $f(t) \in \mathrm{RV}(\rho)$ is written as $f(t) = t^{\rho} g(t)$ with $g(t) \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. If

$$\lim_{t\to\infty} \frac{f(t)}{t^{\rho}} = \lim_{t\to\infty} g(t) = \mathrm{const} > 0,$$

then $f(t)$ is said to be a *trivial* regularly varying function of index ρ and it is denoted by $f(t) \in \mathrm{tr} - \mathrm{RV}(\rho)$. Otherwise, $f(t)$ is said to be a *nontrivial* regularly varying function of index ρ and it is denoted by $f(t) \in \mathrm{ntr} - \mathrm{RV}(\rho)$.

The reader is referred to N.H. Bingham et al. [1] and E. Seneta [17] for the most complete exposition of theory of regular variation and its application to various branches of mathematical analysis.

Since the class of classical Karamata functions is not sufficient to properly describe the asymptotic behavior of positive solutions of the self-adjoint differential equation

$$(p(t)x'(t))' + q(t)x(t) = 0\,,$$

Jaroš and Kusano introduced in [3] the class of generalized Karamata functions with the following definition.

Let $R(t)$ be a positive function which is continuously differentiable on (a, ∞) and satisfies $R'(t) > 0,\ t > a$ and $\lim_{t\to\infty} R(t) = \infty$.

Definition 3.2. A measurable function $f : (a, \infty) \to (0, \infty)$ for some $a > 0$ is said to be *regularly varying of index $\rho \in \mathbb{R}$ with respect to $R(t)$ if $f \circ R^{-1}$ is defined for all large t and is regularly varying function of index ρ in the sense of Karamata, where R^{-1} denotes the inverse function of R.*

The symbol $\mathrm{RV}_R(\rho)$ is used to denote the totality of regularly varying functions of index $\rho \in \mathbb{R}$ with respect to $R(t)$. The symbol SV_R is often used for $\mathrm{RV}_R(0)$. It is easy to see that if $f(t) \in \mathrm{RV}_R(\rho)$, then $f(t) = R(t)^{\rho} g(t),\ g(t) \in \mathrm{SV}_R$. If

$$\lim_{t\to\infty} \frac{f(t)}{R(t)^{\rho}} = \lim_{t\to\infty} g(t) = \mathrm{const} > 0,$$

then $f(t)$ is said to be a *trivial* regularly varying function of index ρ with respect to $R(t)$ and it is denoted by $f(t) \in \mathrm{tr}-\mathrm{RV}_R(\rho)$. Otherwise, $f(t)$ is said to be a *nontrivial* regularly varying function of index ρ with respect to $R(t)$ and it is denoted by $f(t) \in \mathrm{ntr}-\mathrm{RV}_R(\rho)$. Also, from Definition 3.2 it follows that $f \in \mathrm{RV}_R(\rho)$ if and only if it is written in the form $f(t) = g(R(t)),\ g(t) \in \mathrm{RV}(\rho)$. It is clear that $\mathrm{RV}(\rho) = \mathrm{RV}_t(\rho)$. We emphasize that there exists a function which is regularly varying in generalized sense, but is not regularly varying in the sense of Karamata, so that, roughly speaking, the class of generalized Karamata functions is larger than that of classical Karamata functions.

To help the reader we present here some elementary properties of generalized regularly varying functions.

Proposition 3.1. (i) *If* $g_1(t) \in \mathrm{RV}_R(\sigma_1)$, *then* $(g_1(t))^\alpha \in \mathrm{RV}_R(\alpha\sigma_1)$ *for any* $\alpha \in \mathbb{R}$.

(ii) *If* $g_i(t) \in \mathrm{RV}_R(\sigma_i),\ i = 1, 2,$ *then* $g_1(t) + g_2(t) \in \mathrm{RV}_R(\sigma),\ \sigma = \max(\sigma_1, \sigma_2)$.

(iii) *If* $g_i(t) \in \mathrm{RV}_R(\sigma_i),\ i = 1, 2,$ *then* $g_1(t)g_2(t) \in \mathrm{RV}_R(\sigma_1 + \sigma_2)$.

(iv) *If* $g_i(t) \in \mathrm{RV}_R(\sigma_i),\ i = 1, 2$ *and* $g_2(t) \to \infty$ *as* $t \to \infty$, *then* $g_1(g_2(t)) \in \mathrm{RV}_R(\sigma_1\sigma_2)$.

(v) *If* $l(t) \in \mathrm{SV}_R$, *then for any* $\varepsilon > 0$,

$$\lim_{t\to\infty} R(t)^\varepsilon l(t) = \infty, \qquad \lim_{t\to\infty} R(t)^{-\varepsilon} l(t) = 0.$$

Next, we present a fundamental result (see [3]), called *Generalized Karamata integration theorem*, which will be used throughout the paper and play a central role in establishing our main results.

Proposition 3.2. (Generalized Karamata integration theorem). *Let* $f(t) \in \mathrm{SV}_R$. *Then,*

(i) *If* $\alpha > -1$,

$$\int_a^t R'(s)R(s)^\alpha f(s)\, ds \sim \frac{R(t)^{\alpha+1} f(t)}{\alpha+1}, \quad t \to \infty;$$

(ii) *If* $\alpha < -1$,

$$\int_t^\infty R'(s)\, R(s)^\alpha\, f(s)\, ds \sim -\frac{R(t)^{\alpha+1} f(t)}{\alpha+1}, \quad t \to \infty;$$

(iii) *If* $\alpha = -1$,

$$\int_a^t R'(s)\, R(s)^{-1} f(s)\, ds \in \mathrm{SV}_R \quad \textit{and} \quad \int_t^\infty R'(s)\, R(s)^{-1} f(s)\, ds \in \mathrm{SV}_R.$$

4. Existence of Positive Intermediate Solutions

In this section we prove the existence of solutions of type (I) and (II) of equation (E) under assumption that coefficients $p(t)$ and $q(t)$ are positive continuous functions.

Theorem 4.1. *Let* $p(t),\ q(t) \in C[a,\infty)$. *If* (2.9) *holds and if*

$$\int_a^\infty t \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt = \infty, \tag{4.1}$$

then equation (E) has a positive solution $x(t)$ *such that* $1 \prec x(t) \prec t,\ t \to \infty$.

Proof. Choose $t_0 \geq \max\{1, a\}$ such that

$$2^{\frac{\beta}{\alpha}} \int_{t_0}^\infty \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, s^\beta q(s)\, ds \right)^{\frac{1}{\alpha}} dt \leq 1. \tag{4.2}$$

Define the set

$$\mathcal{X}_1 = \{x \in C[t_0,\infty) : 1 \leq x(t) \leq 2t,\ t \geq t_0\}, \tag{4.3}$$

and the operator $\mathcal{G} : \mathcal{X}_1 \to C[t_0,\infty)$

$$\mathcal{G}x(t) := 1 + \int_{t_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r)\, q(u)\, x(u)^\beta\, du \right)^{\frac{1}{\alpha}} dr ds, \quad t \geq t_0. \tag{4.4}$$

It is clear that $\mathcal{X}_1$ is a closed convex subset of the locally convex space $C[t_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$. Using (4.2)- (4.4), we see that $x \in \mathcal{X}_1$ implies

$$\begin{aligned} 1 \leq \mathcal{G}x(t) &\leq 1 + 2^{\frac{\beta}{\alpha}} \int_{t_0}^t \int_{t_0}^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r)\, q(u)\, u^\beta\, du \right)^{\frac{1}{\alpha}} dr\, ds \\ &\leq 1 + t \leq 2t, \quad t \geq t_0. \end{aligned}$$

This means that $\mathcal{G}$ maps $\mathcal{X}_1$ into itself. Furthermore, it can be shown that $\mathcal{G}$ is a continuous map such that $\mathcal{G}(\mathcal{X}_1)$ is relatively compact in $C[t_0,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a function $x_1 \in \mathcal{X}_1$ satisfying the integral equation $x_1(t) = \mathcal{G}x_1(t)$ for $t \geq t_0$. It follows that $x_1(t)$ is a solution of (E) on $[t_0, \infty)$. It is easy to see that $x_1(t)$ has the following asymptotic properties:

$$\lim_{t\to\infty} x_1(t) \geq \lim_{t\to\infty} \int_{t_0}^{t}\int_{s}^{\infty} \left(\frac{1}{p(r)} \int_{r}^{\infty} (u-r)\, q(u)\, du\right)^{\frac{1}{\alpha}} dr\, ds = \infty$$

and

$$0 \leq \lim_{t\to\infty} \frac{x_1(t)}{t} = \lim_{t\to\infty} \int_{t}^{\infty} \left(\frac{1}{p(s)} \int_{s}^{\infty} (r-s)\, q(r)\, x_1(r)^{\beta}\, dr\right)^{\frac{1}{\alpha}} ds$$
$$\leq 2^{\frac{\beta}{\alpha}} \lim_{t\to\infty} \int_{t}^{\infty} \left(\frac{1}{p(s)} \int_{s}^{\infty} (r-s) q(r)\, r^{\beta}\, dr\right)^{\frac{1}{\alpha}} ds = 0,$$

which means that $x_1(t)$ satisfies $1 \prec x_1(t) \prec t,\ t \to \infty$, that is, $x_1(t)$ is an intermediate solution of type (I) of (E). ■

Theorem 4.2. *Let* $p(t),\ q(t) \in C[a,\infty)$. *If* (2.11) *holds and if*

$$\int_{a}^{\infty} t\, q(t)\, \psi_3(t)^{\beta}\, dt = \infty, \tag{4.5}$$

then equation (E) has a positive solution $x(t)$ *such that* $\psi_3(t) \prec x(t) \prec \psi_4(t)$, $t \to \infty$.

Proof. Choose $t_0 \geq \max\{1, a\}$ such that

$$2^{\frac{\beta}{\alpha}} \int_{t_0}^{\infty} q(t)\, \psi_4(t)^{\beta}\, dt \leq 1. \tag{4.6}$$

Define the set

$$\mathcal{X}_2 = \{x \in C[t_0, \infty) : \psi_3(t) \leq x(t) \leq 2^{\frac{1}{\alpha}} \psi_4(t),\ t \geq t_0\}, \tag{4.7}$$

and the integral operator $\mathcal{H} : \mathcal{X}_2 \to C[t_0, \infty)$

$$\mathcal{H}x(t) := \int_{t_0}^{t} (t-s) \left[\frac{1}{p(s)} \left(1 + \int_{t_0}^{s}\int_{r}^{\infty} q(u)\, x(u)^{\beta}\, du\, dr\right)\right]^{\frac{1}{\alpha}} ds, \quad t \geq t_0. \tag{4.8}$$

It is clear that $\mathcal{X}_2$ is a closed convex subset of the locally convex space $C[t_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$. Using (4.6)-(4.8), we see that $x \in \mathcal{X}_2$ implies

$$\psi_3(t) \leq \mathcal{H}x(t) \leq \int_{t_0}^{t} (t-s) \left[\frac{1}{p(s)} \left(1 + 2^{\frac{\beta}{\alpha}} \int_{t_0}^{s}\int_{t_0}^{\infty} q(u)\, \psi_4(u)^{\beta}\, du\, dr\right)\right]^{\frac{1}{\alpha}} ds$$
$$\leq \int_{t_0}^{t} (t-s) \left(\frac{1+s}{p(s)}\right)^{\frac{1}{\alpha}} ds \leq 2^{\frac{1}{\alpha}} \psi_4(t), \quad t \geq t_0.$$

This means that $\mathcal{H}$ maps $\mathcal{X}_2$ into itself. Furthermore, it can be shown that $\mathcal{H}$ is a continuous map such that $\mathcal{H}(\mathcal{X}_2)$ is relatively compact in $C[t_0,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a function $x_2 \in \mathcal{X}_2$ satisfying the integral equation $x_2(t) = \mathcal{H}x_2(t)$ for $t \geq t_0$. It follows that $x_2(t)$ is a solution of (E) on $[t_0,\infty)$. It is easy to see that $x_2(t)$ has the following asymptotic properties:

$$\begin{aligned}\lim_{t\to\infty}\frac{x_2(t)}{\psi_3(t)} &= \lim_{t\to\infty}\left(1+\int_{t_0}^{t}\int_{s}^{\infty} q(r)\,x_2(r)^{\beta}\,dr\,ds\right)^{\frac{1}{\alpha}} \\ &\geq \lim_{t\to\infty}\left(\int_{t_0}^{t}\int_{s}^{\infty} q(r)\,\psi_3(r)^{\beta}\,dr\,ds\right)^{\frac{1}{\alpha}} = \infty\end{aligned}$$

and

$$\begin{aligned}0 \leq \lim_{t\to\infty}\frac{x_2(t)}{\psi_4(t)} &= \left(\lim_{t\to\infty}\frac{1+\int_{t_0}^{t}\int_{s}^{\infty} q(r)\,x_2(r)^{\beta}\,dr\,ds}{t}\right)^{\frac{1}{\alpha}} \\ &= \left(\lim_{t\to\infty}\int_{t}^{\infty} q(s)\,x_2(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}} \leq \left(2^{\frac{\beta}{\alpha}}\lim_{t\to\infty}\int_{t}^{\infty} q(s)\,\psi_4(s)^{\beta}\,ds\right)^{\frac{1}{\alpha}} = 0,\end{aligned}$$

which means that $x_2(t)$ satisfies $\psi_3(t) \prec x_2(t) \prec \psi_4(t),\ t\to\infty$, that is, $x_2(t)$ is an intermediate solution of type (II) of (E). ■

