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AN HARDY ESTIMATE FOR COMMUTATORS OF PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. Let T be a pseudo-differential operator whose symbol belongs to the Hörmander class $S^m_{\rho,\delta}$ with $0 \leq \delta < 1, 0 < \rho \leq 1, \delta \leq \rho$ and $-(n+1) < m \leq -(n+1)(1-\rho)$. In present paper, we prove that if b is a locally integrable function satisfying

$$\sup_{\text{balls } B\subset\mathbb{R}^n} \frac{\log(e+1/|B|)}{(1+|B|)^\theta}\frac{1}{|B|}\int_B \left|f(x)-\frac{1}{|B|}\int_B f(y)dy\right|dx<\infty$$

for some $\theta \in [0,\infty)$, then the commutator $[b,T]$ is bounded on the local Hardy space $h^1(\mathbb{R}^n)$ introduced by Goldberg [9].

As a consequence, when $\rho = 1$ and $m = 0$, we obtain an improvement of a recent result by Yang, Wang and Chen [21].

1. INTRODUCTION

Let T be a Calderón-Zygmund operator. A classical result of Coifman, Rochberg and Weiss (see [6]), states that the commutator $[b,T]$, defined by $[b,T](f) = bTf - T(bf)$, is continuous on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, when $b \in BMO(\mathbb{R}^n)$. Unlike the theory of Calderón-Zygmund operators, the proof of this result does not rely on a weak type $(1,1)$ estimate for $[b,T]$. In fact, it was shown in [13, 18] that, in general, the linear commutator fails to be of weak type $(1,1)$ and fails to be of type (H^1,L^1), when b is in $BMO(\mathbb{R}^n)$. Instead, an endpoint theory was provided for this operator.

Let T be a pseudo-differential operator which is formally defined as

$$Tf(x)=\int_{\mathbb{R}^n}\sigma(x,\xi)e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi,\ f\in\mathcal{S}(\mathbb{R}^n),$$

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where $\hat{f}$ denotes the Fourier transform of f and $\sigma(x,\xi)$ is a symbol in the Hörmander class $S^m_{\rho,\delta}$ for some $m,\rho,\delta\in\mathbb{R}$ (see Section 2). Remark that T is a Calderón-Zygmund operator if the symbol $\sigma(x,\xi)$ satisfies some additional assumptions (cf. [12]). In analogy with the classical results in the setting of Calderón-Zygmund operators, when $b\in BMO(\mathbb{R}^n)$, the boundedness of $[b,T]$ on Lebesgue spaces $L^p(\mathbb{R}^n), 1<p<\infty$, have been established, see for example [2, 5, 16, 19]. We refer to [8, 11, 15] for some similar results in the setting of metric measure spaces. It is well-known that under certain conditions of m,ρ,δ, the operator T is bounded on $h^1(\mathbb{R}^n)$ and bounded on $bmo(\mathbb{R}^n)$ (cf. [9, 10, 22, 23]). A natural question is that can one find functions b for which $[b,T]$ is bounded on $h^1(\mathbb{R}^n)$? Recently, some endpoint results have obtained by Yang, Wang and Chen [21]. More precisely, in [21], the authors proved the following.

Theorem A. *Let $b\in LMO_\infty(\mathbb{R}^n)$. Suppose that T is a pseudo-differential operator with symbol $\sigma(x,\xi)$ in the Hörmander class $S^0_{1,\delta}$ with $0\le\delta<1$. Then,*

(i) $[b,T]$ is bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

(ii) $[b,T]$ is bounded from $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.

Our main theorem is as follows.

Theorem 1.1. Let $b\in LMO_\infty(\mathbb{R}^n)$. Suppose that T is a pseudo-differential operator with symbol $\sigma(x,\xi)$ in the Hörmander class $S^m_{\rho,\delta}$ with $0\le\delta<1, 0<\rho\le 1, \delta\le\rho$ and $-(n+1)<m\le-(n+1)(1-\rho)$. Then,

(i) $[b,T]$ is bounded from $h^1(\mathbb{R}^n)$ into itself.

(ii) $[b,T]$ is bounded from $bmo(\mathbb{R}^n)$ into itself.

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. For any measurable set $A\subset\mathbb{R}^n$, denote by $|A|$ the Lebesgue measure of A.

