

## GENERALIZATIONS OF THE HAHN-BANACH THEOREM REVISITED

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**Abstract.** In this paper, based on the extended versions of the Farkas lemma for convex systems introduced recently in [9], we establish an extended version of a so called Hahn-Banach-Lagrange theorem introduced by Stephan Simons in [22]. This generalized version of the Hahn-Banach-Lagrange theorem holds in locally convex Hausdorff topological vector spaces under a Slater-type constraint qualification condition and with the relaxing of the lower semi-continuity of some functions involved and the closedness of the constrained sets. The version, in turn, yields extended versions of the Mazur-Orlicz theorem, the sandwich theorem, and the Hahn-Banach theorem concerning extended sublinear functions. It is then shown that all the generalized versions of the Farkas lemma for cone-convex/sublinear-convex systems in [9] and the new extended Hahn-Banach-Lagrange theorem just obtained are equivalent together. A class of composite problems involving sublinear-convex mappings is considered at the end of the paper. Here the main results of the paper are applied to get a strong duality result and optimality conditions for the class of problems. Moreover, a formula for the conjugate of the supremum of a family (possibly infinite, not lower semi-continuous) of convex functions is then derived from the duality result to show the generality and the significance of the class of problems in consideration.

### 1. INTRODUCTION

In the recent years, an extended version of the Hahn-Banach theorem, called the Hahn-Banach-Lagrange theorem, was introduced by S. Simons [22-24] in a form that is suitable for dealing with many problems of Lagrange type and covers several well-known theorems in mathematics such as the sandwich theorem, the Mazur-Orlicz theorem.

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In [7], the authors developed two new versions of the Farkas lemma, one for cone-convex systems and another for systems which are convex with respect to an extended sublinear function. They also proved that these versions of the Farkas lemma are equivalent to each other and are also equivalent to an extended, topological version of the Hahn-Banach-Lagrange theorem that substantially extends the earlier version of S. Simons. Particularly, it extends the celebrated Hahn-Banach theorem to the case where the sublinear function appeared in this theorem possesses extended real values, the case where this celebrated theorem failed (see [7, Theorem 4.1]). All the results in [7] were established under very weak qualification conditions. However, in [7] the functions and mappings involved are assumed to be lower semi-continuous (lsc for short), the sets are assumed to be closed, conditions that are not always satisfied for many problems in optimization. It is worth observing that there have been several generalized versions of the Hahn-Banach theorem in variant circumstances (including non convex cases) and under different constraint qualification conditions (see, e.g., [1, 2, 7, 9, 19, 26]) or extended versions without convexity nor any constraint qualification condition such as the ones in [6]. It should be emphasized also that there have been many incorrect results on the extension of the Hahn-Banach theorem published in the last four decades (see [6, Section 1]).

In the recent paper [9], some new versions of the Farkas lemma were established under Slater-type constraint qualification conditions, which are stronger than the ones used in [7]. These versions, however, are liberated from the lower semi-continuity of the functions involved and the closedness of the constrained sets. It is shown in [9] that these new results are successfully applied to study a class of composite problems with the presence of sublinear-convex mappings. This class of problems possibly causes difficulties in treating since the non-convexity of the sublinear-convex mappings may lead to the appearance of the non-convexity of some parts of Lagrangian functions associated to such problems (see [9, Example 4.1]). However, the class subsumes a wide range of optimization problems such as general convex cone-constrained optimization problems [8, 11, 16], penalty problems associated to convex programming, nonlinearly constrained best approximation problems [17]. The study of this class of problems also leads to certain extensions of the Fenchel duality theorem and also some convex separation theorem in normed spaces as shown in [9].

The present paper can be considered as a counter part of [7] and also, a continuation of [9]. More concretely, following the approach proposed in [7] we established an extended version of the Hahn-Banach-Lagrange theorem (in short, HBL theorem) in locally convex Hausdorff topological vector spaces under a Slater-type constraint qualification condition and in the absence of the lower semi-continuity and the closedness of functions (except the sublinear one as in [9]) and constrained sets involved. Moreover, we show that the two versions of the Farkas lemma established in [9] and the extended Hahn-Banach-Lagrange theorem obtained in this paper are equivalent to-

gether. Several extended versions of the Mazur-Orlicz theorem, the sandwich theorem, and also the Hahn-Banach theorem for extended sublinear functions are also derived as consequences of this extended HBL theorem. It is shown that the results obtained can be applied to get a duality result and optimality conditions for some class of composite problems involving sublinear-convex mappings. Moreover, a formula for the conjugate of the supremum of a family (possibly infinite, not lower semi-continuous) of convex functions is then derived from the duality result to show the generality and the significance of the class of problems in consideration.

The organization of the paper is as follows: Section 2 presents notations and preliminaries that will be used in the sequel. This section also recalls two versions of the Farkas lemma for convex systems established in [9], which will be the base for obtaining the main results of this paper. Section 3 establishes the extended Hahn-Banach-Lagrange theorem with some of its corollaries. We show also in this section that the two versions of the Farkas lemma in [9] (Theorems 2.1 and 2.2) and the extended HBL theorem obtained in this paper are actually equivalent together. In Section 4, as consequences of the extended HBL theorem, we get extensions of the Mazur-Orlicz theorem, the sandwich theorem, and of the celebrated Hahn-Banach theorem with extended sublinear functions. The last section, Section 5, devotes to some applications of the main results obtained in Sections 3 and 4 to optimization problems and to convex analysis. Concretely, a strong duality result and necessary and sufficient conditions for optimality for a class of composite problems involving sublinear-convex mappings are established, and as a consequence of the duality result, a formula for the conjugate function of the supremum of a family (possibly infinite) of convex (not necessarily lower semi-continuous) functions is derived.

## 2. NOTATIONS AND PRELIMINARY RESULTS

### 2.1. Notations and preliminaries

In this paper we consider two locally convex Hausdorff topological vector spaces (l.c.H.t.v.s.)  $X$  and  $Y$ , with their topological dual spaces  $X^*$  and  $Y^*$ , respectively. For a set  $A \subset X$ ,  $\text{int}A$  denotes the **interior** of  $A$  in  $X$ . Moreover, the **indicator function** of the set  $A$  is denoted by  $i_A$ , i.e.,  $i_A(x) = 0$  if  $x \in A$ ,  $i_A(x) = +\infty$  if  $x \in X \setminus A$ .

Given  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , we denote by  $\text{dom}f := \{x \in X : f(x) < +\infty\}$  the effective domain of  $f$ , and say that  $f$  is proper if  $\text{dom}f \neq \emptyset$ . The epigraph of  $f$  is  $\text{epi}f := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$ .

