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SPECIAL LAGRANGIAN 4-FOLDS WITH $SO(2) \rtimes S_3$-SYMMETRY IN COMPLEX SPACE FORMS

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Abstract. In this article we obtain a classification of special Lagrangian submanifolds in complex space forms subject to a pointwise $SO(2) \rtimes S_3$-symmetry on the second fundamental form. The algebraic structure of this form has been obtained by Marianty Ionel in [8]. However, the classification of special Lagrangian submanifolds in $\mathbb{C}^4$ having this $SO(2) \rtimes S_3$ symmetry in [8] is incomplete. In this paper we give a complete classification of such submanifolds, and extend the classification to special Lagrangian submanifolds of arbitrary complex space forms with a pointwise $SO(2) \rtimes S_3$-symmetry in the second fundamental form.

1. Introduction

A space (N, J, g) is called a Hermitian manifold with complex structure J and Riemannian metric g, if $g(JX, JY) = g(X, Y)$ for all X and Y. The $(0, 2)$-tensor $\omega(X, Y) = g(X, JY)$ is its symplectic form. If ω is closed, then (N, J, g) is said to be a Kähler manifold. In this case the Levi-Civita connection D of g satisfies $D\omega = 0$ as well, see [12]. A complex space form is a Kähler manifold for which the curvature tensor is given by

$$R(X, Y)Z = \epsilon\,(X \wedge Y + JX \wedge JY + 2g(X, JY)J)\,Z, \tag{1}$$

where ϵ is a real constant and $X \wedge Y$ is defined as

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Every complete, simply connected complex space form of dimension n with constant holomorphic sectional curvature 4ϵ is isometric to one of the following manifolds:

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Professor Franki Dillen passed away on April 17, 2013 unfortunately. Professor Franki Dillen's many friends, colleagues, students, and members of his family rejoice in his contributions in mathematics and society and mourn the passing of this wise and modest man.

(1) the standard complex space $\mathbb{C}^n$ when $\epsilon = 0$,

(2) the complex projective space $\mathbb{C}P^n(4\epsilon)$ when $\epsilon > 0$,

(3) the complex hyperbolic space $\mathbb{C}H^n(4\epsilon)$ when $\epsilon < 0$.

Because we consider submanifolds of a complex space form locally, we can restrict ourselves to those ambient spaces. By rescaling, we can even assume that $\epsilon = 0, 1, -1$.

A Lagrangian submanifold M of a Kähler manifold (N, J, g) is a submanifold such that ω vanishes identically on M and the (real) dimension of M is half the (complex) dimension of N, see [1]. This implies that J induces an orthogonal isomorphism between the tangent and the normal bundle on the submanifold. The Gauss formula is given by

$$D_X Y = \nabla_X Y + h(X, Y) = \nabla_X Y + JA(X, Y),$$

where $A = -Jh$ defines a symmetric $(1, 2)$-tensor on the submanifold, and the Weingarten formula is given by

$$D_X(JY) = J(\nabla_X Y) - A(X, Y).$$

It is easy to see that the cubic form C, defined by

$$C(X, Y, Z) = g(A(X, Y), Z)$$

is totally symmetric. We say that A is g-symmetric. For Lagrangian submanifolds of complex space forms, the equations of Gauss and Codazzi simplify to

$$R(X, Y)Z = \epsilon\,(X \wedge Y)\,Z + [A_X, A_Y]\,Z, \tag{2}$$

$$\nabla A \;\text{ is symmetric.} \tag{3}$$

The following theorem holds, see [4] and [6].

Theorem 1.1. *Suppose (M^n, g) is a Riemannian manifold equipped with a symmetric and g-symmetric $(1, 2)$-tensor A such that* (2) *and* (3) *are satisfied for some constant ϵ. Then for every point $p \in M$ there exists a neighborhood U and a Lagrangian isometric immersion $\phi : U \to N^{2n}(4\epsilon)$ into the complex space form $N^{2n}(4\epsilon)$ such that g and JA are induced as first and second fundamental form. Such an immersion is unique up to isometries of the ambient space.*

We focus on a particular form of A assuming that there is a pointwise G-symmetry of A (or equivalently of the cubic form C), where G is a subgroup of the special orthogonal group $SO(n)$. We say that A has pointwise G-symmetry at p if for all tangent vectors X, Y in p, and all $O \in G$ the relation $A(OX, OY) = OA(X, Y)$ holds (or equivalently $C(OX, OY, OZ) = C(X, Y, Z)$ for all X, Y, Z). Furthermore, we impose a minimality condition on A at p, so for every X at p, we assume that

$Tr(A_X) = 0$. Here, A_X is the linear map which maps Y to $A(X,Y)$. These manifolds are interesting, since in $\mathbb{C}^n$ the minimal Lagrangian submanifolds are locally precisely the special Lagrangian submanifolds of $\mathbb{C}^n$ as introduced by Harvey and Lawson [7]. If a special Lagrangian submanifold of $\mathbb{C}^3$ has G-symmetry at every point, for the same group G, then a classification result for the dimension equal to 3 has been obtained by Bryant [2]. An explicit classification for special (we also use the word "special" for "minimal" in case $\epsilon \neq 0$) Lagrangian 3-folds of complex space forms with pointwise symmetric cubic form is not yet done, but can be easily obtained from a similar classification for affine spheres in [14].

In the present paper we consider the 4-dimensional case. In particular we consider special Lagrangian 4-folds in complex space forms with pointwise symmetry on the cubic form. The shape of the $(1,2)$-tensor A, invariant under subgroups of $SO(4)$, has been described by M. Ionel in [8]. In the same article, the author classifies special Lagrangian 4-folds of $\mathbb{C}^4$ according to their symmetry groups. However, the classification in case the symmetry group is given by $SO(2) \rtimes S_3$ in that article is incomplete; several possible subcases including the most general one is omitted. In the present article, we give a complete classification of all special Lagrangian 4-folds in any complex space form having this particular symmetry. This settles the problem for $SO(2) \rtimes S_3$-symmetry for all ϵ. The classification for other symmetry groups remains open if $\epsilon \neq 0$.