The paper is organized as follows. In Section 2, we give some notations and preliminaries about the spaces of BMO type, Hardy spaces and pseudo-differential operators. Section 3 is devoted to prove Theorem 1.1. An appendix will be given in Section 4.

2. Some Preliminaries and Notations

As usual, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of test functions on $\mathbb{R}^n$, $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions, and $C^\infty_c(\mathbb{R}^n)$ the space of C^∞-functions with compact support.

Let m,ρ and δ be real numbers. A symbol in the Hörmander class $S^m_{\rho,\delta}$ will be a smooth function $\sigma(x,\xi)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$, satisfying the estimates

$$|D^\alpha_x D^\beta_\xi \sigma(x,\xi)|\le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|},\quad \alpha,\beta\in\mathbb{N}^n.$$

We say that an operator T is a pseudo-differential operator associated with the symbol $\sigma(x,\xi) \in S^m_{\rho,\delta}$ if it can be written as

$$Tf(x) = \int_{\mathbb{R}^n} \sigma(x,\xi) e^{2\pi i x\cdot\xi} \hat{f}(\xi) d\xi, \; f \in \mathcal{S}(\mathbb{R}^n),$$

where $\hat{f}$ denotes the Fourier transform of f. Denote by $\mathscr{L}^m_{\rho,\delta}$ the class of pseudo-differential operators whose symbols are in $S^m_{\rho,\delta}$.

Let $0 < \rho \le 1$, $0 \le \delta < 1$ and $m \in \mathbb{R}$. It is well-known (see [10, Proposition 3.1]) that if $T \in \mathscr{L}^m_{\rho,\delta}$ with the symbol $\sigma(x,\xi)$, then T has the distribution kernel $K(x,y)$ given by

$$K(x,y) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^n} e^{2\pi i (x-y)\cdot\xi} \sigma(x,\xi)\psi(\epsilon\xi) d\xi,$$

where $\psi \in C^\infty_c(\mathbb{R}^n)$ satisfies $\psi(\xi) \equiv 1$ for $|\xi| \le 1$, the limit is taken in $\mathcal{S}'(\mathbb{R}^n)$ and does not depend on the choice of ψ.

The following useful estimates of the kernels are due to Alvarez and Hounie [1, Theorem 1.1].

Proposition 2.1. *Let $0 < \rho \le 1$, $0 \le \delta < 1$ and $T \in \mathscr{L}^m_{\rho,\delta}$. Then, the distribution kernel $K(x,y)$ of T is smooth outside the diagonal $\{(x,x) : x \in \mathbb{R}^n\}$. Moreover,*

(i) *For any $\alpha, \beta \in \mathbb{N}^n$, $N > 0$,*

$$\sup_{|x-y|\ge 1} |x-y|^N |D^\alpha_x D^\beta_y K(x,y)| \le C(\alpha,\beta,N).$$

(ii) *If $M \in \mathbb{N}$ satisfies $M + m + n > 0$, then*

$$\sup_{|\alpha+\beta|=M} |D^\alpha_x D^\beta_y K(x,y)| \le C(M) \frac{1}{|x-y|^{\frac{M+m+n}{\rho}}}, \quad x \ne y.$$

Here and in what follows, for any ball $B \subset \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we denote

$$f_B := \frac{1}{|B|} \int_B f(x) dx.$$

Let $0 \le \theta < \infty$. Following Bongioanni, Harboure and Salinas [3], we say that a locally integrable function f is in $BMO_\theta(\mathbb{R}^n)$, if

$$\|f\|_{BMO_\theta} := \sup_B \frac{1}{(1+r_B)^\theta |B|} \int_B |f(y) - f_B| dy < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. We then define

$$BMO_\infty(\mathbb{R}^n) = \cup_{\theta\ge 0} BMO_\theta(\mathbb{R}^n). \tag{2.1}$$

A locally integrable function f is said to belongs $LMO_\theta(\mathbb{R}^n)$ if

$$\|f\|_{LMO_\theta} := \sup_B \frac{\log(e+1/r_B)}{(1+r_B)^\theta}\frac{1}{|B|}\int_B |f(y)-f_B|dy < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. We define

$$LMO_\infty(\mathbb{R}^n) = \cup_{\theta\geq 0} LMO_\theta(\mathbb{R}^n). \tag{2.2}$$

Let ϕ be a Schwartz function satisfying $\int_{\mathbb{R}^n}\phi(x)dx = 1$. According to Goldberg [9], we define $h^1(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{h^1} := \|\mathfrak{m}_\phi f\|_{L^1} < \infty,$$

where $\mathfrak{m}_\phi f(x) := \sup_{0<t\leq 1}|f*\phi_t(x)|$ with $\phi_t(x) := t^{-n}\phi(t^{-1}x)$.