The **Legendre-Fenchel conjugate** of the function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is the function  $f^* : X^* \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$  defined by

$$f^*(x^*) = \sup_{x \in X} (\langle x^*, x \rangle - f(x)), \quad \forall x^* \in X^*.$$

Let  $K$  be a closed convex cone in  $Y$ . Then  $K$  defines on  $Y$  a partial order by

$$y_1 \leq_K y_2 \text{ if } y_2 - y_1 \in K.$$

We add to  $Y$  a **greatest element** with respect to  $\leq_K$ , denoted by  $\infty_K$ ; *i.e.*, in the space  $Y^\bullet = Y \cup \{\infty_K\}$  we have  $y \leq_K \infty_K$  for every  $y \in Y^\bullet$ . The following conventions with respect to the operations in  $Y^\bullet$  will be made:  $y + \infty_K = \infty_K + y = \infty_K$  for all  $y \in Y^\bullet$ , and  $\alpha \infty_K = \infty_K$  if  $\alpha \geq 0$ . Moreover, we set  $\langle y^*, \infty_K \rangle = +\infty$  for all  $y^* \in Y^*$ . The dual cone of  $K$ , denoted by  $K^+$ , is defined by

$$K^+ := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0 \text{ for all } y \in K\}.$$

A mapping  $h : X \rightarrow Y^\bullet$  is called **(extended)  $K$ -convex** if

$$\begin{aligned} x_1, x_2 \in X, \mu_1, \mu_2 > 0, \mu_1 + \mu_2 = 1 \\ \Rightarrow h(\mu_1 x_1 + \mu_2 x_2) \leq_K \mu_1 h(x_1) + \mu_2 h(x_2), \end{aligned}$$

where " $\leq_K$ " is the binary relation (generated by  $K$ ) extended to  $Y^\bullet$ . By the **domain** of  $h$ , we mean the set  $\text{dom}h := \{x \in X : h(x) \in Y\}$ . Let

$$\text{epi}_K h := \{(x, y) \in X \times Y : y \in h(x) + K\}.$$

It is clear that  $h$  is  $K$ -convex if and only if  $\text{epi}_K h$  is convex. Moreover, for any  $y^* \in Y^*$  and  $g : X \rightarrow Y^\bullet$  the function  $y^* \circ g$  is defined (on  $X$ ) as follows:

$$(y^* \circ g)(x) = \begin{cases} \langle y^*, g(x) \rangle & \text{if } x \in \text{dom}g, \\ +\infty & \text{otherwise.} \end{cases}$$

Now let  $S : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ . Then  $S$  is called **(extended) sublinear** if it satisfies

$$S(y + y') \leq S(y) + S(y'), \text{ and } S(\lambda y) = \lambda S(y), \forall y, y' \in Y, \forall \lambda > 0.$$

We set, by convention,  $S(0_Y) = 0$ . Such a function  $S$  can be extended to  $Y^\bullet$  by setting  $S(\infty_K) = +\infty$  and following all other conventions related to the operations in  $Y^\bullet$  defined above.

It is worth observing that such an extended sublinear function  $S : Y \rightarrow \mathbb{R} \cup \{+\infty\}$  allows us to introduce in  $Y$  a **binary relation** which is reflexive and transitive:

$$(2.1) \quad y_1 \leq_S y_2 \text{ if } y_1 \leq_K y_2, \text{ where } K := \{y \in Y : S(-y) \leq 0\}.$$

This means that

$$(2.2) \quad y_1 \leq_S y_2 \iff S(y_1 - y_2) \leq 0, \forall y_1, y_2 \in Y.$$

Moreover, we also have [9]

$$(2.3) \quad S(y_1 - y_2) \leq 0 \iff S(y + y_1) \leq S(y + y_2) \forall y \in Y.$$

Similarly, the definition of the relation  $\leq_S$  (by 2.1) can be extended to  $Y^\bullet$  by the way as mentioned above. Concretely, we set  $y \leq_S \infty_K$  for all  $y \in Y^\bullet$ . Taking the convention  $S(\infty_K) = +\infty$  into account, the extension of the relation  $\leq_S$  to  $Y^\bullet$  is in accordance with (2.2)-(2.3).

Given an (extended) sublinear function  $S : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ , a mapping  $h : X \rightarrow Y^\bullet$  is said to be **(extended)  $S$ -convex** (or, convex with respect to a sublinear function, or sublinear-convex, see [7]) if for all  $x_1, x_2 \in X$ ,  $\mu_1, \mu_2 > 0$ ,  $\mu_1 + \mu_2 = 1$ , one has

$$h(\mu_1 x_1 + \mu_2 x_2) \leq_S \mu_1 h(x_1) + \mu_2 h(x_2).$$

The notion  $S$ -convex was used in [24] and was generalized to extended sublinear function in [7].

It is worth observing that, as mentioned in [23, Remark 1.10], " $S$ -convex can mean different things under different circumstances" such as, when  $Y = \mathbb{R}$ , if  $S(y) := |y|$ ,  $S(y) := y$ ,  $S(y) := -y$ , or  $S(y) = 0$ , respectively, then " $S$ -convex" means "affine", "convex", "concave" or "arbitrary", respectively.

It can be verified easily that if  $h$  is  $S$ -convex then  $h$  is  $K$ -convex with  $K := \{y \in Y : S(-y) \leq 0\}$ . Conversely, if  $h$  is  $K$ -convex with some convex cone  $K$  then  $h$  is (extended)  $S$ -convex with  $S = i_{-K}$  (see [7], [9]).

**2.2. Preliminary results: generalized Farkas lemma for convex systems**

In this subsection, we will recall some versions of extended Farkas lemma for cone-convex systems and for sublinear-convex systems introduced in [9]. These results relaxed the earlier versions of the Farkas lemma proved in [7] in the sense that all the assumptions on the lower semi-continuity (lsc for short) of functions and mappings, the closedness of sets involved are removed. However, the constraint qualification conditions used in [9] are the Slater-type ones, which are stronger than the "closedness-type conditions" used in [7].

**Theorem 2.1.** ([9, Farkas lemma for cone-convex systems]). *Let  $X, Y$  be l.c.H.t.v.s.,  $C$  be a nonempty convex subset of  $X$ ,  $K$  be a closed convex cone in  $Y$ ,  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex function and  $g : X \rightarrow Y^\bullet$  be a  $K$ -convex mapping, and  $\beta \in \mathbb{R}$ . Assume that the Slater condition holds, i.e.,*

$$(SC1) \quad \exists \bar{x} \in (dom f) \cap C \text{ such that } g(\bar{x}) \in -\text{int } K.$$

*Then the following statements are equivalent:*

- (i)  $x \in C, g(x) \in -K \implies f(x) \geq \beta$ ,
- (ii) *there exists  $y^* \in K^+$  such that*

$$f + y^* \circ g \geq \beta \text{ on } C.$$

**Remark 2.1.** As mentioned in [9], the equivalence between (i) and (ii) in Theorem 2.1 was proved in several other works (see the recent survey paper [8] and references therein) under different constraint qualification conditions (see, e.g., [7], [13], [15]-[17]). For these mentioned versions, it is always required the closedness of the convex set  $C$ , the lower semi-continuity of  $f$  and  $y^* \circ g$  for all  $y^* \in K^+$ . Theorem 2.1, on the other side, assumes the usual Slater constraint qualification (which may be a bit stronger than the ones used in the mentioned papers) to get the benefit of removing all the mentioned assumptions on functions, mappings and constrained sets.