5. Asymptotic Behavior of Intermediate Regularly Varying Solutions

In what follows it is always assumed that functions $p(t)$ and $q(t)$ are generalized regularly varying of index η and σ with respect to $R(t)$, which is defined with

$$R(t) = \int_a^t \left(\frac{s}{p(s)}\right)^{\frac{1}{\alpha}} ds, \tag{5.1}$$

and expressed with

$$p(t) = R(t)^{\eta}\,l_p(t),\ l_p(t)\in \mathrm{SV}_R \text{ and } q(t) = R(t)^{\sigma}\,l_q(t),\ l_q(t)\in\mathrm{SV}_R, \tag{5.2}$$

and the intermediate solutions $x(t) \in \mathrm{RV}_R(\rho)$ of (E) are represented as

$$x(t) = R(t)^{\rho}\,l_x(t),\ l_x(t) \in \mathrm{SV}_R. \tag{5.3}$$

From (5.1) and (5.2) we have that

$$t^{\frac{1}{\alpha}} = R'(t)R(t)^{\frac{\eta}{\alpha}}\,l_p(t)^{\frac{1}{\alpha}}. \tag{5.4}$$

Integrating (5.4) from a to t and using the generalized Karamata integration theorem (Proposition 3.2) we have

$$\frac{t^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha}+1} \sim \frac{R(t)^{\frac{\eta}{\alpha}+1}}{\frac{\eta}{\alpha}+1}\, l_p(t)^{\frac{1}{\alpha}}, \quad t \to \infty,$$

implying

$$t \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{-\frac{\alpha}{\alpha+1}} R(t)^{\frac{\alpha+\eta}{\alpha+1}}\, l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty. \tag{5.5}$$

From above relations we get

$$R'(t) \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{-\frac{1}{\alpha+1}} R(t)^{\frac{1-\eta}{\alpha+1}}\, l_p(t)^{-\frac{1}{\alpha+1}}, \quad t \to \infty. \tag{5.6}$$

We can rewrite (5.6) in the form

$$1 \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{\frac{1}{\alpha+1}} R'(t)\, R(t)^{\frac{\eta-1}{\alpha+1}}\, l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty. \tag{5.7}$$

First, express the conditions (1.1) in the terms of regular variation. Using (5.2), (5.5) and (5.7) we have

$$\int_a^t \frac{ds}{p(s)^{\frac{1}{\alpha}}} \sim \left(\frac{\alpha+\eta}{\alpha+1}\right)^{\frac{1}{\alpha+1}} \int_a^t R'(s)\, R(s)^{-\frac{\alpha+\eta}{\alpha(\alpha+1)}}\, l_p(s)^{-\frac{1}{\alpha(\alpha+1)}}\, ds, \quad t \to \infty.$$

For conditions (1.1) to hold it is necessary that $\alpha^2 - \eta \geq 0$. In what follows we limit ourselves to the case where

$$\alpha^2 - \eta > 0 \tag{5.8}$$

excluding the other possibilities because of computational difficulty. We introduce the notation:

$$m_1(\alpha,\eta) = \frac{\alpha+\eta}{\alpha+1}, \quad m_2(\alpha,\eta) = \frac{2\alpha^2+\alpha\eta-\eta}{\alpha(\alpha+1)}, \quad m_3(\alpha,\eta) = \frac{2\alpha+\eta+1}{\alpha+1}. \tag{5.9}$$

It is clear that $0 < m_1(\alpha,\eta) < m_2(\alpha,\eta) < m_3(\alpha,\eta) = m_1(\alpha,\eta)+1$. In proofs of our main results constants $m_i(\alpha,\eta),\ i=1,2,3$ will be abbreviated to m_i.

Now, we state a lemma which will be frequently used in our later discussions. The proof of this lemma follows directly using (5.7) and the generalized Karamata integration theorem.

Lemma 5.1. *Let* $f(t) = R(t)^{\mu} L_f(t),\ L_f(t) \in \mathrm{SV}_R$. *Then,*

(i) *If* $\mu + m_1(\alpha,\eta) > 0$,

$$\int_a^t f(s)\,ds \sim \frac{m_1(\alpha,\eta)^{\frac{1}{\alpha+1}}}{\mu + m_1(\alpha,\eta)}\, R(t)^{\mu+m_1(\alpha,\eta)}\, L_f(t)\, l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty;$$

(ii) *If* $\mu + m_1(\alpha,\eta) < 0$,

$$\int_t^\infty f(s)\,ds \sim \frac{m_1(\alpha,\eta)^{\frac{1}{\alpha+1}}}{-(\mu + m_1(\alpha,\eta))}\, R(t)^{\mu+m_1(\alpha,\eta)}\, L_f(t)\, l_p(t)^{\frac{1}{\alpha+1}}, \quad t \to \infty;$$

(iii) *If* $\mu + m_1(\alpha,\eta) = 0$, *then*

$$\int_a^t f(s)\ ds \sim m_1(\alpha,\eta)^{\frac{1}{\alpha+1}} \int_a^t R'(s)R(s)^{-1}\, L_f(s) l_p(s)^{\frac{1}{\alpha+1}}\, ds \in SV_R$$

and

$$\int_t^\infty f(s)\ ds \sim m_1(\alpha,\eta)^{\frac{1}{\alpha+1}} \int_t^\infty R'(s)R(s)^{-1}\, L_f(s) l_p(s)^{\frac{1}{\alpha+1}}\, ds \in SV_R.$$

In order to make an in depth analysis of intermediate solutions of type (I) and (II) of (E) we need a fair knowledge of the structure of the functions $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ and $\psi_4(t)$ regarded as generalized regularly varying functions. It is clear that $\psi_1(t) \in \mathrm{SV}_R$. From (5.5) it follows that $\psi_2(t) \in \mathrm{RV}_R\,(m_1(\alpha,\eta))$. Using (5.2) and applying Lemma 5.1 twice, we get

$$\begin{aligned} \psi_3(t) &\sim \int_a^t\!\!\int_a^s R(r)^{-\frac{\eta}{\alpha}}\, l_p(r)^{-\frac{1}{\alpha}}\, dr ds \\ &\sim \frac{m_1(\alpha,\eta)^{\frac{2}{\alpha+1}}}{m_2(\alpha,\eta)(m_2(\alpha,\eta) - m_1(\alpha,\eta))}\, R(t)^{m_2(\alpha,\eta)}\, l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}},\ t \to \infty, \end{aligned} \tag{5.10}$$

which shows that $\psi_3(t) \in \mathrm{RV}_R\,(m_2(\alpha,\eta))$. Further, another application of Lemma 5.1 yields

$$\psi_4(t) \sim \int_a^t R(s)\,ds \sim \frac{m_1(\alpha,\eta)^{\frac{1}{\alpha+1}}}{m_3(\alpha,\eta)}\, R(t)^{m_3(\alpha,\eta)}\, l_p(t)^{\frac{1}{\alpha+1}},\ t \to \infty, \tag{5.11}$$

implying $\psi_4(t) \in \mathrm{RV}_R\,(m_3(\alpha,\eta))$.

5.1. Regularly varying intermediate solutions of type (I)

The first subsection is devoted to the study of the existence and asymptotic behavior of generalized regularly varying solutions of type (I) of equation (E) with $p(t)$ and $q(t)$

satisfying (5.2). Since $\psi_1(t) \prec x(t) \prec \psi_2(t),\ t \to \infty$, the regularity index ρ of $x(t)$ must satisfy

$$0 \le \rho \le m_1(\alpha,\eta).$$

If $\rho = 0$, then since $x(t) = l_x(t) \to \infty,\ t \to \infty$, $x(t)$ is a member of $\mathrm{ntr} - \mathrm{SV}_R$, while if $\rho = m_1(\alpha,\eta)$, then since $x(t)/R(t)^{m_1(\alpha,\eta)} = l_x(t) \to 0,\ t \to \infty$, $x(t)$ is a member of $\mathrm{ntr} - \mathrm{RV}_R(m_1(\alpha,\eta))$. If $0 < \rho < m_1(\alpha,\eta)$, then $x(t)$ is a member of $\mathrm{RV}_R(\rho)$ and satisfies $x(t) \to \infty$ and $x(t)/R(t)^{m_1(\alpha,\eta)} \to 0$ as $t \to \infty$. Thus the set of all generalized regularly varying solutions of type (I) is naturally divided into the three disjoint classes

$$\mathrm{ntr} - \mathrm{SV}_R \ \text{ or } \ \mathrm{RV}_R(\rho) \ \text{ with } \ \rho \in (0\ ,\ m_1(\alpha,\eta)) \,\text{or}\, \mathrm{ntr} - \mathrm{RV}_R\,(m_1(\alpha,\eta))\,. \tag{5.12}$$

Our aim is to establish necessary and sufficient conditions for each of the above classes to have a member and furthermore to show that the asymptotic behavior of all members of each class is governed by a unique explicit formula describing the growth order at infinity accurately.

5.1.1. Main results

Theorem 5.1. *Let* $p(t) \in \mathrm{RV}_R(\eta),\ q(t) \in \mathrm{RV}_R(\sigma)$. *Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{SV}_R$ *satisfying* (I) *if and only if*

$$\sigma = -2\alpha - \eta \quad \textit{and} \quad \int_a^\infty t \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt = \infty. \tag{5.13}$$

The asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim X_1(t),\ t \to \infty$, *where*

$$X_1(t) = \left(\frac{\alpha-\beta}{\alpha} \int_a^t s \left(\frac{1}{p(s)} \int_s^\infty (r-s)\, q(r)\, dr \right)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha-\beta}}. \tag{5.14}$$

Theorem 5.2. *Let* $p(t) \in \mathrm{RV}_R(\eta),\ q(t) \in \mathrm{RV}_R(\sigma)$. *Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{RV}_R(\rho)$ *with* $\rho \in (0\ ,\ m_1(\alpha,\eta))$ *if and only if*

$$-2\alpha - \eta < \sigma < -\alpha - (\beta+1) m_1(\alpha,\eta), \tag{5.15}$$

in which case ρ *is given by*

$$\rho = \frac{\sigma + 2\alpha + \eta}{\alpha - \beta} \tag{5.16}$$

and the asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim X_2(t),\ t \to \infty$, *where*

$$X_2(t) = \left(\left(\frac{m_1(\alpha,\eta)}{\alpha} \right)^2 \frac{p(t) q(t) R(t)^{2\alpha}}{\rho^\alpha\,(m_1(\alpha,\eta)-\rho)^\alpha\,(m_2(\alpha,\eta)-\rho)\,(m_3(\alpha,\eta)-\rho)} \right)^{\frac{1}{\alpha-\beta}}. \tag{5.17}$$

Theorem 5.3. *Let* $p(t) \in \mathrm{RV}_R(\eta),\ q(t) \in \mathrm{RV}_R(\sigma)$. *Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{RV}_R\,(m_1(\alpha, \eta))$ *satisfying* (I) *if and only if*

$$\sigma = -\alpha-(\beta+1)m_1(\alpha,\eta) \text{ and } \int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, s^\beta\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt < \infty. \tag{5.18}$$

The asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim X_3(t),\ t \to \infty$, *where*

$$X_3(t) = t\,\left(\frac{\alpha-\beta}{\alpha} \int_t^\infty \left(\frac{1}{p(s)} \int_s^\infty (r-s) r^\beta\, q(r)\, dr\right)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}}. \tag{5.19}$$

5.1.2. Preparatory results

Let $x(t)$ be a solution of (E) on $[t_0, \infty)$ such that $1 \prec x(t) \prec t$ as $t \to \infty$. Since

$$\begin{aligned} &\lim_{t\to\infty} (p(t)|x''(t)|^{\alpha-1}x''(t))' = \lim_{t\to\infty} p(t)|x''(t)|^{\alpha-1}x''(t) = \lim_{t\to\infty} x'(t) = 0, \\ &\lim_{t\to\infty} x(t) = \infty, \end{aligned} \tag{5.20}$$

integrating of equation (E) first three times on $[t, \infty)$ and then once on $[t_0, t]$ gives

$$x(t) = x(t_0) + \int_{t_0}^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r) q(u) x(u)^\beta\, du\right)^{\frac{1}{\alpha}} dr ds,\ t \ge t_0, \tag{5.21}$$

and implies the integral asymptotic relation

$$x(t) \sim \int_b^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r) q(u) x(u)^\beta\, du\right)^{\frac{1}{\alpha}} dr\, ds, \quad t \to \infty, \tag{5.22}$$

for any $b \ge a$. Conversely, if $x(t)$ is a positive continuous function satisfying (5.21) and $\lim_{t\to\infty} x(t) = \infty$, then it is a solution of (E) such that $1 \prec x(t) \prec t,\ t \to \infty$. Our main tools in establishing precise asymptotic forms of intermediate positive solutions will be Schauder-Tychonoff fixed point theorem combined with theory of regular variation. To that end, the closed convex subsets $\mathcal{X}$ of $C[t_0, \infty)$ which should be chosen in such a way that appropriate integral operator $\mathcal{F}$ is a continuous self-map on $\mathcal{X}$ and send it into a relatively compact subset of $C[t_0, \infty)$, will be now found by means of regularly varying functions satisfying the integral asymptotic relation (5.22). Therefore, first we show that regularly varying functions $X_i(t)$, $i = 1, 2, 3$ defined, respectively by (5.14), (5.17),(5.19), satisfy integral asymptotic relation (5.22).

Lemma 5.2. *Suppose that* (5.13) *holds. Function* $X_1(t) \in \mathrm{ntr} - \mathrm{SV}_R$ *given by* (5.14) *satisfies the asymptotic relation* (5.22) *for any* $b \ge a$.