Given $1 < q \leq \infty$, a function a is called an (h^1, q)-atom related to the ball $B = B(x_0, r)$ if $r \leq 2$ and

(i) supp $a \subset B$,

(ii) $\|a\|_{L^q} \leq |B|^{1/q-1}$,

(iii) if $0 < r < 1$, then $\int_{\mathbb{R}^n} a(x)dx = 0$.

The following useful fact is due to Yang and Zhou [24, Proposition 3.2] (see also [4, 22, 23]).

Proposition 2.2. *Let $1 < q < \infty$. If T is a bounded linear operator on $L^q(\mathbb{R}^n)$ satisfying $\|Ta\|_{h^1} \leq C$ for all (h^1, q)-atoms a, then T can be extended to a bounded linear operator on $h^1(\mathbb{R}^n)$.*

It is well-known (see [9]) that the dual space of $h^1(\mathbb{R}^n)$ is $bmo(\mathbb{R}^n)$, namely, the space of locally integrable functions f such that

$$\|f\|_{bmo} := \sup_{B\in\mathcal{D}}\frac{1}{|B|}\int_B |f(x)-f_B|dx + \sup_{B\in\mathcal{D}^c}\frac{1}{|B|}\int_B |f(x)|dx < \infty,$$

where $\mathcal{D} = \{B(x_0,r) \subset \mathbb{R}^n : 0 < r < 1\}$ and $\mathcal{D}^c = \{B(x_0,r)\subset\mathbb{R}^n : r \geq 1\}$.

Denote by $vmo(\mathbb{R}^n)$ the closure of $C_c^\infty(\mathbb{R}^n)$ in the space $bmo(\mathbb{R}^n)$. Thanks to [7, Theorem 9], we have the following.

Theorem B. *The dual of the space $vmo(\mathbb{R}^n)$ is the space $h^1(\mathbb{R}^n)$.*

The following result is due to Hounie and Kapp [10, Theorem 4.1].

Theorem C. *Let $T \in \mathscr{L}^m_{\rho,\delta}$ with $0 \leq \delta < 1, 0 < \rho \leq 1, \delta \leq \rho$ and $m \leq -n(1-\rho)/2$. Then, T is bounded on $h^1(\mathbb{R}^n)$.*

3. Proof of Theorem 1.1

Here and in what follows, for any ball $B = B(x_0, r)$ and $k \in \mathbb{N}$, we denote

$$2^k B := B(x_0, 2^k r).$$

In order to prove Theorem 1.1, we need the following three technical lemmas.

Lemma 3.1. Let $1 \leq q < \infty$ and $0 \leq \theta < \infty$. Then,

(i) There exists a constant $C = C(q, \theta) > 0$ such that

$$\Big(\frac{1}{|2^k B|}\int_{2^k B}|f(y) - f_B|^q\Big)^{1/q} \leq Ck(1 + 2^k r)^{2\theta}\|f\|_{BMO_\theta}$$

for all $f \in BMO_\theta(\mathbb{R}^n)$, $k \geq 1$ and for all balls $B = B(x_0, r) \subset \mathbb{R}^n$.

(ii) There exists a constant $C = C(q, \theta) > 0$ such that

$$\Big(\frac{1}{|2^k B|}\int_{2^k B}|f(y) - f_B|^q\Big)^{1/q} \leq C\frac{k(1 + 2^k r)^{2\theta}}{\log\left(e + \frac{1}{2^k r}\right)}\|f\|_{LMO_\theta}$$

for all $f \in LMO_\theta(\mathbb{R}^n)$, $k \geq 1$ and for all balls $B = B(x_0, r) \subset \mathbb{R}^n$.

Lemma 3.2. Let $1 < q < \infty$ and $T \in \mathscr{L}^m_{\rho,\delta}$ with $0 < \rho \leq 1$, $0 \leq \delta < 1$, $-n-1 < m \leq -(n+1)(1-\rho)$. Then, for each $N > 0$, there exists $C = C(N) > 0$ such that

$$\|Ta\|_{L^q(2^{k+1}B \setminus 2^k B)} \leq C\frac{2^{-ck}}{(1 + 2^k r)^N}|2^k B|^{1/q-1}$$

holds for all (h^1, q)-atom a related to the ball $B = B(x_0, r)$ and for all $k = 1, 2, 3, \ldots$, where $c = \min\{1, \frac{1+n+m}{\rho}\}$.