**Theorem 2.2.** ([9, Farkas lemma for sublinear-convex systems]). *Let  $X, Y$  be l.c.H.t.v.s.,  $C$  be a nonempty convex subset of  $X$ ,  $S : Y \rightarrow \mathbb{R} \cup \{+\infty\}$  be an lsc sublinear function,  $g : X \rightarrow Y^\bullet$  be an  $S$ -convex mapping, and let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $\psi : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  be proper convex functions. Assume that the interior-type constraint qualification condition*

$$(SC2) \quad \exists \bar{a} \in C \cap (\text{dom} f), \exists \bar{\alpha} \in \mathbb{R} \text{ such that } (\bar{\alpha}, +\infty) \cap (\text{dom} \psi) \neq \emptyset$$

$$\text{and } g(\bar{a}) \in \text{int}\{y \in Y : S(y) \leq \bar{\alpha}\}$$

*holds. Then the following statements are equivalent:*

- (a)  $x \in C, \alpha \in \mathbb{R}, (S \circ g)(x) \leq \alpha \implies f(x) + \psi(\alpha) \geq 0,$
- (b) *there exist  $\gamma \geq 0$  and  $y^* \in Y^*$  such that  $y^* \leq \gamma S$  on  $Y$  and*

$$(2.4) \quad f + y^* \circ g \geq \psi^*(\gamma) \text{ on } C.$$

We close this section by recalling an elementary result that will be useful when applying the extended Hahn-Banach-Lagrange theorem to optimization problems in Section 6.

**Lemma 2.1.** ([9]). *Let  $X, Y$  be l.c.H.v.s.,  $S : Y \rightarrow \mathbb{R} \cup \{+\infty\}$  be an extended, lsc sublinear function, and  $g : X \rightarrow Y^\bullet$  be an  $S$ -convex mapping. The following assertions are true:*

- (i) *The function  $S \circ g$  is convex,*
- (ii) *If  $y^* \in Y^*$  and  $y^* \leq \gamma S$  for some  $\gamma \in \mathbb{R}_+$  then  $y^* \circ g$  is convex,*
- (iii) *If  $\kappa : Y \rightarrow \mathbb{R}$  is convex and  $S$ -increasing (i.e.,  $y_1, y_2 \in Y, y_1 \leq_S y_2$  implies  $\kappa(y_1) \leq \kappa(y_2)$ ) then  $\kappa \circ g$  is convex.*

### 3. EXTENDED HAHN-BANACH-LAGRANGE THEOREM

In this section we will establish a generalized version of the so-called Hahn-Banach-Lagrange theorem, introduced by S. Simons in [22] (see also [23], [24]) which was known as an extension of the celebrated Hahn-Banach theorem. Our results lead to

extensions of other fundamental results in mathematics such as extensions of the sandwich theorem, the Mazur-Orlicz theorem, and also of the Hahn-Banach theorem itself, which will be given in the next section. At the end of this section, it is shown that the Extended Hahn-Banach-Lagrange theorem, Theorem 3.1 and the two Farkas-type results in [9]: Theorems 2.1, 2.1 are equivalent together.

In the following, we keep maintaining the assumptions that  $X, Y$  are locally convex Hausdorff topological vector spaces and  $C$  is a nonempty convex subset (not necessarily closed) of  $X$ .

### 3.1. Extended Hahn-Banach-Lagrange theorem

**Theorem 3.1.** ([Extended Hahn-Banach-Lagrange theorem]). *Let  $S : Y \rightarrow \mathbb{R} \cup \{+\infty\}$  be an lsc sublinear function,  $g : X \rightarrow Y^\bullet$  be an  $S$ -convex mapping, and let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex function. Assume that the following condition holds:*

$$(SC3) \quad \exists \bar{a} \in C \cap (\text{dom} f), \exists \bar{\alpha} \in \mathbb{R} \text{ s.t. } g(\bar{a}) \in \text{int}\{y \in Y : S(y) \leq \bar{\alpha}\}.$$

*Then the following statements are equivalent:*

- (i)  $\inf_C [f + S \circ g] \in \mathbb{R}$ ,
- (ii) *there exists  $y^* \in Y^*$  such that  $y^* \leq S$  on  $Y$  and*

$$\inf_C [f + y^* \circ g] = \inf_C [f + S \circ g] \in \mathbb{R}.$$

*Proof.* Let  $\psi(\lambda) = \lambda$  for all  $\lambda \in \mathbb{R}$ . Note that  $\psi^*(\gamma) = 0$  if  $\gamma = 1$  and  $\psi^*(\gamma) = +\infty$  if  $\gamma \neq 1$ .

It is clear that only the implication [(i)  $\implies$  (ii)] needs to prove. The converse one is straightforward. Assume that (i) holds, i.e.,  $\beta := \inf_C [f + S \circ g] \in \mathbb{R}$ . Then  $f + S \circ g \geq \beta$  on  $C$ . It follows that if  $x \in C, \alpha \in \mathbb{R}, (S \circ g)(x) \leq \alpha$  then

$$f(x) + \psi(\alpha) = f(x) + \alpha \geq f(x) + (S \circ g)(x) \geq \beta.$$

Thus, if we set  $\tilde{f} := f - \beta$  then

$$x \in C, \alpha \in \mathbb{R}, (S \circ g)(x) \leq \alpha \implies \tilde{f}(x) + \psi(\alpha) \geq 0,$$

i.e., (a) in Theorem 2.2 holds where  $\tilde{f}$  plays the role of  $f$ . Moreover, the condition (SC2) holds in this case (see the definition of  $\psi$ ). Theorem 2.2 now yields the existence of  $\gamma \geq 0$  and  $y^* \in Y^*$  such that  $y^* \leq \gamma S$  on  $Y$  and

$$(3.1) \quad \tilde{f} + y^* \circ g \geq \psi^*(\gamma) \text{ on } C.$$

It follows from (SC3) that  $(\text{dom } f) \cap (\text{dom } g) \cap C \neq \emptyset$ , and hence there exists  $\bar{a} \in C$  such that

$$\tilde{f}(\bar{a}) + (y^* \circ g)(\bar{a}) < +\infty.$$

The last inequality and (3.1) yield  $\psi^*(\gamma) = 0$  and hence,  $\gamma = 1$ . Consequently, one has  $y^* \leq S$  and  $f(x) + (y^* \circ g)(x) \geq \beta = \inf_C [f + S \circ g]$  for all  $x \in C$ . Finally,

$$f(x) + (S \circ g)(x) \geq f(x) + (y^* \circ g)(x) \geq \inf_C [f + S \circ g] \text{ for all } x \in C.$$

Taking infimum over all  $x \in C$  in the last inequalities, we arrive at

$$\inf_C [f + y^* \circ g] = \inf_C [f + S \circ g] \in \mathbb{R}.$$

The proof is complete. ■

It is clear that even when the sublinear function  $S$  possesses only real values, the constraint qualification condition (SC3) does not hold automatically and this is quite clear because (SC3) depends on the topology in  $Y$ . It can not be satisfied in case the given topology is so coarse. Fortunately, for some wide classes of spaces (including all barreled spaces such as Frechet spaces, Banach spaces, Hilbert spaces, etc.) this condition holds for free, as we will see in the next lemma. We first recall that the topological vector space  $Y$  is a barreled space if every absorbing, convex, and closed subset of  $Y$  is a neighbourhood of  $0_Y \in Y$  (see [25, p.9]).