Proof. Let (5.13) holds. First note that $\sigma = -2\alpha - \eta$ satisfies $\sigma + m_1 = -\alpha m_3$ and $\sigma + 2m_1 = -\alpha m_2$. We integrate $q(t) = R(t)^\sigma l_q(t)$ twice on $[t, \infty)$. Applying Lemma 5.1 twice and using (5.2) we obtain

$$\int_t^\infty (s-t)q(s)\,ds \sim \frac{m_1^{\frac{2}{\alpha+1}}}{\alpha^2 m_2 m_3} R(t)^{\sigma+2m_1} l_p(t)^{\frac{2}{\alpha+1}} l_q(t), \quad t \to \infty,$$

from which it readily follows, using (5.5), that

$$t\left(\frac{1}{p(t)}\int_t^\infty (s-t)q(s)\,ds\right)^{\frac{1}{\alpha}} \sim \left(\frac{m_1^{2-\alpha}}{\alpha^2 m_2 m_3}\right)^{\frac{1}{\alpha}} R'(t)R(t)^{-1} l_p(t)^{\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}}, \quad t \to \infty,$$

where (5.7) has been used in the last step. Integration of the last relation on $[a, t]$ then yields

$$\begin{aligned} &\int_a^t s\left(\frac{1}{p(s)}\int_s^\infty (r-s)\,q(r)\,dr\right)^{\frac{1}{\alpha}} ds \\ &\sim \left(\frac{m_1^{2-\alpha}}{\alpha^2 m_2 m_3}\right)^{\frac{1}{\alpha}} \int_a^t R'(s)R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha}}\, l_q(s)^{\frac{1}{\alpha}}\, ds,\ t \to \infty, \end{aligned} \tag{5.23}$$

so that

$$X_1(t) \sim \left(\frac{(\alpha-\beta)m_1^{\frac{2-\alpha}{\alpha}}}{\alpha^{1+\frac{2}{\alpha}}\,(m_2\, m_3)^{\frac{1}{\alpha}}}\int_a^t R'(s)\,R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha}}\, l_q(s)^{\frac{1}{\alpha}}\, ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty.$$

This shows that $X_1(t) \in \mathrm{SV}_R$. Next, we integrate $q(t)X_1(t)^\beta$ twice on $[t, \infty)$. Applying Lemma 5.1 as above, we see that

$$\left(\int_t^\infty (s-t)\,q(s)\,X_1(s)^\beta\,ds\right)^{\frac{1}{\alpha}} \sim \left(\frac{m_1^{\frac{2}{\alpha+1}}}{\alpha^2\, m_2 m_3}\right)^{\frac{1}{\alpha}} R(t)^{\frac{\sigma+2m_1}{\alpha}} l_p(t)^{\frac{2}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}} X_1(t)^{\frac{\beta}{\alpha}},$$

as $t \to \infty$. Integrating the above relation multiplied by $p(t)^{-\frac{1}{\alpha}}$ first on $[t, \infty)$ and then on $[b, t]$, for any $b \geq a$, we conclude via Lemma 5.1 that

$$\begin{aligned} &\int_b^t\int_s^\infty\left(\frac{1}{p(r)}\int_r^\infty (u-r)q(u)X_1(u)^\beta du\right)^{\frac{1}{\alpha}} dr ds \sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}}\left(\frac{m_1^{2-\alpha}}{\alpha^2 m_2 m_3}\right)^{\frac{1}{\alpha-\beta}} \\ &\times \int_b^t R'(s)\,R(s)^{-1} l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}}\left(\int_a^s R'(r)\,R(r)^{-1} l_p(r)^{\frac{1}{\alpha}} l_q(r)^{\frac{1}{\alpha}} dr\right)^{\frac{\beta}{\alpha-\beta}} ds \\ &= \left(\frac{(\alpha-\beta)m_1^{\frac{2-\alpha}{\alpha}}}{\alpha^{1+\frac{2}{\alpha}}\,(m_2 m_3)^{\frac{1}{\alpha}}}\int_a^t R'(s)R(s)^{-1} l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}} ds\right)^{\frac{\alpha}{\alpha-\beta}} = X_1(t), \quad t \to \infty. \end{aligned}$$

This proves that $X_1(t)$ satisfies the asymptotic relation (5.22) for any $b \geq a$. ■

Lemma 5.3. *Suppose that* (5.15) *holds and let* ρ *be defined by* (5.16)*. Function* $X_2(t) \in \mathrm{RV}(\rho)$ *given by* (5.17) *satisfies the asymptotic relation* (5.22) *for any* $b \geq a$.

Proof. Note that the function $X_2(t)$ given by (5.17) can be expressed in the form

$$X_2(t) \sim \lambda^{-\frac{1}{\alpha-\beta}} \left(\frac{m_1}{\alpha}\right)^{\frac{2}{\alpha-\beta}} R(t)^{\rho} \, (l_p(t)\, l_q(t))^{\frac{1}{\alpha-\beta}}, \; t \to \infty, \tag{5.24}$$

where

$$\lambda = \rho^{\alpha} \, (m_1 - \rho)^{\alpha} \, (m_2 - \rho) \, (m_3 - \rho) \, .$$

Using (5.24) and (5.16) and applying Lemma 5.1 twice, we find that

$$\int_t^{\infty} \int_s^{\infty} q(r)\, X_2(r)^{\beta}\, drds$$
$$\sim \frac{\lambda^{-\frac{\beta}{\alpha-\beta}} \left(\frac{m_1}{\alpha}\right)^{\frac{2\beta}{\alpha-\beta}} m_1^{\frac{2}{\alpha+1}}}{\alpha^2(m_2-\rho)(m_3-\rho)} R(t)^{\alpha(\rho-m_2)} \, (l_p(t) l_q(t))^{\frac{\beta}{\alpha-\beta}} \, l_q(t) l_p(t)^{\frac{2}{\alpha+1}},$$

as $t \to \infty$. We now multiply the last relation by $1/p(t)$, raise to the exponent $1/\alpha$ and integrate it first on $[t, \infty)$ and then on $[b, t]$ for any $b \geq a$. As a result of application of Lemma 5.1 twice, we obtain for $t \to \infty$

$$\int_b^t \int_s^{\infty} \left(\frac{1}{p(r)} \int_r^{\infty} (u-r)\, q(u)\, X_2(u)^{\beta}\, du\right)^{\frac{1}{\alpha}} drds$$
$$\sim \frac{\lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \left(\frac{m_1}{\alpha}\right)^{\frac{2\beta}{\alpha(\alpha-\beta)}} m_1^{\frac{2}{\alpha(\alpha+1)}} \, m_1^{\frac{2}{\alpha+1}}}{\rho(m_1-\rho)(\alpha^2(m_2-\rho)(m_3-\rho))^{\frac{1}{\alpha}}}$$
$$\times R(t)^{\rho} \, (l_p(t) l_q(t))^{\frac{\beta}{\alpha(\alpha-\beta)}} \, l_q(t)^{\frac{1}{\alpha}} \, l_p(t)^{\frac{2}{\alpha(\alpha+1)}} \, l_p(t)^{-\frac{1}{\alpha}} \, l_p(t)^{\frac{2}{\alpha+1}} = X_2(t).$$

This completes the proof of Lemma 5.3. ■

Lemma 5.4. *Suppose that* (5.18) *holds. Function* $X_3(t) \in \mathrm{ntr} - \mathrm{RV}_R\,(m_1(\alpha, \eta))$ *given by* (5.19) *satisfies the asymptotic relation* (5.22) *for any* $b \geq a$.

Proof. Let (5.18) holds. Using (5.2) and (5.5) and applying Lemma 5.1 we see that

$$\int_t^{\infty} s^{\beta} q(s) ds \sim m_1^{\frac{-\alpha\beta}{\alpha+1}} \int_t^{\infty} R(s)^{\sigma+\beta m_1} l_p(s)^{\frac{\beta}{\alpha+1}} l_q(s) ds$$
$$\sim \frac{m_1^{\frac{1-\alpha\beta}{\alpha+1}}}{-(\sigma+(\beta+1)m_1)} R(t)^{\sigma+(\beta+1)m_1} l_p(t)^{\frac{\beta+1}{\alpha+1}} l_q(t) = \frac{m_1^{\frac{1-\alpha\beta}{\alpha+1}}}{\alpha} R(t)^{-\alpha} l_p(t)^{\frac{\beta+1}{\alpha+1}} \, l_q(t),$$

as $t \to \infty$, from which it follows that

$$\left(\frac{1}{p(t)}\int_t^\infty\int_s^\infty r^\beta q(r)drds\right)^{\frac{1}{\alpha}} \sim \frac{m_1^{\frac{2-\alpha\beta}{\alpha(\alpha+1)}}}{(\alpha^2(m_2-m_1))^{\frac{1}{\alpha}}} R(t)^{m_1-m_2-\frac{\eta}{\alpha}} l_p(t)^{\frac{-\alpha+\beta+1}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}}$$

$$\sim \frac{m_1^{\frac{2-\alpha\beta+\alpha}{\alpha(\alpha+1)}}}{(\alpha^2(m_2-m_1))^{\frac{1}{\alpha}}} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}}\, l_q(t)^{\frac{1}{\alpha}},$$

as $t \to \infty$, where we use (5.7) in the last step. Integrating the above on $[t,\infty)$ we obtain

$$\int_t^\infty \left(\frac{1}{p(s)}\int_s^\infty\int_r^\infty u^\beta q(u)dudr\right)^{\frac{1}{\alpha}} ds$$

(5.25)

$$\sim \frac{m_1^{\frac{2-\alpha\beta+\alpha}{\alpha(\alpha+1)}}}{(\alpha^2(m_2-m_1))^{\frac{1}{\alpha}}} \int_t^\infty R'(s)R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} ds, \quad t\to\infty.$$

This, combined with (5.5) and (5.19), gives the following expression for $X_3(t)$:

$$X_3(t) \sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{m_1^{2-\alpha}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha-\beta}} R(t)^{m_1}\, l_p(t)^{\frac{1}{\alpha+1}}$$

$$\times \left(\int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, ds\right)^{\frac{\alpha}{\alpha-\beta}} \in \mathrm{RV}_R\,(m_1)\,,\ t\to\infty.$$

Next, we integrate $q(t)\, X_3(t)^\beta$ twice on $[t,\infty)$, multiply by $1/p(t)$ and raise the result to the exponent $1/\alpha$. Since $q(t)X_3(t)^\beta \in \mathrm{RV}_R(\sigma+m_1\beta) = \mathrm{RV}_R(-\alpha-m_1)$ (cf. (5.18)), repeated application of Lemma 5.1, with the help of (5.7), yields

$$\left(\frac{1}{p(t)}\int_t^\infty\int_s^\infty q(r)X_3(r)^\beta drds\right)^{\frac{1}{\alpha}} \sim \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \left(\frac{m_1^{\frac{2-\alpha\beta+\alpha}{\alpha+1}}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha-\beta}}$$

$$\times R'(t)R(t)^{-1} l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}} \left(\int_t^\infty R'(s)R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} ds\right)^{\frac{\beta}{\alpha-\beta}},$$

as $t \to \infty$. Integrating the above relation first on $[t,\infty)$ and then on $[b,t]$ for any fixed

$b \geq a$, we conclude via Lemma 5.1 that

$$\begin{aligned}
&\int_b^t \int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty (u-r) q(u) X_3(u)^\beta du \right)^{\frac{1}{\alpha}} dr ds \\
\sim &\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{m_1^{2-\alpha}}{\alpha^2(m_2-m_1)} \right)^{\frac{1}{\alpha-\beta}} \\
&\times R(t)^{m_1} l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^\infty R'(s)\, R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha-\beta}} = X_3(t),
\end{aligned}$$

as $t \to \infty$. This completes the proof of Lemma 5.4. ■

After the construction of intermediate solutions with the help of the Schauder-Tychonoff fixed point theorem, to finish the proof of the "if" part of our main results we prove the regularity of those solutions using the generalized L'Hospital rule (see [2]):

Lemma 5.5. *Let* $f, g \in C^1[T, \infty)$. *Let*

$$\lim_{t\to\infty} g(t) = \infty \ \ \textit{and} \ \ g'(t) > 0 \ \ \textit{for all large } t. \tag{5.26}$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \leq \liminf_{t\to\infty} \frac{f(t)}{g(t)} \leq \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leq \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

If we replace (5.26) *with condition*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \ \ \textit{and} \ \ g'(t) < 0 \ \ \textit{fro all large } t,$$

then the same conclusion holds.

5.1.3. Proofs of main results

PROOF OF THE "ONLY IF" PART OF THEOREMS 5.1, 5.2 AND 5.3: Suppose that (E) has a type-(I) intermediate solution $x(t) \in \mathrm{RV}_R(\rho)$ on $[t_0, \infty)$ with $\rho \in [0, m_1]$. For such solution we have (5.20). From

$$\begin{aligned}
&-\left(p(t)(-x''(t))^\alpha\right)' \\
&= \int_t^\infty q(s) x(s)^\beta\, ds \sim \int_t^\infty R(s)^{\sigma+\beta\rho} l_q(s) l_x(s)^\beta\, ds,\ t \to \infty,
\end{aligned} \tag{5.27}$$

the convergence of the last integral in (5.27) means that $\sigma + \beta\rho + m_1 \leq 0$. But the possibility $\sigma + \beta\rho + m_1 = 0$ is precluded, because if this were the case the last integral in (5.27) would be an SV_R- function, which is not integrable on $[t_0, \infty)$ by (i) of Lemma 5.1. This would contradict the fact that the left-hand side of (5.27) is integrable on

$[t_0,\infty)$. It follows that $\sigma+\beta\rho+m_1<0$. Then, integration of (5.27) on $[t,\infty)$ with application of Lemma 5.1 gives

$$\begin{aligned}&p(t)(-x''(t))^{\alpha}\\ &\sim \frac{m_1^{\frac{1}{\alpha+1}}}{-(\sigma+\beta\rho+m_1)}\int_t^{\infty} R(s)^{\sigma+\beta\rho+m_1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^{\beta}\, ds,\ t\to\infty.\end{aligned}\tag{5.28}$$

Noting that the integral in (5.28) is convergent, we conclude that $\sigma+\beta\rho+2m_1\le 0$. But the equality is not allowed here. In fact, if the equality holds, then the right- hand side of (5.28) is SV_R-function denoted by $h(t)$ so that

$$-x''(t)\sim\left(\frac{h(t)}{p(t)}\right)^{\frac{1}{\alpha}} = R(t)^{-\frac{\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha}} h(t)^{\frac{1}{\alpha}},\quad t\to\infty.$$