The $SO(2) \rtimes S_3$-symmetry implies that A can be expressed as

$$\begin{aligned}
&A(X_1,X_1) = rX_1, && A(X_1,X_2) = -rX_2, A(X_1,X_3) = 0,\ A(X_1,X_4) = 0,\\
&A(X_2,X_1) = -rX_2, && A(X_2,X_2) = -rX_1, A(X_2,X_3) = 0,\ A(X_2,X_4) = 0,\\
&A(X_3,X_1) = 0, && A(X_3,X_2) = 0, \qquad A(X_3,X_3) = 0,\ A(X_3,X_4) = 0,\\
&A(X_4,X_1) = 0, && A(X_4,X_2) = 0, \qquad A(X_4,X_3) = 0,\ A(X_4,X_4) = 0,
\end{aligned} \tag{4}$$

in a well-chosen local orthonormal frame $\{X_1, X_2, X_3, X_4\}$. In this expression r is a strictly positive function. The $SO(2)$-symmetry is given by the free rotation in the $\{X_3, X_4\}$ plane and the S_3-symmetry is essentially obtained by rotations over an angle $2\pi/3$ in the $\{X_1, X_2\}$ plane and reflections in the $\{X_1, X_4\}$ plane. We can remark that the form of A is exactly that of Lagrangian submanifolds attaining equality in Chen's inequality, see [5] and [6].

In order to list the different possible subcases, we introduce distributions

$$\mathcal{N}_1 = span\{X_1, X_2\}, \quad \mathcal{N}_+ = span\{X_1, X_2, [X_1, X_2]\}, \quad \mathcal{N}_2 = span\{X_3, X_4\}.$$

We will see that $\mathcal{N}_2$ is always integrable. We obtain:

(1) If $\mathcal{N}_1 = \mathcal{N}_+$, then the submanifold is a double warped product $\mathbb{R} \times_f \mathbb{R} \times_g N^2$ where N^2 is a minimal Lagrangian submanifold in an appropriate space form.

(2) If $\mathcal{N}_1 \subsetneq \mathcal{N}_+$ and $\mathcal{N}_+$ is integrable, then the submanifold is a single warped product $\mathbb{R} \times_f N^3$ where N^3 is a special Lagrangian 3-fold with a pointwise S_3-symmetry of the second fundamental form in an appropriate space form.

(3) If the smallest integrable distribution containing $\mathcal{N}_1$ is TM, then for this final case, we do not obtain an explicit expression for the immersion, but we will rewrite the equations (7) as derivatives of certain complex functions using a complex coordinate on the submanifold. Equations similar to (10) in [3], but with a modified term, appear. Here, techniques will be used similar to those in [9].

When we consider the different cases, we will assume the defining conditions hold on an open neighborhood of the considered point.

2. Preliminaries

2.1. Complex space forms.

We briefly recall the basic properties of $\mathbb{C}^n$ and show how Lagrangian submanifolds of $\mathbb{C}P^n$ and $\mathbb{C}H^n$ can be lifted to subsets of $\mathbb{C}^{n+1}$.

Consider the complex vector space $\mathbb{C}^n$. Its elements can be written as n-tuples of complex numbers, so they are given as

$$\vec{z} = (z_1, \cdots, z_n), \quad z_j = x_j + iy_j, \quad x_j, y_j \in \mathbb{R}.$$

Through the map

$$\phi : \mathbb{C}^n \to \mathbb{R}^{2n} : (z_1, \cdots, z_n) \to (x_1, y_1, \cdots, x_n, y_n)$$

the space $\mathbb{C}^n$ is a real $2n$-dimensional manifold. The multiplication with the imaginary unit i translates to a linear map on $\mathbb{R}^{2n}$ given as

$$i\,(x_1, y_1, \cdots, x_n, y_n) = (-y_1, x_1, \cdots, -y_n, x_n)\,.$$

and its derivative J is given as

$$J\partial_{x_k} = \partial_{y_k},$$
$$J\partial_{y_k} = -\partial_{x_k}.$$

This squares to $-I$ and thus defines a complex structure on $\mathbb{C}^n$. On $\mathbb{C}^n$ there is also a Hermitian form given by

$$s(\vec{z}, \vec{w}) = \sum_{j=1}^{n} z_j \bar{w}_j = \sum_{j=1}^{n} (x_j u_j + y_j v_j) - i \sum_{j=1}^{n} (x_j v_j - y_j u_j).$$

The real part, which can be denoted as $\langle \vec{z}, \vec{w} \rangle$ defines the Euclidean scalar product on $\mathbb{R}^{2n}$ and induces a natural Riemannian metric on $\mathbb{C}^n$. We can see that J is an isometry and the induced Kähler form, which also coincides with the imaginary part of the Hermitian form, is closed. These structures make $\mathbb{C}^n$ into a flat Kähler manifold.

The manifold $\mathbb{C}P^n$ can be modeled as the quotient S^{2n+1}/S^1, where

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | \sum_{i=0}^{n} |z_i|^2 = 1\}.$$

The equivalence is given by

$$\vec{z} \sim \vec{w} \Leftrightarrow \exists \phi \in \mathbb{R}\ \forall j \in \{0, \cdots, n\} : z_j = e^{i\phi} w_j.$$

So the unit sphere S^{2n+1} is the preimage of the Hopf fibration

$$\pi : S^{2n+1} \to \mathbb{C}P^n : \vec{z} \to [\vec{z}]\,.$$

On $S^{2n+1} \subset \mathbb{C}^{n+1}$ the complex structure J induces a contact structure and the standard metric on $\mathbb{C}^{n+1}$ induces a Riemannian metric. The metric on $\mathbb{C}P^n$ that makes π a Riemannian submersion has constant holomorphic sectional curvature 4. An immersion $\phi : M \to S^{2n+1}$ is then said to be C-totally real or horizontal if $i\phi$ is orthogonal to the submanifold. It can be shown that every minimal C-totally real submanifold of S^{2n+1} can be projected onto a special Lagrangian submanifold of $\mathbb{C}P^n$ through π and conversely that a special Lagrangian submanifold in $\mathbb{C}P^n$ has a 1-parameter family of mutually isometric horizontal lifts as a minimal C-totally real submanifold in S^{2n+1}. So in order to classify special Lagrangian submanifolds in $\mathbb{C}P^n$, we can consider minimal C-totally real submanifolds in $S^{2n+1} \subset \mathbb{C}^{n+1}$, see [13]. For those submanifolds, the Gauss identity is given as

$$D_X Y = \nabla_X Y + JA(X, Y) - \langle X, Y\rangle \phi, \tag{5}$$

where D is the Levi Civita connection of $\mathbb{C}^{n+1}$.