Lemma 3.3. Let $T \in \mathscr{L}^m_{\rho,\delta}$ with $0 < \rho \leq 1$, $0 \leq \delta < 1, -n-1 < m \leq -(n+1)(1-\rho)$. Then the following two statements hold:

(i) If $b \in BMO_\theta(\mathbb{R}^n)$ for some $\theta \in [0, \infty)$, then there exists a constant $C > 0$ such that for every $(h^1, 2)$-atom a related to the ball $B = B(x_0, r)$,

$$\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{BMO_\theta}.$$

(ii) If $b \in LMO_\theta(\mathbb{R}^n)$ for some $\theta \in [0, \infty)$, then there exists a constant $C > 0$ such that for every $(h^1, 2)$-atom a related to the ball $B = B(x_0, r)$,

$$\log(e + 1/r)\|(b - b_B)Ta\|_{L^1} \leq C\|b\|_{LMO_\theta}.$$

The proof of Lemma 3.1 can be found in [14, Lemmas 5.3 and 6.6] as the special cases. Now let us give the proofs for Lemmas 3.2 and 3.3.

Proof of Lemma 3.2 If $1 < r \leq 2$, then for every $x \in 2^{k+1}B \setminus 2^k B$ and $y \in B = B(x_0, r)$, we have $|x-y| \geq |x-x_0| - |y-x_0| \geq 2^k r - r \geq 1$. Hence, by (i) of Proposition 2.1 and the Hölder inequality,

$$\begin{aligned} |Ta(x)| = \left|\int_{\mathbb{R}^n} K(x,y)a(y)dy\right| &\leq \int_B |K(x,y)||a(y)|dy \\ &\leq C\int_B \frac{1}{|x-y|^{N+n+1}}|a(y)|dy \\ &\leq C\frac{1}{|x-x_0|^{N+n+1}}\|a\|_{L^q}|B|^{1-1/q} \\ &\leq C\frac{1}{(2^k r)^{N+n+1}} \end{aligned}$$

for all $x \in 2^{k+1}B \setminus 2^k B$. This implies that

$$\begin{aligned} \|Ta\|_{L^q(2^{k+1}B\setminus 2^k B)} &\leq C\frac{1}{(2^k r)^{N+n+1}}|2^{k+1}B \setminus 2^k B|^{1/q} \\ &\leq C\frac{1}{2^k r}\frac{1}{(1+2^k r)^N}|2^k B|^{1/q-1} \\ &\leq C\frac{2^{-ck}}{(1+2^k r)^N}|2^k B|^{1/q-1}. \end{aligned}$$

In the case of $0 < r \leq 1$, we have $\int_B a(y)dy = 0$. Thus, for every $x \in 2^{k+1}B \setminus 2^k B$, from $1+n+m > 0$, Proposition 2.1(ii) yields

$$\begin{aligned} |Ta(x)| = \left|\int_{\mathbb{R}^n} K(x,y)a(y)dy\right| &\leq \int_B |K(x,y) - K(x,x_0)||a(y)|dy \\ &\leq C\int_B \frac{|y-x_0|}{|x-x_0|^{\frac{1+n+m}{\rho}}}|a(y)|dy \\ &\leq C\frac{r}{(2^k r)^{\frac{1+n+m}{\rho}}}, \end{aligned} \tag{3.1}$$

where we used the fact that $|x-\xi| \sim |x-x_0|$ if $\xi \in B$. Let us now consider the following two cases:

(a) If $(2^k - 1)r \geq 1$, then, by using Proposition 2.1(i), it is similar to the case $1 < r \leq 2$ that for every $x \in 2^{k+1}B \setminus 2^k B$,

$$\begin{aligned} |Ta(x)| &\leq C\frac{1}{(2^k r)^{N+n+\frac{1+n+m}{\rho}}} \\ &\leq C\frac{2^{-ck}}{(2^k r)^{N+n}}. \end{aligned}$$