But then, the integrability of $x''(t)$ on $[t_0,\infty)$ implies that $m_1-\frac{\eta}{\alpha}=\frac{\alpha^2-\eta}{\alpha(\alpha+1)}\le 0$, which contradicts the assumption (5.8). Thus it holds $\sigma+\beta\rho+2m_1<0$. Applying Lemma 5.1 in (5.28) first and then multiplying by $1/p(t)$ and raising the result on $1/\alpha$, using (5.7) we obtain

$$\begin{aligned}&-x''(t)\sim\\ &\frac{m_1^{\frac{2}{\alpha(\alpha+1)}}}{((\sigma+\beta\rho+m_1)(\sigma+\beta\rho+2m_1))^{\frac{1}{\alpha}}} R(t)^{\frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}} l_p(t)^{\frac{1-\alpha}{\alpha(\alpha+1)}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}},\end{aligned}\tag{5.29}$$

as $t\to\infty$. The integrability of $x''(t)$ on $[t_0,\infty)$ implies that $\frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+m_1\le 0$. We distinguish the two cases:

$$\text{(a)}\quad \frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+m_1=0 \qquad \text{(b)}\quad \frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+m_1<0.$$

Assume that (a) holds. Since $\sigma+\beta\rho+m_1=-\alpha$ and $\sigma+\beta\rho+2m_1=\alpha(m_1-m_2)$, integration of (5.29) first on $[t,\infty)$, then on $[t_0,t]$, with application of Lemma 5.1, shows that

$$\begin{aligned}x(t)&\sim\left(\frac{m_1^{2-\alpha}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}} R(t)^{m_1} l_p(t)^{\frac{1}{\alpha+1}}\\ &\quad\times\int_t^{\infty} R'(s)\, R(s)^{-1} l_p(s)^{\frac{1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} ds\\ &\sim t\left(\frac{m_2^{\frac{\alpha+2}{\alpha+1}}}{\alpha^2(m_2-m_1)}\right)^{\frac{1}{\alpha}}\\ &\quad\times\int_t^{\infty} R'(s)R(s)^{-1} l_p(s)^{\frac{1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}} l_x(s)^{\frac{\beta}{\alpha}} ds\in\mathrm{RV}_R(m_1),\end{aligned}\tag{5.30}$$

as $t \to \infty$.

Assume next that (b) holds. Integrating (5.29) on $[t,\infty)$, then on $[t_0,t]$, we find via Lemma 5.1 that

$$x(t) \sim \left(\frac{m_1^{\frac{\alpha+2}{\alpha+1}}}{(\sigma+\beta\rho+m_1)\,(\sigma+\beta\rho+2m_1)}\right)^{\frac{1}{\alpha}} \frac{\alpha}{-(\sigma+\beta\rho+(\alpha+2)m_1-\eta)} \tag{5.31}$$
$$\times \int_{t_0}^{t} R(s)^{\frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+m_1} l_p(s)^{\frac{1}{\alpha(\alpha+1)}} l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}} ds,\ t\to\infty.$$

Because of the divergence of the last integral (note that $x(t)\to\infty, t\to\infty$), it follows that

$$\frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+2m_1=\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}\geq 0.$$

We distinguish the two cases:

$$\text{(b.1)}\quad \frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}=0 \quad\text{and}\quad \text{(b.2)}\quad \frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}>0.$$

Assume that (b.1) holds. Then, (5.31) shows that $x(t)\in \mathrm{SV}_R$, that is, $\rho=0$, and hence $\sigma=-2\alpha-\eta$. Since

$$\sigma+\beta\rho+m_1=-\alpha m_3,\ \sigma+\beta\rho+2m_1=-\alpha m_2,\ \sigma+\beta\rho+(\alpha+2)m_1-\eta=-\alpha m_1,$$

(5.31) reduce to

$$x(t)\sim \left(\frac{m_1^{2-\alpha}}{\alpha^2 m_2 m_3}\right)^{\frac{1}{\alpha}} \int_{t_0}^{t} R'(s)\,R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha}} l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\, ds \in \mathrm{SV}_R,\ t\to\infty. \tag{5.32}$$

Assume that (b.2) holds. Applying Lemma 5.1 to the integral in (5.31), we get

$$x(t)\sim \left(\frac{m_1^{2}}{(\sigma+\beta\rho+m_1)\,(\sigma+\beta\rho+2m_1)}\right)^{\frac{1}{\alpha}} \frac{\alpha}{-(\sigma+\beta\rho+(\alpha+2)m_1-\eta)} \tag{5.33}$$
$$\times \frac{\alpha}{\sigma+\beta\rho+2\alpha+\eta} R(t)^{\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}}\, l_p(t)^{\frac{1}{\alpha}}\, l_q(t)^{\frac{1}{\alpha}}\, l_x(t)^{\frac{\beta}{\alpha}},\quad t\to\infty,$$

which implies that $x(t)\in \mathrm{RV}_R(\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha})$.

Let us now suppose that $x(t)$ is a type-(I) solution of (E) belonging to $\mathrm{ntr}-\mathrm{SV}_R$. From the above observations this is possible only when the case (b.1) holds, in which case $\rho=0$, $\sigma=-2\alpha-\eta$ and $x(t)=l_x(t)$ must satisfy the asymptotic behavior (5.32) as $t\to\infty$. Put

$$\mu(t)=H\int_{t_0}^{t} R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}}l_q(s)^{\frac{1}{\alpha}}\,l_x(s)^{\frac{\beta}{\alpha}}\,ds,\qquad H=\left(\frac{m_1^{2-\alpha}}{\alpha^2 m_2 m_3}\right)^{\frac{1}{\alpha}}.$$

Noting that

$$\mu'(t)=H\,R'(t)\,R(t)^{-1}\,l_p(t)^{\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\,l_x(t)^{\frac{\beta}{\alpha}}\sim H\,R'(t)\,R(t)^{-1}\,l_p(t)^{\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}}\,\mu(t)^{\frac{\beta}{\alpha}},$$

as $t\to\infty$, we obtain the differential asymptotic relation

$$\mu(t)^{-\frac{\beta}{\alpha}}\,\mu'(t)\sim HR'(t)\,R(t)^{-1}l_p(t)^{\frac{1}{\alpha}}\,l_q(t)^{\frac{1}{\alpha}},\ t\to\infty. \tag{5.34}$$

Since the left-hand side of (5.34) is not integrable on $[t_0,\infty)$ (note that $x(t)\to\infty$ as $t\to\infty$ and so $\mu(t)\to\infty$ as $t\to\infty$), so is the right-hand side, which in view of (5.23), means that

$$\int_a^\infty t\left(\frac{1}{p(t)}\int_t^\infty (s-t)\,q(s)\,ds\right)^{\frac{1}{\alpha}}\,dt=\infty.$$

We now integrate (5.34) from t_0 to t to obtain

$$x(t)\sim\mu(t)\sim\left(\frac{\alpha-\beta}{\alpha}\,H\int_{t_0}^{t}R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{1}{\alpha}}l_q(s)^{\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty,$$

which, in view of (5.23), is equivalent to

$$x(t)\sim\left(\frac{\alpha-\beta}{\alpha}\int_a^t s\left(\frac{1}{p(s)}\int_s^\infty (r-s)\,q(r)\,dr\right)^{\frac{1}{\alpha}}\,ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t\to\infty.$$

Thus it has been shown that $x(t)\sim X_1(t),\ t\to\infty$, where $X_1(t)$ is given by (5.14). This proves the "only if" part of Theorem 5.1.

Next, suppose that $x(t)$ is a solution of (E) belonging to $\mathrm{RV}_R(\rho)$, $\rho\in(0,m_1)$. This is possible only when (b.2) holds, in which case $x(t)$ must satisfy the asymptotic relation (5.33). Therefore,

$$\rho=\frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha}\quad\Rightarrow\quad\rho=\frac{\sigma+2\alpha+\eta}{\alpha-\beta},$$

which justifies (5.16) and combined with $\rho\in(0,m_1)$ determines that the range of σ is

$$-2\alpha-\eta<\sigma<-\alpha-(\beta+1)m_1.$$

Since

$$\sigma + \beta\rho + m_1 = \alpha(\rho - m_3), \qquad \sigma + \beta\rho + 2m_1 = \alpha(\rho - m_2),$$
$$\sigma + \beta\rho + (\alpha+2)m_1 - \eta = \alpha(\rho - m_1), \qquad \sigma + \beta\rho + 2\alpha + \eta = \alpha\rho,$$

we conclude from (5.33) that $x(t)$ enjoys the asymptotic behavior $x(t) \sim X_2(t),\ t \to \infty$, where $X_2(t)$ is given by (5.17). This proves the "only if" part of the Theorem 5.2.

Finally, suppose that $x(t)$ is a type-(I) intermediate solution of (E) belonging to $\mathrm{ntr} - \mathrm{RV}_R(m_1)$. Then, the case (a) is the only possibility for $x(t)$, which means that $\sigma = -\alpha - (\beta+1)m_1$ and (5.30) is satisfied by $x(t)$. Using $x(t) = R(t)^{m_1}\, l_x(t)$, (5.30) can be expressed as

$$l_x(t) \sim K\, l_p(t)^{\frac{1}{\alpha+1}} \int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\, ds,\ t \to \infty, \tag{5.35}$$

where $K = \left(m_1^{2-\alpha}/\alpha^2(m_2 - m_1)\right)^{\frac{1}{\alpha}}$. Define $\nu(t)$ by

$$\nu(t) = \int_t^\infty R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, l_x(s)^{\frac{\beta}{\alpha}}\, ds.$$

Then, noting that $l_x(t) \sim K\, l_p(t)^{\frac{1}{\alpha+1}}\, \nu(t),\ t \to \infty$, one can transform (5.35) into the following differential asymptotic relation for $\nu(t)$:

$$-\nu(t)^{-\frac{\beta}{\alpha}}\, \nu'(t) \sim K^{\frac{\beta}{\alpha}}\, R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta+1}{\alpha(\alpha+1)}}\, l_q(t)^{\frac{1}{\alpha}},\ \ t \to \infty. \tag{5.36}$$

From (5.30), since $\lim_{t\to\infty} x(t)/t = 0$, we have $\lim_{t\to\infty} \nu(t) = 0$, implying that the left-hand side of (5.36) is integrable over $[t_0, \infty)$, so is the right-hand side. This, in view of (5.25), implies the convergence of the integral

$$\int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, s^\beta\, q(s)\, ds\right)^{\frac{1}{\alpha}}\, dt.$$

Integrating (5.36) on $[t, \infty)$ and combining the result with (5.35), we find that

$$x(t) \sim K^{\frac{\alpha}{\alpha-\beta}}\, R(t)^{m_1}\, l_p(t)^{\frac{1}{\alpha+1}} \left(\frac{\alpha-\beta}{\alpha} \int_t^\infty R'(s)\, R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha(\alpha+1)}}\, l_q(s)^{\frac{1}{\alpha}}\, ds\right)^{\frac{\alpha}{\alpha-\beta}},$$

as $t \to \infty$, which due to (5.25) gives $x(t) \sim X_3(t),\ t \to \infty$, where $X_3(t)$ is given by (5.19). This proves the "only if" part of the proof of Theorem 5.3. ■

PROOF OF THE "IF" PART OF THEOREMS 5.1, 5.2 AND 5.3: Suppose that (5.13) or (5.15) or (5.18) holds. From Lemmas 5.2, 5.3 and 5.4 it is known that $X_i(t)$, $i = 1,2,3$, defined by (5.14), (5.17) and (5.19) satisfy the asymptotic relation (5.22) for any $b \geq a$. We perform the simultaneous proof for $X_i(t),\ i = 1,2,3$ so the subscripts $i = 1, 2, 3$ will be deleted in the rest of the proof. By (5.22) there exists $T_0 > a$ such that

$$\int_{T_0}^{t}\int_{s}^{\infty}\left(\frac{1}{p(r)}\int_{r}^{\infty}(u-r)\,q(u)\,X(u)^{\beta}\,du\right)^{\frac{1}{\alpha}} drds \le 2X(t),\ \ t \ge T_0. \tag{5.37}$$

Let such a T_0 be fixed. We may assume that $X(t)$ is increasing on $[T_0,\infty)$. Since (5.22) holds with $b = T_0$, there exists $T_1 > T_0$ such that

$$\int_{T_0}^{t}\int_{s}^{\infty}\left(\frac{1}{p(r)}\int_{r}^{\infty}(u-r)\,q(u)\,X(u)^{\beta}\,du\right)^{\frac{1}{\alpha}} drds \ge \frac{X(t)}{2},\ \ t \ge T_1. \tag{5.38}$$

Choose positive constants m and M so that

$$m^{1-\frac{\beta}{\alpha}} \le \frac{1}{2},\quad M^{1-\frac{\beta}{\alpha}} \ge 4,\quad 2m\,X(T_1) \le M\,X(T_0). \tag{5.39}$$

Define the integral operator

$$\mathcal{G}x(t) = x_0 + \int_{T_0}^{t}\int_{s}^{\infty}\left(\frac{1}{p(r)}\int_{r}^{\infty}(u-r)\,q(u)\,x(u)^{\beta}\,du\right)^{\frac{1}{\alpha}} drds,\quad t \ge T_0, \tag{5.40}$$

where x_0 is a constant such that

$$m\,X(T_1) \le x_0 \le \frac{M}{2}\,X(T_0), \tag{5.41}$$

and let it act on set

$$\mathcal{X} = \{x \in C[T_0,\infty) : m\,X(t) \le x(t) \le M\,X(t),\ t \ge T_0\}. \tag{5.42}$$

It is clear that $\mathcal{X}$ is a closed, convex subset of the locally convex space $C[T_0,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0,\infty)$.

It can be shown that $\mathcal{G}$ is a continuous self-map on $\mathcal{X}$ and that the set $\mathcal{G}(\mathcal{X})$ is relatively compact in $C[T_0,\infty)$.