Similarly, the space $\mathbb{C}H^n$ can be modeled as H^{2n+1}/S^1, where

$$H^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}_1^{n+1} | |z_0|^2 - \sum_{i=1}^{n} |z_i|^2 = 1\}.$$

The equivalence relationship determined by S^1 is the same as the one used in the projective space. The ambient space $\mathbb{C}_1^{n+1}$ is essentially the space $\mathbb{C}^{n+1}$, but equipped with the scalar product

$$\langle \vec{z}, \vec{w}\rangle_1 = \Re \left(\sum_{j=1}^{n} z_j \bar{w}_j - z_0 \bar{w}_0 \right).$$

The complex structure is still obtained through multiplication with the imaginary unit i and induces a Kähler structure on $\mathbb{C}_1^{n+1}$. This metric induces a Lorentzian metric on H^{2n+1} and a metric of constant holomorphic sectional curvature -4 on $\mathbb{C}H^n$. Similar to the projective case C-totally real submanifolds $\phi : M \to H^{2n+1}$ can be defined having $i\phi$ as a normal. Each minimal C-totally real submanifold corresponds to the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}H^n$. The Gauss identity is given as

$$D_X Y = \nabla_X Y + JA(X,Y) + \langle X, Y\rangle \phi, \tag{6}$$

where D is the Levi Civita connection of $\mathbb{C}_1^{n+1}$.

2.2. Structure equations.

We can return briefly to the equations (2) and (3). We can choose an orthogonal frame $\{X_1, X_2, X_3, X_4\}$ corresponding to (4) and define the components Γ_{ij}^k and A_{ij}^k as

$$\nabla_{X_i} X_j = \sum_{k=1}^{4} \Gamma_{ij}^k X_k,$$
$$A(X_i, X_j) = \sum_{k=1}^{4} A_{ij}^k X_k.$$

Then the equations (2) and (3) can be rewritten as

$$\begin{aligned} X_i\left(\Gamma_{jk}^l\right) - X_j\left(\Gamma_{ik}^l\right) &= \epsilon\left(\delta_{jk}\delta_i^l - \delta_{ik}\delta_j^l\right) + A_{jk}^r A_{ir}^l - A_{ik}^r A_{jr}^l \\ &\quad + \Gamma_{ik}^r \Gamma_{jr}^l - \Gamma_{jk}^r \Gamma_{ir}^l + \Gamma_{rk}^l\left(\Gamma_{ij}^r - \Gamma_{ji}^r\right), \end{aligned} \tag{7}$$

$$\begin{aligned} &X_i\left(A_{jk}^l\right) - X_j\left(A_{ik}^l\right) \\ &= \left(\Gamma_{ij}^r - \Gamma_{ji}^r\right) A_{rk}^l + \Gamma_{ik}^r A_{jr}^l - \Gamma_{jk}^r A_{ir}^l - \Gamma_{ir}^l A_{jk}^r + \Gamma_{jr}^l A_{ik}^r, \end{aligned} \tag{8}$$

where we have used the Einstein convention. We split the connection ∇ into its components and write

$$\begin{array}{rl|rl} \nabla_{X_1} X_1 &= a_1 X_2 + a_2 X_3 + a_3 X_4, & \nabla_{X_1} X_2 &= -a_1 X_1 + a_4 X_3 + a_5 X_4, \\ \nabla_{X_2} X_1 &= b_1 X_2 + b_2 X_3 + b_3 X_4, & \nabla_{X_2} X_2 &= -b_1 X_1 + b_4 X_3 + b_5 X_4, \\ \nabla_{X_3} X_1 &= c_1 X_2 + c_2 X_3 + c_3 X_4, & \nabla_{X_3} X_2 &= -c_1 X_1 + c_4 X_3 + c_5 X_4, \\ \nabla_{X_4} X_1 &= d_1 X_2 + d_2 X_3 + d_3 X_4, & \nabla_{X_4} X_2 &= -d_1 X_1 + d_4 X_3 + d_5 X_4, \end{array}$$

$$\begin{array}{rl|rl} \nabla_{X_1} X_3 &= -a_2 X_1 - a_4 X_2 + a_6 X_4, & \nabla_{X_1} X_4 &= -a_3 X_1 - a_5 X_2 - a_6 X_3, \\ \nabla_{X_2} X_3 &= -b_2 X_1 - b_4 X_2 + b_6 X_4, & \nabla_{X_2} X_4 &= -b_3 X_1 - b_5 X_2 - b_6 X_3, \\ \nabla_{X_3} X_3 &= -c_2 X_1 - c_4 X_2 + c_6 X_4, & \nabla_{X_3} X_4 &= -c_3 X_1 - c_5 X_2 - c_6 X_3, \\ \nabla_{X_4} X_3 &= -d_2 X_1 - d_4 X_2 + d_6 X_4, & \nabla_{X_4} X_4 &= -d_3 X_1 - d_5 X_2 - d_6 X_3. \end{array}$$

Equation (8) induces linear relations between the components, independent of the ambient space. The Gauss equations give further information about ∇ but use differential equations and depend on the ambient space form.