Therefore,

$$\begin{aligned}\|Ta\|_{L^q(2^{k+1}B\setminus 2^kB)} &\le C\frac{2^{-ck}}{(2^kr)^{N+n}}|2^{k+1}B\setminus 2^kB|^{1/q}\\ &\le C\frac{2^{-ck}}{(1+2^kr)^N}|2^kB|^{1/q-1}.\end{aligned}$$

(b) If $(2^k-1)r<1$, then since $m\le -(n+1)(1-\rho)$, (3.1) yields

$$\begin{aligned}\|Ta\|_{L^q(2^{k+1}B\setminus 2^kB)} &\le C\frac{r}{(2^kr)^{\frac{1+n+m}{\rho}}}|2^{k+1}B\setminus 2^kB|^{1/q}\\ &\le C\frac{1}{2^k}\frac{1}{(2^kr)^n}|2^kB|^{1/q}\\ &\le C\frac{2^{-ck}}{(1+2^kr)^N}|2^kB|^{1/q-1},\end{aligned}$$

which ends the proof of Lemma 3.2. ■

Proof of Lemma 3.3. (i) Since $r\le 2$, by the Hölder inequality, the L^2-boundedness of T, Lemmas 3.1(i) and 3.2, we get

$$\begin{aligned}&\|(b-b_B)Ta\|_{L^1}\\ &= \|(b-b_B)Ta\|_{L^1(2B)}+\sum_{k=1}^{\infty}\|(b-b_B)Ta\|_{L^1(2^{k+1}B\setminus 2^kB)}\\ &\le \|b-b_B\|_{L^2(2B)}\|Ta\|_{L^2(2B)}+\sum_{k=1}^{\infty}\|b-b_B\|_{L^2(2^{k+1}B\setminus 2^kB)}\|Ta\|_{L^2(2^{k+1}B\setminus 2^kB)}\\ &\le C|2B|^{1/2}\|b\|_{BMO_\theta}\|a\|_{L^2}\\ &\quad +C\sum_{k=1}^{\infty}(k+1)(1+2^{k+1}r)^{2\theta}|2^{k+1}B|^{1/2}\|b\|_{BMO_\theta}\frac{2^{-ck}}{(1+2^kr)^{2\theta}}|2^kB|^{-1/2}\\ &\le C\|b\|_{BMO_\theta}+C\sum_{k=1}^{\infty}k2^{-ck}\|b\|_{BMO_\theta}\\ &\le C\|b\|_{BMO_\theta},\end{aligned}$$

where $c=\min\{1,\frac{1+n+m}{\rho}\}>0$.

(ii) Setting $\varepsilon=c/2$ with $c=\min\{1,\frac{1+n+m}{\rho}\}>0$, it is easy to check that there exists a positive constant $C=C(\varepsilon)$ such that

$$\log(e+kt)\le Ck^{\varepsilon}\log(e+t)$$

for all $k \geq 1, t > 0$. As a consequence, we get

$$\log\left(e + \frac{1}{r}\right) \leq C2^{\varepsilon k} \log\left(e + \frac{1}{2^k r}\right)$$

for all $k \geq 1$. This, together with the Hölder inequality, Lemmas 3.1(i) and 3.2, gives

$$\begin{aligned}
&\log(e+1/r)\|(b-b_B)Ta\|_{L^1} \\
&= \log(e+1/r)\|(b-b_B)Ta\|_{L^1(2B)} \\
&\quad + \sum_{k=1}^{\infty} \log(e+1/r)\|(b-b_B)Ta\|_{L^1(2^{k+1}B\setminus 2^kB)} \\
&\leq \log(e+1/r)\|b-b_B\|_{L^2(2B)}\|Ta\|_{L^2(2B)} \\
&\quad + \sum_{k=1}^{\infty} \log(e+1/r)\|b-b_B\|_{L^2(2^{k+1}B\setminus 2^kB)}\|Ta\|_{L^2(2^{k+1}B\setminus 2^kB)} \\
&\leq C\log(e+1/r)\frac{|2B|^{1/2}}{\log(e+1/(2r))}\|b\|_{LMO_\theta}\|a\|_{L^2} \\
&\quad + C\sum_{k=1}^{\infty} 2^{\varepsilon k}\log\left(e+\frac{1}{2^k r}\right)\frac{(k+1)(1+2^{k+1}r)^{2\theta}}{\log\left(e+\frac{1}{2^{k+1}r}\right)}|2^{k+1}B|^{1/2} \\
&\qquad \|b\|_{LMO_\theta}\frac{2^{-ck}}{(1+2^k r)^{2\theta}}|2^kB|^{-1/2} \\
&\leq C\|b\|_{LMO_\theta} + C\sum_{k=1}^{\infty} k2^{-\varepsilon k}\|b\|_{LMO_\theta} \\
&\leq C\|b\|_{LMO_\theta},
\end{aligned}$$

where we used the facts that $r \leq 2$ and $c = 2\varepsilon$. ∎

We are now ready to prove the main theorem.