(i) $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$. Let $x(t) \in \mathcal{X}$. Using (5.37), (5.39), (5.41) and (5.42) we get

$$\begin{aligned}\mathcal{G}x(t) &\le \frac{M}{2}\,X(T_0) + M^{\frac{\beta}{\alpha}}\int_{T_0}^{t}\int_{s}^{\infty}\left(\frac{1}{p(r)}\int_{r}^{\infty}(u-r)\,q(u)\,X(u)^{\beta}\,du\right)^{\frac{1}{\alpha}} drds\\ &\le \frac{M}{2}\,X(t) + 2M^{\frac{\beta}{\alpha}}\,X(t) \le \frac{M}{2}\,X(t) + \frac{M}{2}\,X(t) = M\,X(t),\quad t \ge T_0.\end{aligned}$$

On the other hand, using (5.38), (5.39), (5.41) and (5.42) we have

$$\mathcal{G}x(t) \ge x_0 \ge m\,X(T_1) \ge m\,X(t),\quad T_0 \le t \le T_1,$$

and

$$\mathcal{G}x(t) \geq m^{\frac{\beta}{\alpha}} \int_{T_0}^{t} \int_{s}^{\infty} \left(\frac{1}{p(r)} \int_{r}^{\infty} (u-r) q(u) X(u)^{\beta} du \right)^{\frac{1}{\alpha}} dr ds$$
$$\geq m^{\frac{\beta}{\alpha}} \frac{X(t)}{2} \geq m X(t), \quad t \geq T_1.$$

This shows that $\mathcal{G}x(t) \in \mathcal{X}$, that is, $\mathcal{G}$ maps $\mathcal{X}$ into itself.

(ii) $\mathcal{G}(\mathcal{X})$ *is relatively compact*. The inclusion $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$ ensures that $\mathcal{G}(\mathcal{X})$ is locally uniformly bounded on $[T_0, \infty)$. From the inequality

$$0 \leq (\mathcal{G}x)'(t) \leq M^{\frac{\beta}{\alpha}} \int_{t}^{\infty} \left(\frac{1}{p(s)} \int_{s}^{\infty} (r-s) q(r) X(r)^{\beta} \, dr \right)^{\frac{1}{\alpha}} ds, \quad t \geq T_0,$$

holding for all $x \in \mathcal{X}$ it follows that $\mathcal{G}(\mathcal{X})$ is locally equicontinuous on $[T_0, \infty)$. Then, the relative compactness of $\mathcal{G}(\mathcal{X})$ follows from the Arzela-Ascoli lemma.

(iii) $\mathcal{G}$ *is continuous on* $\mathcal{X}$. Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging to $x(t)$ uniformly on any compact subinterval of $[T_0, \infty)$. From (5.40) we have

$$|\mathcal{G}x_n(t) - \mathcal{G}x(t)| \leq \int_{T_0}^{t} \int_{s}^{\infty} \frac{1}{p(r)^{\frac{1}{\alpha}}} G_n(r) \, dr \, ds, \quad t \geq T_0,$$

where

$$G_n(t) = \left| \left(\int_{t}^{\infty} (s-t) \, q(s) \, x_n(s)^{\beta} \, ds \right)^{\frac{1}{\alpha}} - \left(\int_{t}^{\infty} (s-t) \, q(s) \, x(s)^{\beta} \, ds \right)^{\frac{1}{\alpha}} \right|.$$

Using the inequality $|x^{\lambda} - y^{\lambda}| \leq |x-y|^{\lambda}, \ x, y \in R^{+}$ holding for $\lambda \in (0,1)$, we see that if $\alpha \geq 1$, then

$$G_n(t) \leq \left(\int_{t}^{\infty} (s-t) q(s) |x_n(s)^{\beta} - x(s)^{\beta}| \, ds \right)^{\frac{1}{\alpha}}.$$

On the other hand, using the mean value theorem, if $\alpha < 1$ we get

$$G_n(t) \leq \frac{1}{\alpha} \left(M^{\beta} \int_{t}^{\infty} (s-t) q(s) X(s)^{\beta} \, ds \right)^{\frac{\alpha-1}{\alpha}} \int_{t}^{\infty} (s-t) q(s) |x_n(s)^{\beta} - x(s)^{\beta}| \, ds.$$

Thus, using that $q(t)|x_n(t)^{\beta} - x(t)^{\beta}| \to 0$ as $n \to \infty$ at each point $t \in [T_0, \infty)$ and $q(t)|x_n(t)^{\beta} - x(t)^{\beta}| \leq M^{\beta} q(t) X(t)^{\beta}$ for $t \geq T_0$, while $q(t) X(t)^{\beta}$ is integrable on $[T_0, \infty)$, the uniform convergence $G_n(t) \to 0$ on $[T_0, \infty)$ follows by the application of the Lebesgue dominated convergence theorem. We conclude that $\mathcal{G}x_n(t) \to \mathcal{G}x(t)$

uniformly on any compact subinterval of $[T_0, \infty)$ as $n \to \infty$, which proves the continuity of $\mathcal{G}$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x(t) \in \mathcal{X}$ of $\mathcal{G}$, which satisfies integral equation

$$x(t) = x_0 + \int_{T_0}^{t} \int_{s}^{\infty} \left(\frac{1}{p(r)} \int_{r}^{\infty} (u-r)\, q(u)\, x(u)^{\beta}\, du \right)^{\frac{1}{\alpha}} dr ds, \quad t \geq T_0. \tag{5.43}$$

Differentiating the above four times shows that $x(t)$ is a solution of (E) on $[T_0, \infty)$, which due to (5.42) is an intermediate solution of type (I). Therefore, the proof of our main results will be completed with the verification that the intermediate solutions of (E) constructed above are actually regularly varying functions with respect to $R(t)$. We define the function

$$J(t) = \int_{T_0}^{t} \int_{s}^{\infty} \left(\frac{1}{p(r)} \int_{r}^{\infty} (u-r)\, q(u)\, X(u)^{\beta}\, du \right)^{\frac{1}{\alpha}} dr ds,\ t \geq T_0,$$

and put

$$l = \liminf_{t\to\infty} \frac{x(t)}{J(t)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{J(t)}.$$

By Lemmas 5.2, 5.3 and 5.4 we have $X(t) \sim J(t),\ t \to \infty$. Since, $x(t) \in \mathcal{X}$, it is clear that $0 < l \leq L < \infty$. We first consider L. Applying Lemma 5.5 four times, we obtain

$$\begin{aligned}
L \ &\leq \limsup_{t\to\infty} \frac{x'(t)}{J'(t)} \leq \limsup_{t\to\infty} \frac{x''(t)}{J''(t)} = \limsup_{t\to\infty} \frac{\left(\int_t^{\infty} (s-t) q(s) x(s)^{\beta}\, ds\right)^{\frac{1}{\alpha}}}{\left(\int_t^{\infty} (s-t) q(s) X(s)^{\beta}\, ds\right)^{\frac{1}{\alpha}}} \\
&= \left(\limsup_{t\to\infty} \frac{\int_t^{\infty} (s-t) q(s) x(s)^{\beta}\, ds}{\int_t^{\infty} (s-t) q(s) X(s)^{\beta}\, ds} \right)^{\frac{1}{\alpha}} \leq \left(\limsup_{t\to\infty} \frac{\int_t^{\infty} q(s) x(s)^{\beta}\, ds}{\int_t^{\infty} q(s) X(s)^{\beta}\, ds} \right)^{\frac{1}{\alpha}} \\
&\leq \left(\limsup_{t\to\infty} \frac{q(t) x(t)^{\beta}}{q(t) X(t)^{\beta}} \right)^{\frac{1}{\alpha}} = \left(\limsup_{t\to\infty} \frac{x(t)}{X(t)} \right)^{\frac{\beta}{\alpha}} = \left(\limsup_{t\to\infty} \frac{x(t)}{J(t)} \right)^{\frac{\beta}{\alpha}} = L^{\frac{\beta}{\alpha}},
\end{aligned}$$

where we have used $X(t) \sim J(t),\ t \to \infty$, in the last step. Since $\beta/\alpha < 1$, the inequality $L \leq L^{\frac{\beta}{\alpha}}$ implies that $L \leq 1$. Similarly, repeated application of Lemma 5.5 to l leads to $l \geq 1$, from which it follows that $L = l = 1$, that is,

$$\lim_{t\to\infty} \frac{x(t)}{J(t)} = 1 \qquad \Longrightarrow \qquad x(t) \sim J(t) \sim X(t),\ t \to \infty.$$

Therefore it is concluded that if $p(t) \in \mathrm{RV}_R(\eta)$ and $q(t) \in \mathrm{RV}_R(\sigma)$, then the type-(I) solution $x(t)$ under consideration is a member of $\mathrm{RV}_R(\rho)$, where

$$\rho = 0 \quad \text{or} \quad \rho = \frac{\sigma + 2\alpha + \eta}{\alpha - \beta} \in (0, m_1) \quad \text{or} \quad \rho = m_1,$$

according to whether the pair (η, σ) satisfies (5.13),(5.15) or (5.18), respectively. Needless to say, any such solution $x(t)$ in $\mathrm{RV}_R(\rho)$ enjoys one and the same asymptotic behavior (5.14), (5.17) or (5.19) according as $\rho = 0,\ \rho \in (0, m_1)$ or $\rho = m_1$. This completes the "if" parts of Theorems 5.1, 5.2 and 5.3. ■

5.2. Regularly varying intermediate solutions of type (II)

Let us turn our attention to the study of intermediate solutions of type (II) of equation (E), that is, those solutions $x(t)$ such that $\psi_3(t) \prec x(t) \prec \psi_4(t)$ as $t \to \infty$. As in the preceding section use is made of the expressions (5.2) and (5.3) for the coefficients $p(t)$, $q(t)$ and the solutions $x(t)$. Since $\psi_3(t) \in \mathrm{RV}_R(m_2(\alpha,\eta))$ and $\psi_4(t) \in \mathrm{RV}_R(m_3(\alpha,\eta))$ (cf. (5.10) and (5.11)), the regularity index ρ of $x(t)$ must satisfy $m_2(\alpha,\eta) \le \rho \le m_3(\alpha,\eta)$. If $\rho = m_2(\alpha,\eta)$, then since $x(t)/R(t)^{m_2(\alpha,\eta)} = l_x(t) \to \infty,\ t \to \infty$, $x(t)$ is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_2(\alpha,\eta))$, while if $\rho = m_3(\alpha,\eta)$, then $x(t)/R(t)^{m_3(\alpha,\eta)} \to 0,\ t \to \infty$, and so $x(t)$ is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_3(\alpha,\eta))$. If $m_2(\alpha,\eta) < \rho < m_3(\alpha,\eta)$, then $x(t)$ belongs to $\mathrm{RV}_R(\rho)$ and clearly satisfies $x(t)/R(t)^{m_2(\alpha,\eta)} \to \infty$ and $x(t)/R(t)^{m_3(\alpha,\eta)} \to 0$ as $t \to \infty$. Therefore, it is natural to divide the the totality of type-(II) intermediate solutions of (E) into the following three classes

$$\begin{aligned}&\mathrm{ntr}-\mathrm{RV}_R(m_2(\alpha,\eta)) \quad \text{or} \quad \mathrm{RV}_R(\rho),\ \rho \in (m_2(\alpha,\eta), m_3(\alpha,\eta))\\ &\text{or} \quad \mathrm{ntr}-\mathrm{RV}_R(m_3(\alpha,\eta)).\end{aligned} \tag{5.44}$$

Our purpose is to show that, for each of the above classes, necessary and sufficient conditions for the membership are establish and that the asymptotic behavior at infinity of all members of each class is determined precisely by a unique explicit formula.

5.2.1. Main results

Theorem 5.4. *Let* $p(t) \in \mathrm{RV}_R(\eta),\ q(t) \in \mathrm{RV}_R(\sigma)$. *Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{ntr}-\mathrm{RV}_R\,(m_2(\alpha,\eta))$ *satisfying* (II) *if and only if*

$$\sigma = -2m_1(\alpha,\eta) - \beta m_2(\alpha,\eta) \quad \textit{and} \quad \int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt = \infty. \tag{5.45}$$

The asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim Y_1(t),\ t \to \infty$, *where*

$$Y_1(t) = \psi_3(t)\,\left(\frac{\alpha-\beta}{\alpha}\int_a^t s\, q(s)\, \psi_3(s)^\beta\, ds\right)^{\frac{1}{\alpha-\beta}}. \tag{5.46}$$

Theorem 5.5. *Let* $p(t) \in \mathrm{RV}_R(\eta),\ q(t) \in \mathrm{RV}_R(\sigma)$. *Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{RV}_R(\rho)$ *with* $\rho \in (m_2(\alpha,\eta)\,,\ m_3(\alpha,\eta))$ *if and only if*

$$-2m_1(\alpha,\eta) - \beta m_2(\alpha,\eta) < \sigma < -m_1(\alpha,\eta) - \beta\, m_3(\alpha,\eta), \tag{5.47}$$

in which case ρ *is given by* (5.16) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula* $x(t) \sim Y_2(t),\ t \to \infty$, *where*

(5.48)
$$Y_2(t) = \left(\left(\frac{m_1(\alpha,\eta)}{\alpha} \right)^2 \frac{p(t)\ q(t)\ R(t)^{2\alpha}}{\rho^\alpha\ (\rho - m_1(\alpha,\eta))^\alpha\ (\rho - m_2(\alpha,\eta))\ (m_3(\alpha,\eta) - \rho)} \right)^{\frac{1}{\alpha-\beta}}.$$

Theorem 5.6. *Let* $p(t) \in \mathrm{RV}_R(\eta),\ q(t) \in \mathrm{RV}_R(\sigma)$. *Equation* (E) *has intermediate solutions* $x(t) \in \mathrm{ntr} - \mathrm{RV}_R\,(m_3(\alpha,\eta))$ *satisfying* (II) *if and only if*

$$\sigma = -m_1(\alpha,\eta) - \beta\, m_3(\alpha,\eta) \quad \text{and} \quad \int_a^\infty q(t)\, \psi_4(t)^\beta\, dt < \infty. \tag{5.49}$$

The asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim Y_3(t),\ t \to \infty$, *where*

$$Y_3(t) = \psi_4(t)\ \left(\frac{\alpha-\beta}{\alpha} \int_t^\infty q(s)\ \psi_4(s)^\beta\ ds \right)^{\frac{1}{\alpha-\beta}}. \tag{5.50}$$

5.2.2. Preparatory results

Let $x(t)$ be a type-(II) intermediate solution of (E) defined on $[t_0,\infty)$. It is known that $\lim_{t\to\infty} (p(t)\, x''(t)^\alpha)' = 0$ and $\lim_{t\to\infty} p(t)x''(t)^\alpha = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x(t) = \infty$. Integrating (E) first from t to ∞ and then three times on $[t_0,t]$, we obtain

$$\begin{aligned} x(t) &= c_0 + c_1(t-t_0) \\ &\quad + \int_{t_0}^t (t-s) \left(\frac{1}{p(s)} \left(c_2 + \int_{t_0}^s \int_r^\infty q(u)x(u)^\beta du dr \right) \right)^{\frac{1}{\alpha}} ds, \end{aligned} \tag{5.51}$$

for $t \ge t_0$, where $c_0 = x(t_0)$, $c_1 = x'(t_0)$ and $c_2 = (p(t)\, x''(t)^\alpha)'|_{t=t_0}$. From (5.51) we easily see that $x(t)$ satisfies the integral asymptotic relation

$$x(t) \sim \int_{t_0}^t (t-s) \left(\frac{1}{p(s)} \int_{t_0}^s \int_r^\infty q(u)x(u)^\beta\, du\, dr \right)^{\frac{1}{\alpha}} ds,\ t \to \infty. \tag{5.52}$$

We first prove that regularly varying functions $Y_i(t),\ i = 1,2,3$ satisfy the integral asymptotic relation (5.52).