Lemma 2.1. *On a special Lagrangian submanifold M having a pointwise $SO(2) \rtimes S_3$-symmetry on the cubic form there exists a frame corresponding to* (4) *such that:*

$$
(9) \qquad
\begin{array}{c|c}
\nabla_{X_1} X_1 = a_1 X_2 + a_2 X_3 + a_3 X_4, & \nabla_{X_1} X_2 = -a_1 X_1 - b_2 X_3, \\
\nabla_{X_2} X_1 = b_1 X_2 + b_2 X_3, & \nabla_{X_2} X_2 = -b_1 X_1 + a_2 X_3 + a_3 X_4, \\
\nabla_{X_3} X_1 = \frac{b_2}{3} X_2, & \nabla_{X_3} X_2 = -\frac{b_2}{3} X_1, \\
\nabla_{X_4} X_1 = 0, & \nabla_{X_4} X_2 = 0, \\
\nabla_{X_1} X_3 = -a_2 X_1 + b_2 X_2 + a_6 X_4, & \nabla_{X_1} X_4 = -a_3 X_1 - a_6 X_3, \\
\nabla_{X_2} X_3 = -b_2 X_1 - a_2 X_2 + b_6 X_4, & \nabla_{X_2} X_4 = -a_3 X_2 - b_6 X_3, \\
\nabla_{X_3} X_3 = c_6 X_4, & \nabla_{X_3} X_4 = -c_6 X_3, \\
\nabla_{X_4} X_3 = d_6 X_4, & \nabla_{X_4} X_4 = -d_6 X_3.
\end{array}
$$

Furthermore, the derivatives of r are given by

$$(X_1 + iX_2)(r) = 3ir(a_1 + ib_1), \tag{10}$$

$$X_3(r) = ra_2, \tag{11}$$

$$X_4(r) = ra_3. \tag{12}$$

Proof. This is just a straightforward application of equation (8). For instance

$$(\nabla_{X_2} A)(X_1, X_1) = X_2(r) X_1 + 3rb_1 X_2 + rb_2 X_3 + rb_3 X_4,$$

$$(\nabla_{X_1} A)(X_2, X_1) = 3ra_1 X_1 - X_1(r) X_2 - ra_4 X_3 - ra_5 X_4.$$

Then the corresponding components of both derivatives are the same. Finally we can set $b_3 = 0$, by rotating the distribution $\mathcal{N}_2$ such that X_3 lies in the direction of $\nabla_{X_1} X_2$, projected on $\mathcal{N}_2$. ∎

It is interesting to note that $\mathcal{N}_2$ is an integrable distribution. The distribution $\mathcal{N}_1$ however is integrable if and only if $b_2 = 0$. Applying (7), we obtain the following result.

Lemma 2.2. *The equations* (7) *on our frame of choice induce a system of differential equations given by:*

$$(X_1 + iX_2)(a_2 - ib_2) = a_3(a_6 + ib_6), \tag{13}$$

$$X_3(a_2 + ib_2) = \epsilon + a_3^2 + (a_2 + ib_2)^2, \tag{14}$$

$$X_4(a_2 + ib_2) = a_3(a_2 + ib_2), \tag{15}$$

$$(X_1 + iX_2)(a_3) = -(a_2 - ib_2)(a_6 + ib_6), \tag{16}$$

$$X_3(a_3) = 0, \tag{17}$$

$$X_4(a_3) = a_3^2 + \epsilon, \tag{18}$$

$$X_1(b_6) - X_2(a_6) = -(a_1 a_6 + b_1 b_6), \tag{19}$$

$$X_3(a_6 + ib_6) = \frac{5}{3} i b_2 (a_6 + ib_6), \tag{20}$$

$$X_4(a_6 + ib_6) = 2a_3(a_6 + ib_6), \tag{21}$$

$$X_1(b_1) - X_2(a_1) = 2r^2 - (\epsilon + a_3^2) - \frac{5}{3} b_2^2 - a_2^2 - a_1^2 - b_1^2, \tag{22}$$

$$X_2(b_1) + X_1(a_1) = -\frac{2}{3} a_2 b_2, \tag{23}$$

$$3X_3(a_1) - X_1(b_2) = 3a_1 a_2 - 2b_1 b_2, \tag{24}$$

$$3X_3(b_1) - X_2(b_2) = 2b_2 a_1 + 3b_1 a_2, \tag{25}$$

$$X_4(a_1 + ib_1) = a_3(a_1 + ib_1) + \frac{b_2}{3}(a_6 + ib_6). \tag{26}$$

Proof. This is also a straightforward application of (7). For example:

$$\begin{aligned} X_1\left(\Gamma_{23}^1\right) - X_2\left(\Gamma_{13}^1\right) &= \Gamma_{13}^r \Gamma_{2r}^1 - \Gamma_{23}^r \Gamma_{1r}^1 + \Gamma_{r3}^1 \Gamma_{12}^r - \Gamma_{r3}^1 \Gamma_{21}^r \\ &= a_3 b_6, \\ X_1\left(\Gamma_{23}^2\right) - X_2\left(\Gamma_{13}^2\right) &= \Gamma_{13}^r \Gamma_{2r}^2 - \Gamma_{23}^r \Gamma_{1r}^2 + \Gamma_{r3}^2 \Gamma_{12}^r - \Gamma_{r3}^2 \Gamma_{21}^r \\ &= -a_3 a_6. \end{aligned}$$

Combining both equations using the usual complex notations leads to (13). The other equations are obtained in a similar way. ■

2.3. Warped Products.

In the analysis that follows, we will often encounter warped products of manifolds. When we consider a warped product of Riemannian manifolds (M_1, g_1) and (M_2, g_2) with warping function e^f, where $f : M_1 \to \mathbb{R}$, we get a Riemannian manifold $(M_1 \times M_2, g_f)$ where $M_1 \times M_2$ as a differentiable manifold is the product of M_1 and M_2 and the metric g_f is given as

$$g_f(X, Y) = g_1(X_1, Y_1) + e^{2f} g_2(X_2, Y_2),$$

where a vector field X is uniquely decomposed into a part X_1 tangent to M_1 and X_2 tangent to M_2. We denote this warped product as $M_1 \times_{e^f} M_2$. The following result can be obtained, see [10].