Proof of Theorem 1.1. (i) Assume that $b \in LMO_\theta(\mathbb{R}^n)$ for some $\theta \in [0, \infty)$. By Proposition 2.2, it is sufficient to show that

$$\|[b, T](a)\|_{h^1} \leq C\|b\|_{LMO_\theta}$$

holds for all $(h^1, 2)$-atoms a related to the ball $B = B(x_0, r)$. To this ends, by Theorem C, we need to prove that

$$\|(b - b_B)a\|_{h^1} \leq C\|b\|_{LMO_\theta} \tag{3.2}$$

and

$$\|(b - b_B)Ta\|_{h^1} \leq C\|b\|_{LMO_\theta}. \tag{3.3}$$

Thanks to Theorem B, to establish (3.2) and (3.3), it is sufficient to prove that

$$\|f(b-b_B)a\|_{L^1} \le C\|b\|_{LMO_\theta}\|f\|_{bmo}$$

and

$$\|f(b-b_B)Ta\|_{L^1} \le C\|b\|_{LMO_\theta}\|f\|_{bmo}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. Indeed, since $f \in C_c^\infty(\mathbb{R}^n)$, it is well-known that $|f_B| \le C\log(e+1/r)\|f\|_{bmo}$. Therefore, by the Hölder inequality and Lemma 3.1(ii),

$$\begin{aligned}
&\|f(b-b_B)a\|_{L^1}\\
\le\ &\|(f-f_B)(b-b_B)a\|_{L^1} + \log(e+1/r)\|f\|_{bmo}\|(b-b_B)a\|_{L^1}\\
\le\ &\|(f-f_B)\chi_B\|_{L^4}\|(b-b_B)\chi_B\|_{L^4}\|a\|_{L^2}\\
&+\log(e+1/r)\|f\|_{bmo}\|(b-b_B)\chi_B\|_{L^2}\|a\|_{L^2}\\
\le\ &C|B|^{1/4}\|f\|_{BMO}|B|^{1/4}\|b\|_{LMO_\theta}|B|^{-1/2} + C\|f\|_{bmo}|B|^{1/2}\|b\|_{LMO_\theta}|B|^{-1/2}\\
\le\ &C\|b\|_{LMO_\theta}\|f\|_{bmo},
\end{aligned}$$

where we used the facts that supp $a \subset B$ and $r \le 2$.

By the Hölder inequality, the L^2-boundedness of T and Lemmas 3.1(ii) and 3.2,

$$\begin{aligned}
&\|(f-f_B)(b-b_B)Ta\|_{L^1}\\
=\ &\|(f-f_B)(b-b_B)Ta\|_{L^1(2B)} + \sum_{k=1}^{\infty}\|(f-f_B)(b-b_B)Ta\|_{L^1(2^{k+1}B\setminus 2^kB)}\\
\le\ &\|f-f_B\|_{L^4(2B)}\|b-b_B\|_{L^4(2B)}\|Ta\|_{L^2}\\
&+\sum_{k=1}^{\infty}\|f-f_B\|_{L^4(2^{k+1}B\setminus 2^kB)}\|b-b_B\|_{L^4(2^{k+1}B\setminus 2^kB)}\|Ta\|_{L^2(2^{k+1}B\setminus 2^kB)}\\
\le\ &C|2B|^{1/4}\|f\|_{BMO}|2B|^{1/4}\|b\|_{LMO_\theta}\|a\|_{L^2}\\
&+C\sum_{k=1}^{\infty}(k+1)|2^{k+1}B|^{1/4}\|f\|_{BMO}\frac{(k+1)(1+2^{k+1}r)^{2\theta}}{\log(e+\frac{1}{2^{k+1}r})}|2^{k+1}B|^{1/4}\\
&\|b\|_{LMO_\theta}\frac{2^{-ck}}{(1+2^kr)^{2\theta}}|2^kB|^{-1/2}\\
\le\ &C\|f\|_{BMO}\|b\|_{LMO_\theta},
\end{aligned}$$