Lemma 5.6. *Suppose that* (5.45) *holds. The function* $Y_1(t)$ *given by* (5.46) *satisfies the asymptotic relation*

$$y(t) \sim \int_b^t (t-s) \left(\frac{1}{p(s)} \int_b^s \int_r^\infty q(u)\, y(u)^\beta\, du\, dr \right)^{\frac{1}{\alpha}} ds,\ t \to \infty, \tag{5.53}$$

for any $b \ge a$.

Proof. Let (5.45) holds. Using (5.2), (5.5) and (5.10), since $\sigma+\beta m_2+m_1=-m_1$, we obtain

$$tq(t)\,\psi_3(t)^\beta \sim \frac{m_1^{\frac{2\beta-\alpha}{\alpha+1}}}{(m_2(m_2-m_1))^\beta}\,R(t)^{-m_1}\,l_p(t)^{\frac{\beta(\alpha-1)+\alpha}{\alpha(\alpha+1)}}\,l_q(t),\quad t\to\infty,$$

so that applying (iii) of Lemma 5.1 we have

$$\begin{aligned}&\int_a^t s\,q(s)\,\psi_3(s)^\beta\,ds\\ &\sim \frac{m_1^{\frac{2\beta-\alpha+1}{\alpha+1}}}{(m_2(m_2-m_1))^\beta}\int_a^t R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\,l_q(s)\,ds,\end{aligned}\tag{5.54}$$

as $t\to\infty$. This, combined with (5.10), gives the following expression for $Y_1(t)$:

$$\begin{aligned}Y_1(t)\sim&\left(\frac{(\alpha-\beta)m_1}{\alpha(m_2(m_2-m_1))^\alpha}\right)^{\frac{1}{\alpha-\beta}}R(t)^{m_2}\,l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}}\\ &\times\left(\int_a^t R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\,l_q(s)\,ds\right)^{\frac{1}{\alpha-\beta}}\in \mathrm{RV}_R(m_2),\quad t\to\infty.\end{aligned}$$

Next, we integrate $q(t)\,Y_1(t)^\beta$ first on $[t,\infty)$, then on $[b,t]$, for any $b\ge a$. Since $q(t)Y_1(t)^\beta\in\mathrm{RV}_R(\beta m_2+\sigma)=\mathrm{RV}_R(-2m_1)$ (cf. (5.45)), application of Lemma 5.1 and (5.7) yields

$$\begin{aligned}&\int_b^t\int_s^\infty q(r)\,Y_1(r)^\beta\,dr\,ds\sim\left(\frac{\alpha-\beta}{\alpha(m_2(m_2-m_1))^\alpha}\right)^{\frac{\beta}{\alpha-\beta}}m_1^{\frac{\alpha(2\beta-\alpha+1)}{(\alpha-\beta)(\alpha+1)}}\\ &\times\int_b^t R'(s)R(s)^{-1}l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}l_q(s)\left(\int_a^s R'(r)R(r)^{-1}l_p(r)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}l_q(r)dr\right)^{\frac{\beta}{\alpha-\beta}}ds\\ &\sim\left(\frac{\alpha-\beta}{\alpha(m_2(m_2-m_1))^\beta}\right)^{\frac{\alpha}{\alpha-\beta}}m_1^{\frac{\alpha(2\beta-\alpha+1)}{(\alpha-\beta)(\alpha+1)}}\left(\int_a^t R'(s)R(s)^{-1}l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}l_q(s)ds\right)^{\frac{\alpha}{\alpha-\beta}},\end{aligned}$$

as $t\to\infty$. Multiply the above by $1/p(t)$, raise the result to the exponent $1/\alpha$ and then integrate twice on $[b,t]$, for any $b\ge a$, we conclude via Lemma 5.1 that

$$\begin{aligned}&\int_b^t(t-s)\left(\frac{1}{p(s)}\int_b^s\int_r^\infty q(u)\,Y_1(u)^\beta\,du dr\right)^{\frac{1}{\alpha}}ds\sim\left(\frac{(\alpha-\beta)m_1}{\alpha(m_2(m_2-m_1))^\alpha}\right)^{\frac{1}{\alpha-\beta}}\\ &\times R(t)^{m_2}\,l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}}\left(\int_a^t R'(s)\,R(s)^{-1}\,l_p(s)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\,l_q(s)\,ds\right)^{\frac{1}{\alpha-\beta}}=Y_1(t),\end{aligned}$$

as $t\to\infty$. This proves that $Y_1(t)$ satisfies the asymptotic relation (5.53). ∎

Lemma 5.7. *Suppose that* (5.47) *holds and let* ρ *be defined by* (5.16)*. The function* $Y_2(t)$ *given by* (5.48) *satisfies the asymptotic relation* (5.53) *for any* $b \geq a$*.*

Proof. Putting $\lambda = \rho^\alpha(\rho - m_1)^\alpha(\rho - m_2)(m_3 - \rho)$, we express $Y_2(t)$ in the form

$$Y_2(t) \sim C\, R(t)^\rho\, l_p(t)^{\frac{1}{\alpha-\beta}} l_q(t)^{\frac{1}{\alpha-\beta}}, \quad C = \left(\frac{1}{\lambda}\left(\frac{m_1}{\alpha}\right)^2\right)^{\frac{1}{\alpha-\beta}}.$$

We integrate $q(t)Y_2(t)^\beta$ first on $[t,\infty)$ and then on $[b,t]$, for any $b \geq a$. Using (5.16) and Lemma 5.1 twice, we see that

$$\begin{aligned}(5.55)\qquad &\int_b^t \int_s^\infty q(r)\, Y_2(r)^\beta\, dr ds\\ &\sim \frac{C^\beta m_1^{\frac{2}{\alpha+1}}}{\alpha^2(m_3-\rho)(\rho-m_2)} R(t)^{\alpha(\rho-m_2)}\, l_p(t)^{\frac{\alpha\beta+2\alpha-\beta}{(\alpha-\beta)(\alpha+1)}}\, l_q(t)^{\frac{\alpha}{\alpha-\beta}},\end{aligned}$$

as $t \to \infty$. Since (5.55) implies

$$\begin{aligned}&\left(\frac{1}{p(t)}\int_b^t \int_s^\infty q(r)Y_2(r)^\beta\, dr ds\right)^{\frac{1}{\alpha}}\\ &\sim \frac{C^{\frac{\beta}{\alpha}}\, m_1^{\frac{2}{\alpha(\alpha+1)}}}{(\alpha^2(m_3-\rho)(\rho-m_2))^{\frac{1}{\alpha}}} R(t)^{\rho-2m_1} l_p(t)^{\frac{2\beta-\alpha+1}{(\alpha-\beta)(\alpha+1)}} l_q(t)^{\frac{1}{\alpha-\beta}},\ t \to \infty,\end{aligned}$$

integrating the last relation twice on $[b,t]$, we conclude that

$$\begin{aligned}&\int_b^t (t-s)\left(\frac{1}{p(s)}\int_b^s \int_r^\infty q(u)\, Y_2(u)^\beta\, du\, dr\right)^{\frac{1}{\alpha}} ds\\ &\sim \frac{C^{\frac{\beta}{\alpha}} m_1^{\frac{2}{\alpha}}}{(\alpha^2(m_3-\rho)(\rho-m_2))^{\frac{1}{\alpha}}(\rho-m_1)\rho} R(t)^\rho l_p(t)^{\frac{1}{\alpha-\beta}}\, l_q(t)^{\frac{1}{\alpha-\beta}} = Y_2(t),\ t \to \infty.\end{aligned}$$

This proves that $Y_2(t)$ satisfies the asymptotic relation (5.53). ∎

Lemma 5.8. *Suppose that* (5.49) *holds. The function* $Y_3(t)$ *given by* (5.50) *satisfies the asymptotic relation* (5.53) *for any* $b \geq a$*.*

Proof. Suppose that (5.49) holds. Using (5.7) and (5.11) we easily see that

$$(5.56)\qquad \int_t^\infty q(s)\psi_4(s)^\beta\, ds \sim \frac{m_1^{\frac{\beta+1}{\alpha+1}}}{m_3^\beta}\int_t^\infty R'(s)R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s)\, ds,\ t \to \infty.$$

Combining the above with (5.50), we obtain the following asymptotic representation for $Y_3(t)$ in terms of $R(t)$, $l_p(t)$ and $l_q(t)$:

$$Y_3(t) \sim \left(\frac{(\alpha-\beta)m_1}{\alpha m_3^{\alpha}}\right)^{\frac{1}{\alpha-\beta}} R(t)^{m_3} l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^{\infty} R'(s)\, R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s) ds\right)^{\frac{1}{\alpha-\beta}},$$

as $t \to \infty$. Integrating the above relation, using (5.7), we compute

$$\begin{aligned}
\int_t^{\infty} q(s)\, Y_3(s)^{\beta}\, ds &\sim \left(\frac{(\alpha-\beta)\, m_1^{\frac{\alpha(\beta+1)}{\beta(\alpha+1)}}}{\alpha\, m_3^{\alpha}}\right)^{\frac{\beta}{\alpha-\beta}} \\
&\times \int_t^{\infty} R'(s)\, R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s) \left(\int_s^{\infty} R'(r) R(r)^{-1} l_p(r)^{\frac{\beta+1}{\alpha+1}} l_q(r) dr\right)^{\frac{\beta}{\alpha-\beta}} ds \\
&= \left(\frac{(\alpha-\beta) m_1^{\frac{\beta+1}{\alpha+1}}}{\alpha m_3^{\beta}}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\int_t^{\infty} R'(s) R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s) ds\right)^{\frac{\alpha}{\alpha-\beta}},\ t \to \infty.
\end{aligned}$$

Next we integrate the above relation on $[b,t],\ b \geq a$, multiply it by $1/p(t)$ and raise the result to the power $1/\alpha$. Then we find that

$$\begin{aligned}
\left(\frac{1}{p(t)} \int_b^t \int_s^{\infty} q(r) Y_3(r)^{\beta} dr ds\right)^{\frac{1}{\alpha}} &\sim \left(\frac{(\alpha-\beta) m_1^{\frac{\beta+1}{\alpha+1}}}{\alpha m_3^{\beta}}\right)^{\frac{1}{\alpha-\beta}} m_1^{-\frac{1}{\alpha+1}} \\
&\times R(t)^{\frac{m_1-\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha+1}} \left(\int_t^{\infty} R'(s)\, R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s) ds\right)^{\frac{1}{\alpha-\beta}} \\
&\sim \left(\frac{(\alpha-\beta) m_1^{\frac{\beta+1}{\alpha+1}}}{\alpha m_3^{\beta}}\right)^{\frac{1}{\alpha-\beta}} R'(t) \left(\int_t^{\infty} R'(s) R(s)^{-1} l_p(s)^{\frac{\beta+1}{\alpha+1}} l_q(s)\, ds\right)^{\frac{1}{\alpha-\beta}},
\end{aligned} \tag{5.57}$$

as $t \to \infty$. Integrating (5.57) twice on $[b,t]$ leads to the desired conclusion that $Y_3(t)$ satisfies the integral asymptotic relation (5.53). ∎

5.2.3. Proof of main results

PROOF OF THE "ONLY IF" PART OF THEOREMS 5.4, 5.5 AND 5.6: Suppose that equation (E) has a type-(II) intermediate solution $x(t) \in \mathrm{RV}_R(\rho),\ \rho \in [m_2, m_3]$, defined on $[t_0, \infty)$. We begin by integrating (E) on $[t,\infty)$. Using (5.2), (5.3) and (5.7), we have

$$(p(t)\, x''(t)^{\alpha})' = \int_t^{\infty} q(s) x(s)^{\beta} ds \sim \int_t^{\infty} R(s)^{\sigma+\beta\rho} l_q(s) l_x(s)^{\beta}\, ds, \quad t \to \infty. \tag{5.58}$$

To proceed further we distinguish the two cases:

$$\text{(a)}\ \sigma + \beta\rho + m_1 = 0 \quad \text{and} \quad \text{(b)}\ \sigma + \beta\rho + m_1 < 0.$$