Theorem 2.1. *Consider a Riemannian manifold* (M, g) *with Levi-Civita connection* ∇ *and suppose that on a neighborhood of* $p \in M$ *there are orthogonal distributions* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *such that*

$$\forall X, Y \in \mathcal{D}_1 (\text{i.e. } X \text{ and } Y \text{ are sections of } \mathcal{D}_1) : \nabla_X Y \in \mathcal{D}_1,$$
$$\forall X, Y \in \mathcal{D}_2 : \nabla_X Y = \tilde{\nabla}_X Y + g(X, Y) H,$$

where $\tilde{\nabla}$ *is the projection of* ∇ *on* $\mathcal{D}_2$ *and* $H \in \mathcal{D}_1$. *Then there exists a function* $f : M \to \mathbb{R}$ *such that on a neighborhood of* p, M *can be written as* $M_1 \times_{e^f} M_2$, *where* M_i *is an integral manifold of* $\mathcal{D}_i$.
If furthermore $H = \lambda H_0$, *where* $\|H_0\| = 1$, *and* $X(\lambda) = 0$ *for every* $X \in \mathcal{D}_2$, *then* $f : M_1 \to \mathbb{R}$ *and* $H = -grad e^f$.

The first part of the theorem constructs a twisted product, the second part reduces this to a warped product. This will be useful in choosing coordinates, since the product structures allows for coordinates to be chosen on each factor separately. In particular, if $\dim(\mathcal{D}_1) = 1$, then any non vanishing vector field in $\mathcal{D}_1$ can be fixed as a useful coordinate vector field on M.

3. Submanifolds in $\mathbb{C}^4$

3.1. The case where $b_2 = 0$.

The assumption that X_3 lies along $\nabla_{X_1} X_2$ becomes redundant since the latter has no $\mathcal{N}_2$ component. Instead, we can choose X_3 in the direction of $\nabla_{X_1} X_1$, projected on $\mathcal{N}_2$. Hence without loss of generality we can assume that $a_3 = 0$. The equations (7) show that in this case either $a_2 = 0$ or $a_6 = b_6 = 0$. Furthermore calculating $\langle R(X_1, X_4) X_1, X_4 \rangle$ and $\langle R(X_2, X_3), X_2, X_4 \rangle$ in (7) yields

$$a_2 d_6 = \epsilon = 0,$$
$$a_2 c_6 = 0.$$

So if $a_2 \neq 0$, then we also obtain $c_6 = d_6 = 0$.

Theorem 3.1. *Consider* M *a special Lagrangian submanifold in* $\mathbb{C}^4$ *having a pointwise* $SO(2) \rtimes S_3$*-symmetry and an orthogonal frame corresponding to* (4). *Suppose that* $\mathcal{N}_1$ *is an integrable distribution and* $\nabla_{X_1} X_1$ *is nowhere contained within this distribution. Then* M *is locally the direct product of* $\mathbb{R}$ *and a Lagrangian cone over a minimal C-totally real submanifold of the unit sphere in* $\mathbb{C}^3$.

Proof. Taking into account every component that vanishes in (9), we find

$$\begin{array}{c|c}
\nabla_{X_1}X_1 = a_1X_2 + a_2X_3 & \nabla_{X_1}X_2 = -a_1X_1, \\
\nabla_{X_2}X_1 = b_1X_2 & \nabla_{X_2}X_2 = -b_1X_1 + a_2X_3, \\
\nabla_{X_3}X_1 = 0 & \nabla_{X_3}X_2 = 0, \\
\nabla_{X_4}X_1 = 0 & \nabla_{X_4}X_2 = 0, \\
\nabla_{X_1}X_3 = -a_2X_1 & \nabla_{X_1}X_4 = 0, \\
\nabla_{X_2}X_3 = -a_2X_2 & \nabla_{X_2}X_4 = 0, \\
\nabla_{X_3}X_3 = 0 & \nabla_{X_3}X_4 = 0, \\
\nabla_{X_4}X_3 = 0 & \nabla_{X_4}X_4 = 0.
\end{array}$$

We find that the distributions $span\{X_4\}$ and $span\{X_1, X_2, X_3\}$ satisfy the conditions for a warped product $\mathbb{R} \times_{e^f} N^3$. But $X_4(f) = 0$, hence f is a constant. M is a standard Riemannian product $\mathbb{R} \times N^3$ and its immersion can be written, up to an isometry as

$$F(t, x) = (t, \psi(x)), \ \psi : N^3 \to \mathbb{C}^4.$$

The immersion ψ is contained in the subspace orthogonal to both X_4 and JX_4, since they both are constant unit normals along N^3. Now it is also obvious that $span\{X_3\}$ and $\mathcal{N}_1$ satisfy the conditions for a warped product. So N^3 can be decomposed as $\mathbb{R} \times_{e^g} N^2$ and $X_3(g) = -a_2$. Then X_3 can be associated with a coordinate s on the manifold and it follows that

$$D_{X_3}X_3 = \frac{\partial^2 F}{\partial s^2} = 0 \Rightarrow F = As + B.$$

Both A and B are independent of (s, t). Calculating (7), one has

$$X_3(a_2) = \frac{\partial a_2}{\partial s} = a_2^2.$$

The solution of this equation, after a translation of the s-coordinate, is given as $a_2 = -\frac{1}{s}$. The derivatives of X_3 to X_1 and X_2 are

$$\begin{aligned}
D_{X_i}X_3 &= \frac{\partial F_* X_i}{\partial s} = A_* X_i \\
&= \frac{1}{s}X_i = A_* X_i + \frac{B_* X_i}{s} \\
&\Rightarrow B_* = 0.
\end{aligned}$$

So B is a constant vector along the submanifold and vanishes when applying a translation. It is easy to see that $X_3 = A$ and is orthogonal to $X_i = sA_*(X_i)$, for $i \in \{1, 2\}$. Hence everywhere along A, the position vector is orthogonal to the tangent space. Thus A has constant length. Calculating the other covariant derivatives yields for example for $i, j \in \{1, 2\}$ that

$$\begin{aligned} A_* X_i &= \frac{F_* X_i}{s}, \\ D_{X_i}(A_* X_j) &= \frac{D_{X_i}(F_* X_j)}{s} = A_*\left(\tilde{\nabla}_{X_i} X_j\right) + J A_*\left(K(X_i, X_j)\right) - \frac{1}{s^2}\delta_{ij}\phi. \end{aligned} \tag{27}$$

Here $\tilde{\nabla}$ is the connection restricted to N^2. Combining this with the other equations in (7), it follows that A is a C-totally real immersion in $S^5 \subset \mathbb{C}^3$. Furthermore, the components a_1 and b_1 have no other restrictions on them except satisfying the Gauss equations for a minimal C-totally real submanifold of S^5. This proves the theorem. ∎

The case $a_2 = 0$ was the only case that was studied in [8]. We can quote the following result from [8].