where we used the facts that $r \le 2$ and $c = \min\{1, \frac{1+n+m}{\rho}\} > 0$. Combining this with (ii) of Lemma 3.3 allow to conclude that

$$\begin{aligned}
\|f(b-b_B)Ta\|_{L^1} &\le \|(f-f_B)(b-b_B)Ta\|_{L^1} + |f_B|\|(b-b_B)Ta\|_{L^1}\\
&\le C\|b\|_{LMO_\theta}\|f\|_{BMO} + C\log(e+1/r)\|f\|_{bmo}\|(b-b_B)Ta\|_{L^1}\\
&\le C\|b\|_{LMO_\theta}\|f\|_{bmo},
\end{aligned}$$

which completes the proof of (i).

(ii) By a symbol calculation (cf. [20, Proposition 0.3.B]), there exists $\sigma^* \in S^m_{\rho,\delta}$ such that T is the conjugate operator of T_{σ^*} whose symbol is σ^*. So (ii) can be viewed as a consequence of (i). This ends the proof of Theorem 1.1. ∎

4. Appendix

The following theorem yields the converse of Theorem 1.1. Although, it can be followed from Theorem 1.2 of Yang, Wang and Chen [21], however we also would like to give a proof here for completeness. Also, it should be pointed out that our approach is different from that of Yang, Wang and Chen.

Theorem 4.1. *Let b be a function in $BMO_\infty(\mathbb{R}^n)$. Suppose that $[b,T]$ is bounded on $h^1(\mathbb{R}^n)$ for all $T \in \mathscr{L}^m_{\rho,\delta}$ with $0 \leq \delta < 1, 0 < \rho \leq 1, \delta \leq \rho$ and $-(n+1) < m \leq -(n+1)(1-\rho)$. Then, $b \in LMO_\infty(\mathbb{R}^n)$.*

Proof. Assume that b is a function in $BMO_\theta(\mathbb{R}^n)$, for some $\theta \in [0,\infty)$, such that $[b,T]$ is bounded on $h^1(\mathbb{R}^n)$ for all $T \in \mathscr{L}^m_{\rho,\delta}$ with $0 \leq \delta < 1, 0 < \rho \leq 1, \delta \leq \rho$ and $-(n+1) < m \leq -(n+1)(1-\rho)$. Then, for any $r_j, j = 1, 2, \ldots, n$, the classical local Riesz transform of Goldberg (see [9] for details), the commutator $[b, r_j]$ is bounded on $h^1(\mathbb{R}^n)$ since $r_j \in \mathscr{L}^0_{1,0}$ (e.g. [10]). Therefore, for every $(h^1, 2)$-atom a related to the ball B, (i) of Lemma 3.3 yields

$$\begin{aligned}\|r_j((b-b_B)a)\|_{L^1} &\leq \|(b-b_B)r_j\|_{L^1} + C\|[b,r_j](a)\|_{h^1}\\ &\leq C\|b\|_{BMO_\theta} + C\|[b,r_j]\|_{h^1\to h^1}.\end{aligned}$$

By the local Riesz transforms characterization (see [9, Theorem 2]), we get

$$\|(b-b_B)a\|_{h^1} \leq C\left(\|b\|_{BMO_\theta} + \sum_{j=1}^n \|[b,r_j]\|_{h^1\to h^1}\right), \tag{4.1}$$

for all $(h^1,2)$-atom a related to the ball B, where the constant C is independent of b and a. We now prove that $b \in LMO_\theta(\mathbb{R}^n)$. To do this, since $b \in BMO_\theta(\mathbb{R}^n)$, it is sufficient to show that

$$\frac{\log(e+1/r)}{(1+r)^\theta}\frac{1}{|B|}\int_B |b(x)-b_B|dx \leq C\left(\|b\|_{BMO_\theta} + \sum_{j=1}^n \|[b,r_j]\|_{h^1\to h^1}\right)$$

holds for all $B = B(x_0, r)$ the ball in $\mathbb{R}^n$ satisfying $0 < r < 1/2$. Indeed, let f be the signum function of $b - b_B$ and $a = (2|B|)^{-1}(f - f_B)\chi_B$. Then it is easy to see that a is an $(h^1, 2)$-atom related to the ball B. We next consider the function

$$g_{x_0,r}(x) = \chi_{[0,r]}(|x-x_0|)\log(1/r) + \chi_{(r,1]}(|x-x_0|)\log(1/|x-x_0|).$$