**Lemma 3.2.** ([Sufficient condition for (SC3)]). *Let  $X$  be a l.c.H.t.v.s.,  $C \subset X$  be a nonempty convex subset,  $Y$  is a nontrivial vector space,  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex function, and let  $S : Y \rightarrow \mathbb{R}$  be a sublinear function and  $g : X \rightarrow Y^\bullet$  be a mapping such that  $C \cap (\text{dom}f) \cap (\text{dom}g) \neq \emptyset$ . Assume that one of the following conditions holds:*

- (a)  $Y$  is a barreled space and  $S$  is lsc,
- (b)  $Y$  is equipped with the finest locally convex topology. <sup>1</sup>

Then the condition (SC3) holds.

*Proof.* (a) If  $Y$  is a barreled space then by [25, Theorem 2.2.20], the function  $S$  is continuous on  $\text{int}(\text{dom}S) = Y$ . Now take  $\bar{a} \in C \cap (\text{dom}f) \cap (\text{dom}g)$ . Then  $S(g(\bar{a})) \in \mathbb{R}$  and  $S(g(\bar{a})) < \bar{\alpha}$  for some  $\bar{\alpha} \in \mathbb{R}$ . We then have  $\bar{\alpha} > \inf_{y \in Y} S(y)$ . As  $S$  is a continuous sublinear function we have (see [25], p.147)

$$g(\bar{a}) \in \{y \in Y : S(y) < \bar{\alpha}\} = \text{int}\{y \in Y : S(y) \leq \bar{\alpha}\}.$$

This shows that (SC3) holds.

(b) In the case when  $Y$  is equipped with the finest locally convex topology, say  $\tau_Y$  (note that  $\tau_Y$  is Hausdorff), then the function  $p$  defined by

$$p(y) := \max\{S(y), S(-y)\}, \quad \forall y \in Y$$

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i.e., the weakest locally convex topology (also Hausdorff) in  $Y$  for which all the semi-norms on  $Y$  are continuous.



is continuous on  $Y$  with respect to  $\tau_Y$  as  $p$  is a semi-norm on  $Y$  [21]. Consequently, it is bounded from above on a neighbourhood of  $0_Y$ , which ensures that  $S$  is also bounded from above on this neighbourhood since  $S \leq p$ , and hence,  $S$  is continuous on  $\text{int}(\text{dom}S) = Y$  (see [25, Theorem 2.2.9]). The rest of the proof is the same as in (a). ■

We now get a version Hahn-Banach-Lagrange theorem for a real-valued sublinear function  $S$  on a barreled space  $Y$ . This result (Corollary 3.1) and the next one (Corollary 3.2), in some sense, justify the name "extended Hahn-Banach-Lagrange theorem" used for the Theorem 3.1.

**Corollary 3.1.** *Let  $X$  be l.c.H.t.v.s.,  $C$  be a nonempty convex subset of  $X$ ,  $Y$  be a barreled space,  $S : Y \rightarrow \mathbb{R}$  be an lsc sublinear function,  $g : X \rightarrow Y$  be an  $S$ -convex mapping, and let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex function satisfying  $C \cap \text{dom}f \neq \emptyset$ . Then the following statements are equivalent:*

- (i)  $\inf_C [f + S \circ g] \in \mathbb{R}$ ,
- (i) there exists  $y^* \in Y^*$  such that  $y^* \leq S$  on  $Y$  and

$$\inf_C [f + y^* \circ g] = \inf_C [f + S \circ g] \in \mathbb{R}.$$

*Proof.* It follows from Lemma 3.2 that the condition (SC3) holds under the assumption of this corollary. The conclusion now follows from Theorem 3.1. ■

The original Hahn-Banach-Lagrange theorem in [23] (for real-valued sublinear functions in vector spaces) now readily follows from Theorem 3.1 as shown in the next corollary.

**Corollary 3.2.** ([23, Algebraic Hahn-Banach-Lagrange theorem]). *Let  $Y$  be a nontrivial vector space,  $S : Y \rightarrow \mathbb{R}$  be a sublinear function,  $C$  be a nonempty convex subset of a vector space  $X$ ,  $g : C \rightarrow Y$  be an  $S$ -convex mapping, and  $f : C \rightarrow \mathbb{R}$  be a convex function. Then there exists a linear functional  $L$  on  $Y$  such that  $L \leq S$  on  $Y$  and*

$$\inf_C [f + L \circ g] = \inf_C [f + S \circ g]$$

*Proof.* Let  $\beta = \inf_C [f + S \circ g]$ . It is clear that  $\beta < +\infty$  since  $\text{dom}f \cap \text{dom}(S \circ g) = C \neq \emptyset$ . If  $\beta = -\infty$  then by [23, Lemma 1.2], there exists a linear functional  $L$  on  $Y$  such that  $L \leq S$  on  $Y$ . Then  $\inf_C [f + L \circ g] \leq \inf_C [f + S \circ g]$  which yields  $\inf_C [f + L \circ g] = \inf_C [f + S \circ g] = -\infty$  and the conclusion of the corollary holds.

So we now can assume that  $\beta \in \mathbb{R}$ . Let us equip  $X, Y$  with the finest locally convex Hausdorff topologies  $\tau_X, \tau_Y$ , respectively. Let also  $\tilde{f} : X \rightarrow \mathbb{R} \cup \{+\infty\}$  and

$\tilde{g} : X \rightarrow Y^\bullet$  be the functions defined by

$$\tilde{f}(x) := \begin{cases} f(x) - \beta & \text{if } x \in C, \\ +\infty & \text{else,} \end{cases}$$

$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } x \in C, \\ \infty_K & \text{else.} \end{cases}$$

Then  $\tilde{f}$  is a proper convex function and  $\tilde{g}$  is an  $S$ -convex mapping. Moreover,  $C \cap (\text{dom} \tilde{f}) \cap (\text{dom} \tilde{g}) = C \neq \emptyset$ . Lemma 3.2 ensures that (SC3) holds for  $\tilde{f}$ ,  $\tilde{g}$ ,  $C$ , and  $S$ . The conclusion now follows from Theorem 3.1 where  $\tilde{f}, \tilde{g}$  play the roles of  $f, g$ , respectively. The proof is complete.  $\blacksquare$