Let case (a) hold. Integration of (5.58) on $[t_0, t]$ yields

$$x''(t) \sim m_1^{\frac{1-\alpha}{\alpha(\alpha+1)}} R(t)^{\frac{m_1-\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha+1}} \left(\int_t^\infty R'(s)R(s)^{-1} l_p(s)^{\frac{1}{\alpha+1}} l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}},$$

as $t \to \infty$. Integrating (5.59) twice over $[t_0, t]$, we obtain via Lemma 5.1 and (5.11) that

$$\begin{aligned} x(t) &\sim \frac{m_1^{\frac{1}{\alpha}}}{m_3} R(t)^{m_3} l_p(t)^{\frac{1}{\alpha+1}} \left(\int_t^\infty R'(s)R(s)^{-1} l_p(s)^{\frac{1}{\alpha+1}} l_q(s) l_x(s)^\beta ds \right)^{\frac{1}{\alpha}} \\ &\sim \psi_4(t) m_1^{\frac{1}{\alpha(\alpha+1)}} \left(\int_t^\infty R'(s)R(s)^{-1} l_p(s)^{\frac{1}{\alpha+1}} l_q(s)\, l_x(s)^\beta ds \right)^{\frac{1}{\alpha}}, t \to \infty. \end{aligned} \tag{5.59}$$

Let case (b) hold. Then, integration of (5.58) on $[t_0, t]$ gives

$$p(t)x''(t)^\alpha \sim \frac{m_1^{\frac{1}{\alpha+1}}}{-(\sigma + \beta\rho + m_1)} \int_{t_0}^t R(s)^{\sigma+\beta\rho+m_1} l_p(s)^{\frac{1}{\alpha+1}} l_q(s) l_x(s)^\beta ds, \tag{5.60}$$

as $t \to \infty$. The divergence of the last integral as $t \to \infty$ implies $\sigma + \beta\rho + 2m_1 \geq 0$. To preform further integration of (5.60) we consider the following two cases separately:

$$\text{(b.1)} \quad \sigma + \beta\rho + 2m_1 = 0; \qquad \text{(b.2)} \quad \sigma + \beta\rho + 2m_1 > 0.$$

Suppose that (b.1) holds. Since $\sigma + \beta\rho + m_1 = -m_1$ and $-\frac{\eta}{\alpha} + m_1 = m_2 - m_1$, integrating (5.60) twice on $[t_0, t]$, we have

$$\begin{aligned} x(t) &\sim \frac{m_1^{\frac{1}{\alpha}}}{m_2(m_2 - m_1)} R(t)^{m_2} l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \\ &\quad \left(\int_{t_0}^t R'(s)R(s)^{-1} l_p(s)^{\frac{2}{\alpha+1}} l_q(s) l_x(s)^\beta ds \right)^{\frac{1}{\alpha}} \\ &\sim \psi_3(t)\, m_1^{\frac{1-\alpha}{\alpha(\alpha+1)}} \left(\int_{t_0}^t R'(s)R(s)^{-1} l_p(s)^{\frac{2}{\alpha+1}} l_q(s)\, l_x(s)^\beta\, ds \right)^{\frac{1}{\alpha}}, \end{aligned} \tag{5.61}$$

as $t \to \infty$, which means that $x(t) \in \mathrm{RV}_R(m_2)$ and that its regularly varying part $l_x(t)$ satisfies the relation

$$\begin{aligned} l_x(t) &\sim \frac{m_1^{\frac{1}{\alpha}}}{m_2(m_2 - m_1)} l_p(t)^{\frac{\alpha-1}{\alpha(\alpha+1)}} \\ &\quad \times \left(\int_{t_0}^t R'(s)R(s)^{-1} l_p(s)^{\frac{2}{\alpha+1}} l_q(s) l_x(s)^\beta ds \right)^{\frac{1}{\alpha}},\ t \to \infty. \end{aligned} \tag{5.62}$$

Suppose that (b.2) holds. Applying first Lemma 5.1 in (5.60), then multiplying by $1/p(t)$, raising the result on $1/\alpha$ and integrating twice from t_0 to t, we obtain

$$\begin{aligned} x(t) \sim & \left(\frac{m_1^2}{-(\sigma+\beta\rho+m_1)\,(\sigma+\beta\rho+2m_1)}\right)^{\frac{1}{\alpha}} \\ & \times \frac{R(t)^{\frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+2m_1} l_p(t)^{\frac{1}{\alpha}} l_q(t)^{\frac{1}{\alpha}} l_x(t)^{\frac{\beta}{\alpha}}}{\left(\frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+m_1\right)\left(\frac{\sigma\beta\rho+2m_1-\eta}{\alpha}+2m_1\right)}, \end{aligned} \tag{5.63}$$

as $t\to\infty$. This implies that $x(t) \in \mathrm{RV}\left(\frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+2m_1\right)$. It is easy to see that

$$m_2 < \frac{\sigma+\beta\rho+2m_1-\eta}{\alpha}+2m_1 = \frac{\sigma+\beta\rho+2\alpha+\eta}{\alpha} < m_3.$$

Now, let $x(t)$ be a type-(II) intermediate solution of (E) belonging to $\mathrm{RV}_R(m_2)$. Then, from the above observations it is clear that only the case (b.1) is admissible, so that $\sigma = -2m_1 - \beta m_2$ and $x(t)$ must satisfy (5.61). Put

$$\mu(t) = \int_{t_0}^{t} R'(s)R(s)^{-1} l_p(s)^{\frac{2}{\alpha+1}} l_q(s)\, l_x(s)^{\beta}\, ds.$$

Then, we can convert (5.62) to the differential asymptotic relation for $\mu(t)$

$$\mu(t)^{-\frac{\beta}{\alpha}}\,\mu'(t) \sim C^{\beta} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\, l_q(t),\ t\to\infty, \tag{5.64}$$

where $C = m_1^{\frac{1}{\alpha}}/m_2(m_2-m_1)$. From (5.61), since $\lim_{t\to\infty} x(t)/\psi_3(t) = \infty$, we have $\lim_{t\to\infty}\mu(t) = \infty$, implying that the left-hand side of (5.64) is not integrable on $[t_0,\infty)$, so is the right-hand side, that is,

$$\int_{t_0}^{\infty} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta(\alpha-1)+2\alpha}{\alpha(\alpha+1)}}\, l_q(t) dt = \infty,$$

which, as shown in the proof of Lemma 5.6 (cf. (5.54)), is equivalent to

$$\int_a^{\infty} t\, q(t)\, \psi_3(t)^{\beta}\, dt = \infty.$$

We now integrate (5.64) on $[t_0, t]$ and in view of (5.54), we obtain

$$\mu(t) \sim m_1^{\frac{\alpha-1}{\alpha+1}} \left(\frac{\alpha-\beta}{\alpha}\int_{t_0}^{t} s\, q(s)\, \psi_3(s)^{\beta}\, ds\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty,$$

and this, combined with (5.61), shows that

$$x(t) \sim \psi_3(t)\, m_1^{\frac{1-\alpha}{\alpha(\alpha+1)}}\, m_1^{\frac{\alpha-1}{\alpha(\alpha+1)}} \left(\frac{\alpha-\beta}{\alpha} \int_a^t s\, q(s)\, \psi_3(s)^{\beta}\, ds \right)^{\frac{1}{\alpha-\beta}} = Y_1(t),$$

as $t \to \infty$. This completes the "only if" part of the Theorem 5.4.

Next, let $x(t)$ be an intermediate solution of (E) belonging to $\mathrm{RV}_R(\rho)$ for some $\rho \in (m_2, m_3)$. Clearly, $x(t)$ falls into the case (b.2) and hence satisfies the asymptotic relation (5.63). This means that

$$\rho = \frac{\sigma + \beta\rho + 2m_1 - \eta}{\alpha} + 2m_1 = \frac{\sigma + \beta\rho + 2\alpha + \eta}{\alpha} \quad \Longrightarrow \quad \rho = \frac{\sigma + 2\alpha + \eta}{\alpha - \beta},$$

verifying that the regularity index ρ is given by (5.16). From the requirement $m_2 < \rho < m_3$ it follows that $-2m_1 - \beta m_2 < \sigma < -m_1 - \beta m_3$, showing that the range of σ is given by (5.47). Since

$$\frac{\sigma + \beta\rho + 2m_1 - \eta}{\alpha} + m_1 = \rho - m_1, \quad \frac{\sigma + \beta\rho + 2m_1 - \eta}{\alpha} + 2m_1 = \rho,$$

$$-(\sigma + \beta\rho + m_1) = \alpha(m_3 - \rho), \quad \sigma + \beta\rho + 2m_1 = \alpha(\rho - m_2),$$

the relation (5.63) can be rewritten as

$$x(t) \sim \left(\frac{m_1^2\, p(t)\, q(t)\, R(t)^{2\alpha}}{\alpha^2 \rho^{\alpha} (\rho - m_1)^{\alpha} (\rho - m_2)(m_3 - \rho)} \right)^{\frac{1}{\alpha}} x(t)^{\frac{\beta}{\alpha}},$$

from which it readily follows that $x(t)$ enjoys the asymptotic behavior (5.48). This proves the "only if" part of the Theorem 5.5.

Finally, let $x(t)$ is a type-(II) intermediate solution of (E) belonging to $\mathrm{RV}_R(m_3)$. Since only the case (a) is possible for $x(t)$, it satisfies (5.59), which implies $\rho = m_3$ and $\sigma = -m_1 - \beta m_3$. Letting

$$\nu(t) = \left(\int_t^{\infty} R'(s)\, R(s)^{-1}\, l_p(s)^{\frac{1}{\alpha+1}}\, l_q(s)\, l_x(s)^{\beta}\, ds \right)^{\frac{1}{\alpha}},$$

and using the relation $l_x(t) \sim (m_1^{\frac{1}{\alpha}}/m_3) l_p(t)^{\frac{1}{\alpha+1}} \nu(t)$, we convert (5.59) into the differential asymptotic relation

$$(5.65) \qquad -\alpha\nu(t)^{\alpha-\beta-1}\, \nu'(t) \sim \frac{m_1^{\frac{\beta}{\alpha}}}{m_3^{\beta}} R'(t)\, R(t)^{-1}\, l_p(t)^{\frac{\beta+1}{\alpha+1}}\, l_q(t), \quad t \to \infty.$$

Since the left-hand side of (5.65) is integrable on $[t_0, \infty)$ (note that $\lim_{t\to\infty} x(t)/\psi_4(t) = 0$ and so $\lim_{t\to\infty} \nu(t) = 0$), so is the right-hand side, that is,

$$\int_{t_0}^{\infty} R'(t)R(t)^{-1} l_p(t)^{\frac{\beta+1}{\alpha+1}} l_q(t)dt < \infty$$

which is equivalent to $\int_a^\infty q(t)\psi(t)^\beta\, dt < \infty$. (See the proof of Lemma 5.8, (5.56).) Integrating (5.65) over $[t, \infty)$, using (5.56), then yields

$$\nu(t) \sim m_1^{-\frac{1}{\alpha(\alpha+1)}} \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\,\psi_4(s)^\beta\, ds\right)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty,$$

and this combined with (5.59) determines the precise asymptotic behavior of $x(t)$ as follows:

$$x(t) \sim \psi_4(t)\, m_1^{\frac{1}{\alpha(\alpha+1)}}\, m_1^{-\frac{1}{\alpha(\alpha+1)}} \left(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\,\psi_4(s)^\beta\, ds\right)^{\frac{1}{\alpha-\beta}} = Y_3(t), \quad t\to\infty.$$

Thus the "only if" part of the Theorem 5.6 has been proved. ■

PROOF OF THE "IF" PART OF THEOREMS 5.4, 5.5 AND 5.6: Suppose that (5.45) or (5.47) or (5.49) holds. From Lemmas 5.6, 5.7 and 5.8 it is known that $Y_i(t)$, $i = 1, 2, 3$, defined by (5.46), (5.48) and (5.50) satisfy the asymptotic relation (5.53). We perform the simultaneous proof for $Y_i(t)$, $i = 1, 2, 3$ so the subscripts $i = 1, 2, 3$ will be deleted in the rest of the proof. By (5.52) there exists $T_0 > a$ such that

$$\int_{T_0}^{t}(t-s)\left(\frac{1}{p(s)}\int_{T_0}^{s}\int_{r}^{\infty} q(u)\,Y(u)^\beta\, du\, dr\right)^{\frac{1}{\alpha}} ds \le 2Y(t),\ \ t\ge T_0.$$

Let such a T_0 be fixed. We may assume that $Y(t)$ is increasing on $[T_0, \infty)$. Since (5.52) holds with $b = T_0$, there exists $T_1 > T_0$ such that

$$\int_{T_0}^{t}(t-s)\left(\frac{1}{p(s)}\int_{T_0}^{s}\int_{r}^{\infty} q(u)\,Y(u)^\beta\, du\, dr\right)^{\frac{1}{\alpha}} ds \ge \frac{Y(t)}{2},\ \ t\ge T_1.$$

Choose positive constants k and K such that

$$k^{1-\frac{\beta}{\alpha}} \le \frac{1}{2},\quad K^{1-\frac{\beta}{\alpha}} \ge 4,\quad 2k\,Y(T_1) \le K\,Y(T_0).$$

Considering the integral operator

$$\mathcal{H}y(t) = y_0 + \int_{T_0}^{t}(t-s)\left(\frac{1}{p(s)}\int_{T_0}^{s}\int_{r}^{\infty} q(u)\,y(u)^\beta\, du\, dr\right)^{\frac{1}{\alpha}} ds,\quad t\ge T_0,$$

where y_0 is a constant such that $k\,Y(T_1) \le y_0 \le \dfrac{K}{2}\,Y(T_0)$, we may verify that $\mathcal{H}$ is continuous self-map on the set

$$\mathcal{Y} = \{y \in C[T_0,\infty) : k\,Y(t) \le y(t) \le K\,Y(t),\ t \ge T_0\},$$

and that $\mathcal{H}$ sends $\mathcal{Y}$ into relatively compact subset of $C[T_0,\infty)$. Thus, $\mathcal{H}$ has a fixed point $y(t) \in \mathcal{Y}$, which generates a solution of equation (E) of type (II) satisfying above inequalities and thus yields that

$$0 < \liminf_{t\to\infty} \frac{y(t)}{Y(t)} \le \limsup_{t\to\infty} \frac{y(t)}{Y(t)} < \infty.$$

Denoting

$$L(t) = \int_a^t (t-s)\left(\frac{1}{p(s)}\int_a^s\int_r^\infty q(u)\,Y(u)^\beta\,du\,dr\right)^{\frac{1}{\alpha}} ds$$

and using $Y(t) \sim L(t),\ t\to\infty$ we get

$$0 < \liminf_{t\to\infty} \frac{y(t)}{L(t)} \le \limsup_{t\to\infty} \frac{y(t)}{L(t)} < \infty.$$

Then, proceeding exactly as in the proof of the "if" part of Theorems 5.1-5.3, with application of Lemma 5.5, we conclude that $y(t) \sim L(t) \sim Y(t),\ t\to\infty$. Therefore, $y(t)$ is a generalized regularly varying solution of (E) with requested regularity index and the asymptotic behavior (5.46), (5.48), (5.50) depending on if $q(t) \in \mathrm{RV}_R(\sigma)$ satisfies, respectively, (5.45) or (5.47) or (5.49). Thus, the "if part" of Theorems 5.4, 5.5 and 5.6 has been proved. ■

6. Corollaries

The final section is concerned with equation (E) whose coefficients $p(t)$ and $q(t)$ are regularly varying functions (in the sense of Karamata). It is natural to expect that such equation may possess intermediate solutions which are regularly varying. Our purpose here is to show that this new problem can be embedded in the framework of generalized regularly varying functions, so that the results of the preceding section provide full information about the existence and the precise asymptotic behavior of regularly varying solutions of (E).