Then, thanks to [17, Lemma 2.5], we have $\|g_{x_0,r}\|_{bmo} \leq C$. Moreover, it is clear that $g_{x_0,r}(b-b_B)a \in L^1(\mathbb{R}^n)$. By (4.1) and $bmo(\mathbb{R}^n) = (h^1(\mathbb{R}^n))^*$,

$$\begin{aligned}
\frac{\log(e+1/r)}{(1+r)^\theta}\frac{1}{|B|}\int_B |b(x)-b_B|dx &\leq 3\log(1/r)\frac{1}{|B|}\int_B |b(x)-b_B|dx \\
&= 6\left|\int_{\mathbb{R}^n} g_{x_0,r}(x)(b(x)-b_B)a(x)dx\right| \\
&\leq C\|g_{x_0,r}\|_{bmo}\|(b-b_B)a\|_{h^1} \\
&\leq C\left(\|b\|_{BMO_\theta} + \sum_{j=1}^n \|[b,r_j]\|_{h^1\to h^1}\right).
\end{aligned}$$

This proves that $b \in LMO_\theta(\mathbb{R}^n)$, moreover,

$$\|b\|_{LMO_\theta} \leq C\left(\|b\|_{BMO_\theta} + \sum_{j=1}^n \|[b,r_j]\|_{h^1\to h^1}\right). \qquad \blacksquare$$

Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$. A function a is called an h^1_b-atom related to the ball $B = B(x_0,r)$ if a is a (h^1,∞)-atom related to the ball $B = B(x_0,r)$, and when $0<r<1$, it also satisfies $\int_{\mathbb{R}^n} a(x)b(x)dx = 0$.

We define $h^1_b(\mathbb{R}^n)$ as the space of finite linear combinations of h^1_b-atoms. As usual, the norm on $h^1_b(\mathbb{R}^n)$ is defined by

$$\|f\|_{h^1_b} = \inf\left\{\sum_{j=1}^N \lambda_j a_j : f = \sum_{j=1}^N \lambda_j a_j\right\}.$$

Given $b \in BMO_\infty(\mathbb{R}^n)$, similar to a result of Pérez [18, Theorem 1.4], we find a subspace of $h^1(\mathbb{R}^n)$ for which $[b,T]$ is bounded from this space into $L^1(\mathbb{R}^n)$. In particular, we have:

Theorem 4.2. *Let* $b \in BMO_\infty(\mathbb{R}^n)$ *and* $T \in \mathscr{L}^m_{\rho,\delta}$ *with* $0 \leq \delta < 1, 0 < \rho \leq 1, \delta \leq \rho$ *and* $-(n+1) < m \leq -(n+1)(1-\rho)$. *Then,* $[b,T]$ *is bounded from* $h^1_b(\mathbb{R}^n)$ *into* $L^1(\mathbb{R}^n)$.

Proof. Assume that $b \in BMO_\theta(\mathbb{R}^n)$ for some $\theta \in [0,\infty)$. It is sufficient to prove that for all h^1_b-atom a related to the ball $B = B(x_0,r)$,

$$\|[b,T](a)\|_{L^1} \leq C\|b\|_{BMO_\theta}. \tag{4.2}$$

Indeed, we first remark that supp $((b-b_B)a) \subset B$ and $\|(b-b_B)a\|_{L^2} \leq C\|b\|_{BMO_\theta}|B|^{1/2}$ by (i) of Lemma 3.1. Moreover, if $0 < r < 1$, then $\int_{\mathbb{R}^n}(b(x)-b_B)a(x)dx =$

$\int_{\mathbb{R}^n} a(x)b(x)dx - b_B \int_{\mathbb{R}^n} a(x)dx = 0$. Therefore, $(b - b_B)a$ is a multiple of an $(h^1, 2)$-atom. So, by (i) of Lemma 3.3 and Theorem C, we get

$$\begin{aligned} \|[b,T](a)\|_{L^1} &\leq \|(b-b_B)Ta\|_{L^1} + \|T((b-b_B)a)\|_{L^1} \\ &\leq C\|b\|_{BMO_\theta}, \end{aligned}$$

which ends the proof of Theorem 4.2. ∎

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