### 3.2. The equivalence between extended Farkas lemmas and the extended Hahn-Banach-Lagrange theorem

It is well-known that the convex Farkas lemma (also called Farkas-Minkovski lemma [14, 14F. Corollary 2]) is equivalent to the celebrated Hahn-Banach theorem [14, Section 14. I]. The proof of this fact was given in [14] not directly but using other intermediate results. Turning back to our situation, we have just established an extended version of Hahn-Banach theorem: The Hahn-Banach-Lagrange theorem, and in [9] the authors got two extended Farkas lemmas. A natural question arises: Are the three mentioned theorems equivalent together? Fortunately and also reasonably, the answer is affirmative. Moreover, it is even better when we are able to give a direct proof for these equivalences. Concretely, we have the claim:

**Claim.** *The two versions of extended Farkas lemma: Theorems 2.1, 2.2, and the extended Hahn-Banach-Lagrange theorem, Theorem 3.1, are equivalent to each other.*

*Proof.* (of the Claim) We observe firstly that:

- The implication *Theorem 2.1*  $\implies$  *Theorem 2.2* was proved in [9],
- The implication *Theorem 2.2*  $\implies$  *Theorem 3.1* was shown in Subsection 3.1, namely, the proof of Theorem 3.1.
- Consequently, we have

$$\textit{Theorem 2.1} \implies \textit{Theorem 2.2} \implies \textit{Theorem 3.1}.$$

To complete the proof we need to prove:

- [Theorem 3.1  $\implies$  Theorem 2.1] Let  $X, Y, C, K, f$ , and  $g$  be as in Theorem 2.1 and assume that the Slater condition (SC1) holds. Let  $S := i_{-K}$ . Then  $S$  is an lsc, extended sublinear function (since  $K$  is a closed convex cone). Moreover, the condition (SC3) follows from (SC1).

On the other hand, if (i) in Theorem 2.1 holds, *i.e.*,

$$x \in C, g(x) \in -K \implies f(x) \geq \beta.$$

This is equivalent to:

$$f(x) + (S \circ g)(x) \geq \beta, \forall x \in C,$$

or equivalently,  $\inf_C [f + S \circ g] \geq \beta$ . Now, Theorem 3.1 yields the existence of  $y^* \in Y^*$  such that  $y^* \leq S$  on  $Y$  and

$$\inf_C [f + y^* \circ g] \geq \beta,$$

which is equivalent to

$$f + y^* \circ g \geq \beta \text{ on } C.$$

It remains to prove that  $y^* \in K^+$  but this follows from the fact that  $y^* \leq S$  on  $Y$ , as one has

$$\langle y^*, y \rangle \leq S(y) = i_{-K}(y) = 0, \forall y \in -K.$$

This ensures that  $y^* \in K^+$  and the implication (i)  $\implies$  (ii) in Theorem 2.1 has been proved. The converse implication is trivial. The proof is complete. ■

#### 4. EXTENDED SANDWICH THEOREM, MAZUR-ORLICZ THEOREM AND HAHN-BANACH THEOREM

The extended Hahn-Banach-Lagrange theorem, Theorem 3.1, in the previous section will lead to extensions of some fundamental theorems in mathematics such as the Mazur-Orlicz theorem, the sandwich theorem, and the Hahn-Banach theorem. These results extend the ones in [23] to the cases where the sublinear functions (appeared in these theorems) are possibly assumed extended real-values.

**Corollary 4.1.** ([Extended sandwich theorem]). *Let  $X$  be an l.c.H.t.v.s.,  $S : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be an lsc, extended sublinear function, and let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex function satisfying  $-f \leq S$  on  $X$ . Assume that*

$$(4.1) \quad \exists \bar{\alpha} \in \mathbb{R} \text{ such that } (\text{dom} f) \cap \text{int}\{x \in X : S(x) \leq \bar{\alpha}\} \neq \emptyset.$$

*Then there exists  $L \in X^*$  such that  $-f \leq L \leq S$  on  $X$ .*

*Proof.* We will show that the conclusion of the corollary follows from Theorem 3.1. Let  $Y \equiv X$ ,  $C = X$ , and  $g(x) := x$  for all  $x \in X$ . Then the mapping  $g$  is  $S$ -convex. Now, (4.1) and the fact that  $-f \leq S$  on  $X$  ensure that  $\inf_C [f + S \circ g] \in \mathbb{R}$ . Moreover, (4.1) guarantees also (SC3) holds for our setting. Theorem 3.1 now yields the existence of a continuous functional  $L \in X^*$  with the properties:  $L \leq S$  on  $X$  and

$$(4.2) \quad \inf_X [f + L] = \inf_X [f + S].$$

As  $-f \leq S$  on  $X$  we have  $\inf_X [f + S] \geq 0$ . Combining this, (4.2) and the fact that  $L \leq S$  on  $X$ , one gets  $-f \leq L \leq S$  on  $X$ . ■

**Corollary 4.2.** ([Extended Mazur-Orlicz theorem]). *Let  $X$  be an l.c.H.t.v.s.,  $S : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be an lsc, extended sublinear function,  $C$  be a nonempty convex subset of  $X$ . Assume that  $\inf_C S > -\infty$  and that the following condition holds*

$$(4.3) \quad \exists \bar{\alpha} \in \mathbb{R} \text{ such that } \text{int}\{x \in X : S(x) \leq \bar{\alpha}\} \neq \emptyset.$$

*Then there exists  $L \in X^*$  such that  $L \leq S$  on  $X$  and*

$$\inf_C L = \inf_C S.$$

*Proof.* An application of Theorem 3.1 to the case where  $X \equiv Y$ ,  $g(x) := x$  for all  $x \in X$ , and  $f \equiv 0$ . ■

**Remark 4.2.** It is worth noticing that the algebraic versions of the sandwich theorem and of the Mazur-Orlicz theorem in [23] are consequences of Corollaries 4.1, and 4.2 (respectively) when taking  $S$  being a sublinear function with values in  $\mathbb{R}$  and the finest locally convex topologies being equipped to the corresponding vector spaces (see also Corollary 3.2).

It is well-known that the celebrated Hahn-Banach theorem fails in the case where the sublinear function appeared in this theorem (the function  $S$  in the next corollary) possesses extended real values (see [23, Remark 2.3]). An extended version of the Hahn-Banach theorem to the mentioned situation (i.e., with extended real valued sublinear functions) [9] will be found again as a direct consequence of Theorem 3.1.