We assume that $p(t)$ and $q(t)$ are regularly varying functions of indices η and σ, respectively, i.e.,

$$p(t) = t^\eta l_p(t), \qquad q(t) = t^\sigma l_q(t), \qquad l_p(t), l_q(t) \in \mathrm{SV}, \tag{6.1}$$

and seek regularly varying solutions $x(t)$ of (E) expressed in the from

$$x(t) = t^\rho l_x(t), \qquad l_x(t) \in \mathrm{SV}. \tag{6.2}$$

We begin by noticing that the condition (1.1) on $p(t)$ are satisfied if $\eta \leq \alpha$. In what follows we assume that $\eta < \alpha$, excluding the case $\eta = \alpha$ because of computational difficulty and the fact that integral $\int_a^\infty dt/p(t) = \int_a^\infty t^{-\frac{\eta}{\alpha}} l_p(t)^{-\frac{1}{\alpha}} dt$ might be either convergent or divergent. Then, it is easy to see that

$$R(t) \in \mathrm{RV}\left(\frac{\alpha+1-\eta}{\alpha}\right) \quad \Longrightarrow \quad R^{-1}(t) \in \mathrm{RV}\left(\frac{\alpha}{\alpha+1-\eta}\right).$$

Therefore, any regularly varying function $f(t) \in \mathrm{RV}(\lambda)$ is considered as a generalized regularly varying function of index $\alpha\lambda/(\alpha+1-\eta)$ with respect to $R(t)$, and conversely any generalized regularly varying function $f(t) \in \mathrm{RV}_R(\lambda^*)$ is regarded as an regularly varying function of index $\lambda = \lambda^*(\alpha+1-\eta)/\alpha$. It follows that

$$p(t) \in \mathrm{RV}_R\left(\frac{\alpha\,\eta}{\alpha+1-\eta}\right),\ q(t) \in \mathrm{RV}_R\left(\frac{\alpha\,\sigma}{\alpha+1-\eta}\right),\ x(t) \in \mathrm{RV}_R\left(\frac{\alpha\,\rho}{\alpha+1-\eta}\right).$$

Put

$$\eta^* = \frac{\alpha\eta}{\alpha+1-\eta}, \quad \sigma^* = \frac{\alpha\sigma}{\alpha+1-\eta}, \quad \rho^* = \frac{\alpha\rho}{\alpha+1-\eta}.$$

Note that (5.8) implies $\alpha^2 - \eta^* > 0$ and that the tree positive constants given by (5.9) are reduced to

$$m_1(\alpha,\eta^*) = \frac{\alpha}{\alpha+1-\eta}, \quad m_2(\alpha,\eta^*) = \frac{2\alpha-\eta}{\alpha+1-\eta}, \quad m_3(\alpha,\eta^*) = \frac{2\alpha-\eta+1}{\alpha+1-\eta}.$$

It turns out therefore that any type-(I) intermediate regularly varying solution of (E) is a member of one of the three classes

$$\mathrm{ntr}-\mathrm{SV}, \quad \mathrm{RV}(\rho),\ \rho \in (0,1), \quad \mathrm{ntr}-\mathrm{RV}(1),$$

while any type-(II) intermediate regularly varying solution belongs to one of the three classes

$$\mathrm{ntr}-\mathrm{RV}\left(\frac{2\alpha-\eta}{\alpha}\right), \quad \mathrm{RV}(\rho),\ \rho \in \left(\frac{2\alpha-\eta}{\alpha}, \frac{2\alpha-\eta+1}{\alpha}\right),$$
$$\mathrm{ntr}-\mathrm{RV}\left(\frac{2\alpha-\eta+1}{\alpha}\right).$$

Based on the above observations we are able to apply results for generalized regularly varying solutions created in Section 4 to the present situation, thereby establishing necessary and sufficient conditions for the existence of intermediate regularly varying solutions of (E) and determining the asymptotic behavior of all such solutions explicitly and accurately. First, we state the results on type-(I) intermediate solutions that can be derived as corollaries of Theorems 5.1, 5.2 and 5.3.

Theorem 6.1. *Assume that* $q(t) \in \mathrm{RV}(\sigma)$*,* $p(t) \in \mathrm{RV}(\eta)$ *and* $\eta < \alpha$ *. Equation* (E) *possess intermediate slowly varying solutions if and only if*

$$\sigma = \eta - 2\alpha - 2 \quad \textit{and} \quad \int_a^\infty t \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt = \infty. \tag{6.3}$$

Any such solution $x(t)$ *enjoys one and the same asymptotic behavior* $x(t) \sim X_1(t), t \to \infty$*, where* $X_1(t)$ *is given by* (5.14).

Theorem 6.2. *Assume that* $q(t) \in \mathrm{RV}(\sigma)$*,* $p(t) \in \mathrm{RV}(\eta)$ *and* $\eta < \alpha$ *. Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}(\rho)$ *with* $\rho \in (0,1)$ *if and only if*

$$\eta - 2\alpha - 2 < \sigma < \eta - \alpha - \beta - 2, \tag{6.4}$$

in which case ρ *is given by*

$$\rho = \frac{\sigma + 2\alpha - \eta + 2}{\alpha - \beta} \tag{6.5}$$

and any such solution $x(t)$ *enjoys one and the same asymptotic behavior*

$$x(t) \sim \left(\frac{t^{2\alpha+2}\, p(t)^{-1}\, q(t)}{\rho^\alpha\, (1-\rho)^\alpha\, (2\alpha - \eta - \alpha\,\rho)\, (2\alpha - \eta + 1 - \alpha\,\rho)} \right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{6.6}$$

Theorem 6.3. *Assume that* $q(t) \in \mathrm{RV}(\sigma)$*,* $p(t) \in \mathrm{RV}(\eta)$ *and* $\eta < \alpha$ *. Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}(1)$ *if and only if*

$$\sigma = \eta - \alpha - \beta - 2 \quad \textit{and} \quad \int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, s^\beta\, q(s)\, ds \right)^{\frac{1}{\alpha}} dt < \infty. \tag{6.7}$$

Any such solution $x(t)$ *enjoys one and the same asymptotic behavior* $x(t) \sim X_3(t), t \to \infty$*, where* $X_3(t)$ *is given by* (5.19).

Similarly, we are able to gain a through knowledge of type-(II) intermediate regularly varying solutions of (E) from Theorems 5.4, 5.5 and 5.6.

Theorem 6.4. *Assume that* $q(t) \in \mathrm{RV}(\sigma)$*,* $p(t) \in \mathrm{RV}(\eta)$ *and* $\eta < \alpha$ *. Equation* (E) *possess intermediate regularly varying solutions of index* $\frac{2\alpha-\eta}{\alpha}$ *if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2 \quad \textit{and} \quad \int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt = \infty. \tag{6.8}$$

The asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim Y_1(t)$*,* $t \to \infty$*, where* $Y_1(t)$ *is given by* (5.46).

Theorem 6.5. *Assume that* $q(t) \in \mathrm{RV}(\sigma)$, $p(t) \in \mathrm{RV}(\eta)$ *and* $\eta < \alpha$. *Equation* (E) *possess intermediate regularly varying solutions belonging to* $\mathrm{RV}(\rho)$ *with* $\rho \in \left(\frac{2\alpha-\eta}{\alpha}, \frac{2\alpha-\eta+1}{\alpha}\right)$ *if and only if*

$$\frac{\beta}{\alpha}\eta - 2\beta - 2 < \sigma < \frac{\beta}{\alpha}\eta - \frac{\beta}{\alpha} - 2\beta - 1, \tag{6.9}$$

in which case ρ *is given by* (6.5) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique formula*

$$x(t) \sim \left(\frac{t^{2\alpha+2}\, p(t)^{-1}\, q(t)}{\rho^{\alpha}\, (\rho-1)^{\alpha}\, (\alpha\,\rho - 2\alpha + \eta)\, (2\alpha - \eta + 1 - \alpha\,\rho)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{6.10}$$

Theorem 6.6. *Assume that* $q(t) \in \mathrm{RV}(\sigma)$, $p(t) \in \mathrm{RV}(\eta)$ *and* $\eta < \alpha$. *Equation* (E) *possess intermediate regularly varying solutions of index* $\frac{2\alpha-\eta+1}{\alpha}$ *if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - \frac{\beta}{\alpha} - 2\beta - 1 \quad \text{and} \quad \int_a^{\infty} q(t)\, \psi_4(t)^{\beta}\, dt < \infty. \tag{6.11}$$

The asymptotic behavior of any such solution $x(t)$ *is governed by the unique formula* $x(t) \sim Y_3(t)$, $t \to \infty$, *where* $Y_3(t)$ *is given by* (5.50).

Above corollaries combined with Theorems 2.1-2.4 enable us to describe in full details the structure of RV-solutions of equation (E) with RV-coefficients. Denote with $\mathcal{R}$ the set of all regularly varying solutions of (E) and define the subsets

$$\mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{RV}(\rho), \ \ \mathrm{tr} - \mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{tr} - \mathrm{RV}(\rho), \ \ \mathrm{ntr} - \mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{ntr} - \mathrm{RV}(\rho).$$

Corollary 6.1. *Let* $q(t) \in \mathrm{RV}(\sigma)$, $p(t) \in \mathrm{RV}(\eta)$ *and* $\eta < \alpha$.

(i) *If* $\sigma < \eta - 2\alpha - 2$, *or* $\sigma = \eta - 2\alpha - 2$ *and* $\int_a^{\infty} t \left(\frac{1}{p(t)} \int_t^{\infty} (s-t)\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}(0) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha-\eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha+1-\eta}{\alpha}\right).$$

(ii) *If* $\sigma = \eta - 2\alpha - 2$ *and* $\int_a^{\infty} t \left(\frac{1}{p(t)} \int_t^{\infty} (s-t)\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt = \infty$, *then*

$$\mathcal{R} = \mathrm{ntr} - \mathcal{R}(0) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha-\eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha+1-\eta}{\alpha}\right).$$

(iii) *If* $\sigma \in (\eta - 2\alpha - 2, \eta - \alpha - \beta - 2)$, *then*

$$\mathcal{R} = \mathcal{R}\left(\frac{\sigma+2\alpha-\eta+2}{\alpha-\beta}\right) \cup \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha-\eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha+1-\eta}{\alpha}\right).$$

(iv) *If* $\sigma = \eta - \alpha - \beta - 2$ *and* $\int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, s^\beta\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{ntr} - \mathcal{R}(1) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(v) *If* $\sigma = \eta - \alpha - \beta - 2$ *and* $\int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty (s-t)\, s^\beta\, q(s)\, ds\right)^{\frac{1}{\alpha}} dt = \infty$, *or* $\sigma \in \left(\eta - \alpha - \beta - 2, \frac{\beta}{\alpha}\eta - 2\beta - 2\right)$, *or* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ *and* $\int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(vi) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ *and* $\int_a^\infty t\, q(t)\, \psi_3(t)^\beta\, dt = \infty$, *then*

$$\mathcal{R} = \mathrm{ntr} - \mathcal{R}\left(\frac{2\alpha - \eta}{\alpha}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(vii) *If* $\sigma \in \left(\frac{\beta}{\alpha}\eta - 2\beta - 2, \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1\right)$, *then*

$$\mathcal{R} = \mathcal{R}\left(\frac{\sigma + 2\alpha - \eta + 2}{\alpha - \beta}\right) \cup \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(viii) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$ *and* $\int_a^\infty q(t)\, \psi_4(t)^\beta\, dt < \infty$, *then*

$$\mathcal{R} = \mathrm{tr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right) \cup \mathrm{ntr} - \mathcal{R}\left(\frac{2\alpha + 1 - \eta}{\alpha}\right).$$

(ix) *If* $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$ *and* $\int_a^\infty q(t)\, \psi_4(t)^\beta\, dt = \infty$, *or* $\sigma > \frac{\beta}{\alpha}\eta - 2\beta - \frac{\beta}{\alpha} - 1$, *then*

$$\mathcal{R} = \varnothing.$$

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Jelena V. Manojlović and Jelena Milošević
University of Niš
Faculty of Science and Mathematics
Department of Mathematics
Višegradska 33, 18000 Niš
Serbia
E-mail: jelenam@pmf.ni.ac.rs
jefimija@pmf.ni.ac.rs