Theorem 3.2. *Consider M a special Lagrangian submanifold in $\mathbb{C}^4$ having a pointwise $SO(2) \rtimes S_3$-symmetry group on the cubic form and an orthogonal frame corresponding to* (4)*. Suppose that $\mathcal{N}_1$ is an integrable distribution and $\nabla_{X_1} X_1$ is contained within this distribution. Then M is locally the direct product of $\mathbb{R}^2$ and a holomorphic curve.*

Remark 3.1. As proved in [8], a special Lagrangian surface in $\mathbb{C}^2$, with complex coordinates $x_1 + iy_1$ and $x_2 + iy_2$ is a holomorphic curve in $\mathbb{C}^2$ with complex coordinates $x_1 - ix_2$ and $y_1 + iy_2$, and conversely.

3.2. The case where $b_2 \neq 0$.

Now the distribution $\mathcal{N}_1$ is no longer integrable. The simplest case one can hope for is that there is a 3-dimensional integrable distribution containing $\mathcal{N}_1$. Such a distribution should contain at least X_3 since

$$[X_1, X_2] \mod \mathcal{N}_1 \parallel X_3.$$

Using the fact that $b_2 \neq 0$, the equations (7) reduce (9) to

$$\begin{array}{c|c} \nabla_{X_1} X_1 = a_1 X_2 + a_2 X_3 + a_3 X_4 & \nabla_{X_1} X_2 = -a_1 X_1 - b_2 X_3, \\ \nabla_{X_2} X_1 = b_1 X_2 + b_2 X_3 & \nabla_{X_2} X_2 = -b_1 X_1 + a_2 X_3 + a_3 X_4, \\ \nabla_{X_3} X_1 = \frac{b_2}{3} X_2 & \nabla_{X_3} X_2 = -\frac{b_2}{3} X_1, \\ \nabla_{X_4} X_1 = 0 & \nabla_{X_4} X_2 = 0, \\ \nabla_{X_1} X_3 = -a_2 X_1 + b_2 X_2 + a_6 X_4 & \nabla_{X_1} X_4 = -a_3 X_1 - a_6 X_3, \\ \nabla_{X_2} X_3 = -b_2 X_1 - a_2 X_2 + b_6 X_4 & \nabla_{X_2} X_4 = -a_3 X_2 - b_6 X_3, \\ \nabla_{X_3} X_3 = a_3 X_4 & \nabla_{X_3} X_4 = -a_3 X_3, \\ \nabla_{X_4} X_3 = 0 & \nabla_{X_4} X_4 = 0. \end{array} \tag{28}$$

In this case, calculating $\langle R(X_1, X_3) X_2, X_4 \rangle$ and $\langle R(X_1, X_4), X_2, X_4 \rangle$ results in

$$\begin{aligned} b_2(c_6 - a_3) &= 0, \\ b_2 d_6 &= 0. \end{aligned}$$

This way, we obtain $c_6 = a_3$ and $d_6 = 0$. It is apparent that the condition that $\mathcal{N}_+$ is integrable is given by $a_6 + ib_6 = 0$. We consider this case first.

Theorem 3.3. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}^4$ with a pointwise $SO(2) \rtimes S_3$-symmetry, such that $\mathcal{N}_1$ is not an integrable distribution, but $\mathcal{N}_+$ is. Then the submanifold is locally either a direct product of $\mathbb{R}$ and a special Lagrangian submanifold in $\mathbb{C}^3$ having a pointwise S_3-symmetry or a Lagrangian cone over a minimal C-totally real submanifold of the unit sphere in $\mathbb{C}^4$ having a pointwise S_3-symmetry.*

Proof. We find according to (28) and (7) that $span\{X_4\}$ and $\mathcal{N}_+$ satisfy the conditions for a warped product. So M can be decomposed as $\mathbb{R} \times_{e^f} N^3$, where $X_4(f) = -a_3$. We can solve

$$X_4(a_3) = \frac{\partial a_3}{\partial t} = a_3^2.$$

This equation has 2 possible solutions.
First, we assume $a_3 = 0$. In this case M is simply the manifold $\mathbb{R} \times N^3$. Hence the immersion, up to isometry, can be given as

$$F(t, s, u, v) = (t, \phi(s, u, v)),$$

where ϕ is a 3-fold immersed in the subspace $\mathbb{C}^3$ orthogonal to X_4 and JX_4. Similar calculations as in (27) show that this can be any special Lagrangian submanifold in $\mathbb{C}^3$, given the presence of an S_3-symmetry in the second fundamental form.
The second solution, after a translation of t, is given by $a_3 = -\frac{1}{t}$. The calculations are similar to the case where $b_2 = 0$ and $a_2 \neq 0$. This gives the required result. ■

The last case in $\mathbb{C}^4$ is the one where there is no integrable distribution containing $\mathcal{N}_1$ other than the whole tangent bundle. In this case, we can no longer rely on an obvious warped product structure. We can attempt to introduce a set of independent coordinates and reduce (7) to a system of PDE's on $\mathbb{C}^4$ using as little functions as possible. We will consider a space form with arbitrary ϵ first and fill in $\epsilon = 0$ afterwards. We now use (13) to (26) to construct a coordinate frame from $\{X_1, X_2, X_3, X_4\}$. Since $\mathcal{N}_2$ is integrable, it is a good idea to choose $X_4 = T$ and $\mu X_3 = S$. Requiring that $[S, T] = 0$ implies that

$$[\mu X_3, X_4] = \mu\,[X_3, X_4] - X_4(\mu)X_3 = -\left(\mu a_3 + X_4(\mu)\right) X_3 = 0.$$

We can find such a μ by taking $\mu = \frac{1}{\sqrt{|\epsilon + a_3^2|}}$. The equation $a_3^2 + \epsilon = 0$ implies that a_3 is a constant and hence $(a_2 - ib_2)(a_6 + ib_6) = 0$. This will correspond to the integrability of either $\mathcal{N}_1$ or $\mathcal{N}_+$. Therefore μ is well defined.