**Corollary 4.3.** ([9, Extended Hahn-Banach theorem]). *Let  $X$  be an l.c.H.t.v.s.,  $S : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be an lsc, extended sublinear function,  $M$  be a subspace of  $X$ ,  $\phi : M \rightarrow \mathbb{R}$  be a linear function satisfying  $\phi \leq S$  on  $M$ . Assume that the following condition holds*

$$(4.4) \quad \exists \bar{\alpha} \in \mathbb{R} : M \cap \text{int}\{x \in X : S(x) \leq \bar{\alpha}\} \neq \emptyset.$$

*Then there exists  $L \in X^*$  such that  $L \leq S$  on  $X$  and  $L|_M = \phi$ , where  $L|_M$  denotes the restriction of  $L$  to the subspace  $M$ .*

*Proof.* The conclusion of the corollary follows directly from Theorem 3.1. Indeed, let  $Y = X$ ,  $C = M$ ,  $g : X \rightarrow X$  defined by  $g(x) := x$  for all  $x \in X$ , and  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$f(x) := \begin{cases} -\phi(x) & \text{if } x \in M, \\ +\infty & \text{else.} \end{cases}$$

Then  $g$  is an  $S$ -convex mapping while  $f$  is a proper, convex function. It is clear that (4.4) ensures that the condition (SC3) holds in this new setting. Observe also that (4.4)

and the fact that  $\phi \leq S$  on  $M$  entail that  $\inf_C [f + S \circ g] = \inf_M [-\phi + S] \in \mathbb{R}$ , which is (i) in Theorem 3.1. By this theorem, there exists  $L \in X^*$  such that  $L \leq S$  on  $X$  and

$$\inf_M [-\phi + L] = \inf_M [-\phi + S] \geq 0 \text{ (as } \phi \leq S \text{ on } M),$$

which means that  $\phi \leq L$  on  $M$ . As  $M$  is subspace and  $\phi, L$  are linear we have  $L|_M = \phi$ . The proof is complete. ■

5. COMPOSITE PROBLEMS INVOLVING SUBLINEAR-CONVEX MAPPINGS WITH APPLICATION

In this section, the extended Hahn-Banach-Lagrange theorem, Theorem 3.1, will be applied to some optimization problem. Concretely, we will see how the mentioned theorem is used to derive easily the duality result and optimality conditions for the class of composite problems involving sublinear-convex mappings. To illustrate the applications of these results, a conjugate formula for the supremum of a family of proper, convex (not necessarily lower semi-continuous) functions will be introduced at the end of the section.

5.1. Composite problems involving sublinear-convex mappings

Let  $X, Y$  be locally convex Hausdorff topological vector spaces,  $C$  be a nonempty convex subset of  $X$ , and  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex function. Let further  $S : Y \rightarrow \mathbb{R} \cup \{+\infty\}$  be an lsc, extended sublinear function, and  $g : X \rightarrow Y^\bullet$  be an  $S$ -convex mapping.

We consider a composite optimization problem involving the  $S$ -convex mapping  $g$  defined as follows:

$$(CP) \quad \inf_{x \in C} \{f(x) + (S \circ g)(x)\}.$$

It is worth observing that (CP) is a convex problem since, by Lemma 2.1, the composite function  $S \circ g$  is convex. However, it is not a convex composite problem since the sublinear-convex mapping  $g$  can be concave, or in general non-convex, depending on the extended sublinear function  $S$  (see Section 2), and so, the methods for convex composite problems (for instance, in [10], [11]) can not be applied to (CP).

We define the dual problem (D) of (CP) as:

$$(D) \quad \sup_{\substack{y^* \in Y^* \\ y^* \leq S}} \inf_{x \in C} \{f(x) + (y^* \circ g)(x)\}.$$

Observe that for any  $y^* \in Y^*$  satisfying  $y^* \leq S$  on  $Y$ , one has

$$f(x) + (y^* \circ g)(x) \leq f(x) + (S \circ g)(x) \quad \forall x \in C,$$

and so, by taking the infimum over all  $x \in C$ , we get

$$(5.1) \quad \inf_{x \in C} \{f(x) + (y^* \circ g)(x)\} \leq \inf_{x \in C} \{f(x) + (S \circ g)(x)\}.$$

As (5.1) holds for all  $y^* \in Y^*$  satisfying  $y^* \leq S$  we have  $\sup(D) \leq \inf(CP)$ , i.e., the weak duality holds.

**Theorem 5.1.** (Strong duality for (CP)). *Assume that  $\inf_{x \in C}(CP) \in \mathbb{R}$  and that (SC3) holds. Then the strong duality holds between (CP) and (D), i.e.,*

$$\max_{\substack{y^* \in Y^* \\ y^* \leq S}} \inf_{x \in C} \{f(x) + (y^* \circ g)(x)\} = \inf_{x \in C} \{f(x) + (S \circ g)(x)\}.$$

*Proof.* Assume that  $\inf_{x \in C}(CP) \in \mathbb{R}$ . Since (SC3) holds, it follows from Theorem 3.1 that there exists  $y^* \in Y^*$  such that  $y^* \leq S$  on  $Y$  and

$$\inf_{x \in C} [f(x) + (y^* \circ g)(x)] = \inf_{x \in C} [f(x) + (S \circ g)(x)].$$

Therefore,

$$\sup(D) \geq \inf_{x \in C} [f(x) + (y^* \circ g)(x)] = \inf_{x \in C} [f(x) + (S \circ g)(x)],$$

which, together with the weak duality, yields  $\sup(D) = \inf(CP)$ . Moreover, the supremum of the dual problem (D) is attained at  $y^* \in Y^*$ . The proof is complete. ■

**Remark 5.1.** It is worth noting that the conclusion of the Theorem 5.1 still holds if the assumption “ $\inf_{x \in C}(CP) \in \mathbb{R}$ ” is replaced by the following one:

$$(5.2) \quad \exists \bar{\beta} \in \mathbb{R} \text{ such that } 0_Y \in \text{int}\{y \in Y : S(y) \leq \bar{\beta}\}.$$

Indeed, as (SC3) holds we have  $\inf(CP) < +\infty$ . If  $\inf(CP) \in \mathbb{R}$ , the conclusion follows from Theorem 5.1. If  $\inf(CP) = -\infty$ , then by the weak duality one gets  $\sup(D) = \inf(CP) = -\infty$ , and in this case any  $y^* \in Y^*$  satisfying  $y^* \leq S$  on  $Y$  is a solution of (D). Note that such an  $y^* \in Y^*$  exists by the extended Hahn-Banach theorem, Corollary 4.3, applied to the case where  $X := Y$ ,  $M := \{0_Y\}$ ,  $\phi := 0$  on  $M$  (note that (4.4) is satisfied as (5.2) does).

We now establish the optimality condition for (CP) which is an easy consequence of Theorem 3.1.