Vector fields U and V can be sought such that every couple out of $\{S, T, U, V\}$ commutes. Such an attempt can be made, writing

$$U + iV = (\rho_1 - i\rho_2)\Big((X_1 + iX_2) + (\alpha_1 + i\beta_1)S + (\alpha_2 + i\beta_2)T\Big) \tag{29}$$

We rename the following expressions:

$$\rho = \rho_1 - i\rho_2,$$
$$\gamma_j = \alpha_j + i\beta_j \qquad j \in \{1, 2\}.$$

After calculating the Lie brackets of these four vector fields, the following conditions on the introduced functions make the vector fields commute:

$$(X_1 - iX_2)(\rho) = (b_1 + ia_1)\,\rho, \tag{30}$$
$$X_3(\rho) = -\left(a_2 + \frac{2}{3}ib_2\right)\rho, \tag{31}$$
$$X_4(\rho) = -a_3\rho, \tag{32}$$
$$X_2(\alpha_1) - X_1(\beta_1) = a_1\alpha_1 + b_1\beta_1 - \frac{2}{\mu}b_2, \tag{33}$$
$$X_3(\gamma_1) = \frac{1}{\mu^2}(X_1 + iX_2)(\mu) + \gamma_1\left(a_2 + \frac{2}{3}ib_2\right), \tag{34}$$
$$X_4(\gamma_1) = -\frac{1}{\mu}(a_6 + ib_6) + a_3\gamma_1, \tag{35}$$
$$X_2(\alpha_2) - X_1(\beta_2) = a_1\alpha_2 + b_1\beta_2, \tag{36}$$
$$X_3(\gamma_2) = (a_6 + ib_6) + \left(a_2 + \frac{2}{3}ib_2\right)\gamma_2, \tag{37}$$
$$X_4(\gamma_2) = a_3\gamma_2. \tag{38}$$

The following result can be obtained.

Lemma 3.1. *Suppose f and g are real valued functions on the manifold satisfying*

$$S(f) = 0, \quad T(f) = -1,$$
$$S(g) = -1, \quad T(g) = 0,$$

and defining

$$X_1(f) = \alpha_2, \quad X_2(f) = \beta_2,$$
$$X_1(g) = \alpha_1, \quad X_2(g) = \beta_1,$$

then the functions α_i and β_i obtained this way satisfy the conditions (33) *to* (38).

It is interesting to see that this way the vector fields

$$\tilde{U} = X_1 + \alpha_1 S + \alpha_2 T,$$
$$\tilde{V} = X_2 + \beta_1 S + \beta_2 T,$$

satisfy $\tilde{U}(f) = \tilde{U}(g) = \tilde{V}(f) = \tilde{V}(g) = 0$. Furthermore $\tilde{U}$ and $\tilde{V}$ are independent of one-another and they span the distribution which is the intersection of the kernel of df and dg. Note that this distribution is indeed 2-dimensional since both forms have a hyperplane as a kernel and these kernels do not coincide, since the 1-forms are linearly independent. Using the dimension theorem, they have a 2-dimensional intersection. Construction (29) is just a complex rotation of these two vector fields in that distribution. This way, it is clear that f and g serve as coordinates s and t conjugate to S and T.

Proof. Apply the relation

$$[X_i, X_j](f) = X_i X_j(f) - X_j X_i(f) = \nabla_{X_i} X_j(f) - \nabla_{X_j} X_i(f)$$

on both functions, using (28). ∎

A suitable function for f is easily found, since $S(a_3) = 0$. Let f be a function of a_3, then

$$X_4(f) = f'(\epsilon + a_3^2) = -1 \Leftrightarrow f' = -\frac{1}{\epsilon + a_3^2}.$$

Hence f can be given by

$$f = -\int \frac{1}{\epsilon + a_3^2} da_3.$$

This also determines γ_2 completely, since using (16) yields

$$\begin{aligned}\gamma_2 &= (X_1 + iX_2)(f) = f'(X_1 + iX_2)(a_3)\\ &= \frac{a_2 a_6 + b_2 b_6}{\epsilon + a_3^2} + i\frac{a_2 b_6 - b_2 a_6}{\epsilon + a_3^2}.\end{aligned}$$

As for the function g, the complex valued function $z = \mu(a_2 + ib_2)$ can be considered and calculations show

$$\begin{aligned}X_4(z) &= -\mu a_3(a_2 + ib_2) + \mu a_3(a_2 + ib_2) = 0,\\ S(z) &= \mu^2 \left(\epsilon + a_3^2 + (a_2 + ib_2)^2\right) = sign\left(\epsilon + a_3^2\right) + z^2.\end{aligned}$$

Rewriting $\tilde{\epsilon} = sign(\epsilon + a_3^2)$, we find that z is useful as long as $z^2 + \tilde{\epsilon} \neq 0$. When $\tilde{\epsilon} = +1$, we have that $z^2 = -\tilde{\epsilon}$ thus $a_2 = 0$ and $|b_2| = \sqrt{\epsilon + a_3^2}$. When $\tilde{\epsilon} = -1$, this occurs when $|a_2| = \sqrt{|\epsilon + a_3^2|}$ and $b_2 = 0$, resulting in $\mathcal{N}_1$ being integrable and the immersion can be given explicitly. We've already considered this for $\epsilon = 0$.