**Theorem 5.2.** *Let  $\bar{x} \in C$  be a feasible point of (CP). Assume that (SC3) holds. Then the following statements are equivalent:*

- (i)  $\bar{x}$  is a solution (CP),
- (ii) there exists  $y^* \in Y^*$  such that  $y^* \leq S$  on  $Y$  and

$$(5.3) \quad \begin{cases} 0_{X^*} \in \partial(f + y^* \circ g + i_C)(\bar{x}) \\ (y^* \circ g)(\bar{x}) = (S \circ g)(\bar{x}). \end{cases}$$

*Proof.* • *Necessity.* If  $\bar{x}$  is a solution of (CP), then one has

$$f(x) + (S \circ g)(x) \geq f(\bar{x}) + (S \circ g)(\bar{x}) =: \beta, \quad \forall x \in C,$$

or, equivalently,

$$\inf_{x \in C} \{f(x) - \beta + (S \circ g)(x)\} \geq 0.$$

This and (SC3) ensures (see Remark 5.1) that  $\inf_{x \in C} \{f(x) - \beta + (S \circ g)(x)\} \in \mathbb{R}$ , which is nothing but (i) of Theorem 3.1 (with  $f - \beta$  playing the role of  $f$ ). By this theorem, there exists  $y^* \in Y^*$  with  $y^* \leq S$  and such that

$$\inf_{x \in C} [f(x) - \beta + (y^* \circ g)(x)] = \inf_{x \in C} [f(x) - \beta + (S \circ g)(x)] \geq 0.$$

The last equality shows that

$$(5.4) \quad f(x) + (y^* \circ g)(x) \geq \beta = f(\bar{x}) + (S \circ g)(\bar{x}), \quad \forall x \in C.$$

Substituting  $x = \bar{x}$  into (5.4), we get

$$(y^* \circ g)(\bar{x}) \geq (S \circ g)(\bar{x}),$$

which together with  $y^* \leq S$  (and hence,  $(y^* \circ g)(\bar{x}) \leq (S \circ g)(\bar{x})$ ), entails

$$(y^* \circ g)(\bar{x}) = (S \circ g)(\bar{x}).$$

Combining this with (5.4) we arrive at

$$f(x) + (y^* \circ g)(x) \geq f(\bar{x}) + (y^* \circ g)(\bar{x}) \quad \forall x \in C,$$

which shows that

$$0_{X^*} \in \partial(f + y^* \circ g + i_C)(\bar{x}).$$

Thus, (ii) is proved.

• *Sufficiency.* Assume that (ii) holds, i.e., there exists  $y^* \in Y^*$  with  $y^* \leq S$  and such that

$$(5.5) \quad \begin{cases} f(x) + (y^* \circ g)(x) \geq f(\bar{x}) + (y^* \circ g)(\bar{x}), \quad \forall x \in C, \\ (y^* \circ g)(\bar{x}) = (S \circ g)(\bar{x}). \end{cases}$$

Since  $y^* \in Y^*$  and  $y^* \leq S$ , we get from (5.5) that

$$f(x) + (S \circ g)(x) \geq f(\bar{x}) + (S \circ g)(\bar{x}) \quad \forall x \in C,$$

which means that  $\bar{x}$  is a solution (CP). The proof is complete. ■

**Remark 5.2.** Note that the (CP) problem is a special case of the problem (P1) in [9] with  $\psi$  being defined as  $\psi(\alpha) = \alpha$  for all  $\alpha \in \mathbb{R}$ . Despite of this, the model (CP) still covers a wide range of important classes of optimization problems such as nonlinearly constrained best approximation problems [17], penalty problems associated to convex programs [18] and many other models introduced in [3] and [5]. It is also worth observing that even the duality theorem and optimality conditions for (CP) in this section can be derived from the results in [9], here they are proved in a different and easier way, illustrating the direct use of the extended Hahn-Banach-Lagrange theorem (Theorem 3.1). We show the usefulness of the duality result for this class of problems in the next subsection where with a special choice of the sublinear-convex mapping  $g$ , the result leads to a conjugate formula for a supremum of a family of convex functions (see also Remarks 5.3, 5.4).

## 5.2. A conjugate formula for the supremum of a family of convex functions

The duality result for (CP) problem, Theorem 5.1, may lead to extensions of results in convex analysis due to the different choices of the mapping  $g$ . As an illustration, we introduce one of such choices of  $g$  which leads to a formula of the conjugate of the supremum of a (possibly infinite) family of convex (not necessarily lsc) functions on locally topological vector spaces.

Let  $X$  be a locally convex Hausdorff topological vector space,  $T$  be an arbitrary (possibly infinite) index set, and  $g_t : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be proper convex (not necessarily lsc) function for all  $t \in T$ . We consider the product space  $\mathbb{R}^T$  endowed with the product topology and denote by  $\mathbb{R}^{(T)}$  the space of real tuples  $\lambda = (\lambda_t)_{t \in T}$  with only finitely many  $\lambda_t \neq 0$ . Observe that  $\mathbb{R}^{(T)}$  is the topological dual of  $\mathbb{R}^T$  (see [12, 20]). We represent by  $\mathbb{R}_+^{(T)}$  the positive cone in  $\mathbb{R}^{(T)}$ , that is

$$\mathbb{R}_+^{(T)} = \{(\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} : \lambda_t \geq 0 \text{ for all } t \in T\}.$$

Note that  $\mathbb{R}_+^{(T)}$  is also the dual cone of the positive cone

$$\mathbb{R}_+^T := \{(\gamma_t)_{t \in T} \in \mathbb{R}^T : \gamma_t \geq 0 \text{ for all } t \in T\}$$

in the product space  $\mathbb{R}^T$ . The supporting set of  $\lambda \in \mathbb{R}^{(T)}$  is  $\text{supp } \lambda := \{t \in T : \lambda_t \neq 0\}$ , and

$$\lambda(u) := \sum_{t \in T} \lambda_t u_t = \sum_{t \in \text{supp } \lambda} \lambda_t u_t \quad \forall u = (u_t)_{t \in T} \in \mathbb{R}^T, \forall \lambda = (\lambda_t) \in \mathbb{R}^{(T)}.$$

The following formula for the conjugate of the supremum function  $\sup_{t \in T} g_t$  comes as a consequence of Theorem 5.1. This result can be considered as a counter part of the one in [4] (see Remark 5.3).



**Proposition 5.3.** *Let  $g_t : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex (not necessarily lsc) function for all  $t \in T$ . Assume that the following condition holds:*

$$(5.6) \quad \exists \bar{x} \in X, \exists \bar{\alpha} \in \mathbb{R} : (g_t(\bar{x}))_{t \in T} \in \text{int} \left\{ (y_t)_{t \in T} \in \mathbb{R}^T : \sup_{t \in T} y_t \leq \bar{\alpha} \right\}.$$

Then for any  $x^* \in X^*$  with  $\left(\sup_{t \in T} g_t\right)^*(x^*) \in \mathbb{R}$ , one has

$$(5.7) \quad \left(\sup_{t \in T} g_t\right)^*(x^*) = \min_{\substack{(\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t = 1}} \left(\sum_{t \in T} \lambda_t g_t\right)^*(x^*).$$

*Proof.* Let  $Y := \mathbb{R}^T$ ,  $g : X \rightarrow Y^\bullet := \mathbb{R}^T \cup \{\infty_{\mathbb{R}_+^T}\}$  be the mapping defined by

$$g(x) = \begin{cases} (g_t(x))_{t \in T} & \text{if } x \in \bigcap_{t \in T} \text{dom} g_t, \\ \infty_{\mathbb{R}_+^T} & \text{otherwise.} \end{cases}$$

Define  $S : \mathbb{R}^T \rightarrow \mathbb{R} \cup \{+\infty\}$  by  $S(y) := \sup_{t \in T} y_t$  for all  $y = (y_t)_{t \in T} \in \mathbb{R}^T$ .

• We first claim that  $S$  is an extended sublinear and lsc function. It is clear that  $S$  is an extended sublinear function. Moreover, for all  $y = (y_t)_{t \in T} \in \mathbb{R}^T$ ,  $S(y) = \sup_{t \in T} y_t = \sup_{t \in T} p_t(y)$ , where  $p_t : \mathbb{R}^T \rightarrow \mathbb{R}$ ,  $t \in T$  is the canonical function, i.e.,  $p_t(y) = y_t$ , which is continuous w.r.t. the product topology on  $\mathbb{R}^T$ . Therefore,  $S$  is an lsc function. We extended  $S$  to  $Y^\bullet := \mathbb{R}^T \cup \{\infty_{\mathbb{R}_+^T}\}$  by setting  $S(\infty_{\mathbb{R}_+^T}) = +\infty$ .

•  $g$  is a proper and  $S$ -convex mapping. Since (5.6) holds,  $\bigcap_{t \in T} \text{dom} g_t \neq \emptyset$  and hence,  $g$  is proper. Moreover,  $g$  is an  $S$ -convex mapping. Indeed, for any  $\mu_1, \mu_2 > 0$ ,  $\mu_1 + \mu_2 = 1$  and any  $x_1, x_2 \in X$  we will verify that

$$(5.8) \quad g(\lambda_1 x_1 + \lambda_2 x_2) \leq_S \lambda_1 g(x_1) + \lambda_2 g(x_2),$$

where “ $\leq_S$ ” is the binary relation associated to the extended sublinear function  $S$  defined by (2.2). We consider the following cases:

If  $x_1, x_2 \in \bigcap_{t \in T} \text{dom} g_t$ , then by the convexity of  $g_t$  for all  $t \in T$  and the fact that  $\bigcap_{t \in T} \text{dom} g_t$  is a convex set, one has

$$\begin{aligned} & S(g(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 g(x_1) - \lambda_2 g(x_2)) \\ &= \sup_{t \in T} \{g_t(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 g_t(x_1) - \lambda_2 g_t(x_2)\} \leq 0, \end{aligned}$$

which means that (5.8) holds.

If  $x_1 \notin \bigcap_{t \in T} \text{dom}g_t$  or  $x_2 \notin \bigcap_{t \in T} \text{dom}g_t$ , then (5.8) holds (note that  $y \leq_S \infty_{\mathbb{R}_+^T}$  for all  $y \in \mathbb{R}^T \cup \{\infty_{\mathbb{R}_+^T}\}$ ).

If  $\lambda_1 x_1 + \lambda_2 x_2 \notin \bigcap_{t \in T} \text{dom}g_t$ , then at least one of the two  $x_1$  and  $x_2$  does not belong to  $\bigcap_{t \in T} \text{dom}g_t$  (as  $\bigcap_{t \in T} \text{dom}g_t$  is a convex set), i.e.,  $g(x_1) = \infty_{\mathbb{R}_+^T}$  or  $g(x_2) = \infty_{\mathbb{R}_+^T}$ , or both. Hence, (5.8) also holds. Thus  $g$  is  $S$ -convex.

• We now apply Theorem 5.1 to the function  $S$ , the mapping  $g$  defined as above,  $C = X$ , and the function  $f : X \rightarrow \mathbb{R}$  with  $f(x) := -\langle x^*, x \rangle$  for all  $x \in X$ . Observe that the condition (5.6) ensures that (SC3) holds.

Now if  $x^* \in X^*$  satisfying  $\left(\sup_{t \in T} g_t\right)^*(x^*) \in \mathbb{R}$  then

$$\inf_{x \in X} \{f(x) + (S \circ g)(x)\} = \inf_{x \in X} \left\{ -\langle x^*, x \rangle + \sup_{t \in T} g_t(x) \right\} = -\left(\sup_{t \in T} g_t\right)^*(x^*) \in \mathbb{R}.$$

It now follows from Theorem 5.1 and the definition of the conjugate functions that

$$\begin{aligned} -\left(\sup_{t \in T} g_t\right)^*(x^*) &= \inf_{x \in X} \left\{ -\langle x^*, x \rangle + \sup_{t \in T} g_t(x) \right\} \\ (5.9) \qquad &= \max_{\substack{\lambda \in \mathbb{R}^{(T)} \\ \lambda(\cdot) \leq S(\cdot)}} \inf_{x \in X} \left\{ -\langle x^*, x \rangle + \sum_{t \in T} \lambda_t g_t(x) \right\} \\ &= -\min_{\substack{\lambda \in \mathbb{R}^{(T)} \\ \lambda(\cdot) \leq S(\cdot)}} \left( \sum_{t \in T} \lambda_t g_t \right)^*(x^*). \end{aligned}$$

To complete the proof, let us fix  $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)}$  such that  $\lambda(y) \leq S(y) = \sup_{t \in T} y_t$  for all  $y = (y_t)_{t \in T} \in \mathbb{R}^T$ . We will show that  $\lambda_t \geq 0$  for all  $t \in T$  and  $\sum_{t \in T} \lambda_t = 1$ .

Indeed, for any  $k \in T$ , take  $u_k = (u_t)_{t \in T}$  satisfying  $u_t = -1$  if  $t = k$  and  $u_t = 0$  for  $t \neq k$ . Then the relation  $\lambda(u_k) \leq S(u_k)$  yields  $\lambda_k \geq 0$  for all  $k \in T$ . Let further  $\bar{u} = (\bar{u}_t)_{t \in T}$ ,  $u^* = (u_t^*)_{t \in T} \in \mathbb{R}^T$  be such that  $\bar{u}_t = 1$  and  $u_t^* = -1$  for all  $t \in T$ . Then the relations  $\lambda(\bar{u}) \leq S(\bar{u})$  and  $\lambda(u^*) \leq S(u^*)$  give us  $\sum_{t \in T} \lambda_t = 1$ . The equality (5.7) now follows from (5.9). The proof is complete. ■

**Remark 5.3.** The conjugate formula (5.7) can be considered as a counter part of the one in [4, page 78] where it was proved by another technique and under a so-called closedness qualification condition and an extra assumption that  $g_t$  is lsc for all  $t \in T$ . Here, in the Proposition 5.3, we assume the condition (5.3), which is stronger than the closedness qualification condition in [4]. However, here the assumption on the lower semi-continuity of  $g_t$  for all  $t \in T$  is removed.

**Remark 5.4.** The procedure used in the proof of Proposition 5.3 paves the way for the study of some general minimax problems and this will be done somewhere else.

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