First we assume that $z^2 \neq -\tilde{\epsilon}$. Then the function g can be calculated as the real part of a function G of z given by

$$S(G) = (\tilde{\epsilon} + z^2)G' = -1 \Leftrightarrow G' = -\frac{1}{\tilde{\epsilon} + z^2}.$$

A function ρ still has to be constructed satisfying (30) to (32). Define a function H as

$$H = \rho^3 r(z^2 + \tilde{\epsilon})\,|\epsilon + a_3^2|.$$

Using (30) to (32) we find that $X(H) = 0$ for all vector fields X. We can scale such that $H = 1$. This defines a function ρ satisfying the necessary conditions.

Using the Frobenius theorem in [11], a coordinate frame on the submanifold is given by

$$\begin{aligned} X_1 + iX_2 &= \frac{U + iV}{\rho} - \gamma_1 S - \gamma_2 T, \\ X_3 &= \frac{1}{\mu} S, \\ X_4 &= T, \end{aligned}$$

where the coordinates (s, t, u, v) correspond to their coordinate vector fields (S, T, U, V). Remember that we assume that $\rho \neq 0$. We can describe the dependence of $a_6 + ib_6$ on (s, t) by writing

$$a_6 + ib_6 = \frac{k_3 + ik_4}{\rho}\sqrt{|a_3^2 + \epsilon|}\left(\bar{z}^2 + \tilde{\epsilon}\right)^{-\frac{1}{2}}.$$

The functions k_3 and k_4 depend solely on (u, v). This expression is obtained from (20) and (21). The rest of the equations in (7) can be rewritten and solved. Applying our method for $\epsilon = 0$, we find after a translation of the coordinates that

$$\begin{aligned} a_3 &= -\frac{1}{t}, \\ x = \frac{\sin(2s)}{\cos(2s) + \cosh(2k_1)} &\Rightarrow a_2 = -\frac{\sin(2s)}{t(\cos(2s) + \cosh(2k_1))}, \\ y = \frac{\sinh(2k_1)}{\cos(2s) + \cosh(2k_1)} &\Rightarrow b_2 = -\frac{\sinh(2k_1)}{t(\cos(2s) + \cosh(2k_1))}, \\ r &= \frac{e^{k_2}}{t\sqrt{\cos(2s) + \cosh(2k_1)}}. \end{aligned}$$

Here, we set $z = x + iy$. The functions k_1 and k_2 depend solely on (u, v). Then we can use (13) to find an expression for γ_1 in terms of the coordinates. Equation (10) can be used to find an expression for a_1 and b_1 in terms of the coordinates. We obtain

$$\gamma_1 = \frac{(k_3 + ik_4)\cos(s - ik_1) + t\left(\frac{\partial k_1}{\partial v} - i\frac{\partial k_1}{\partial u}\right)}{t\rho},$$

$$\begin{aligned}
a_1 = &\frac{2^{\frac{2}{3}} e^{\frac{2}{3}k_2}}{3t^3 \left(\cos(2s) + \cosh(2k_1)\right)^2} \big(t\big((\cos(2s) + \cosh(2k_1))(\rho_1 \frac{\partial k_2}{\partial v} + \rho_2 \frac{\partial k_2}{\partial u}) \\
&+ \sin(2s)(\rho_1 \frac{\partial k_1}{\partial u} - \rho_2 \frac{\partial k_1}{\partial v}) - \sinh(2k_1)(\rho_1 \frac{\partial k_1}{\partial v} + \rho_2 \frac{\partial k_1}{\partial u})\big) \\
&+ \sinh(2k_1)\left(\cos(s)\cosh(k_1)(k_4\rho_2 - k_3\rho_1) + \sin(s)\sinh(k_1)(k_4\rho_1 + k_3\rho_2)\right)\big), \\
b_1 = &\frac{2^{\frac{2}{3}} e^{\frac{2}{3}k_2}}{3t^3 \left(\cos(2s) + \cosh(2k_1)\right)^2} \big(t\big((\cos(2s) + \cosh(2k_1))(\rho_2 \frac{\partial k_2}{\partial v} - \rho_1 \frac{\partial k_2}{\partial u}) \\
&+ \sin(2s)(\rho_2 \frac{\partial k_1}{\partial u} + \rho_1 \frac{\partial k_1}{\partial v}) + \sinh(2k_1)(\rho_1 \frac{\partial k_1}{\partial u} - \rho_2 \frac{\partial k_1}{\partial v})\big) \\
&+ \sinh(2k_1)\left(\sin(s)\sinh(k_1)(k_4\rho_2 - k_3\rho_1) - \cos(s)\cosh(k_1)(k_4\rho_1 + k_3\rho_2)\right)\big).
\end{aligned}$$

Now every function on the submanifold is expressed in terms of (s, t, u, v), possibly indirectly through $\{k_1, k_2, k_3, k_4\}$. Demanding that the other Gauss equations are satisfied gives partial differential equations for k_i, given by

$$\begin{aligned}
\Delta k_1 &= -e^{-\frac{2k_2 - \ln(2)}{3}} \sinh(2k_1), \\
\Delta k_2 &= 3e^{-\frac{2k_2 - \ln(2)}{3}} \left(-e^{2k_2} + \cosh(2k_1)\right), \\
\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} &= -2\coth(k_1)\left(k_3 \frac{\partial k_1}{\partial u} + k_4 \frac{\partial k_1}{\partial v}\right), \\
\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} &= 2\tanh(k_1)\left(k_3 \frac{\partial k_1}{\partial v} - k_4 \frac{\partial k_1}{\partial u}\right).
\end{aligned} \tag{39}$$

These equations can be simplified by considering k_i as functions of a complex coordinate $u + iv$. Let us denote

$$\begin{aligned}
\partial &= \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \\
\bar{\partial} &= \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).
\end{aligned}$$

We define $\omega = \frac{2k_2 - \ln(2)}{3}$ and $\alpha = \frac{\sinh(2k_1)(k_3 - ik_4)}{2}$. We can rewrite (39) as

$$\begin{aligned}
\bar{\partial}\alpha &= -2\bar{\alpha}\partial k_1 csch(2k_1), \\
2\partial\bar{\partial}k_1 &= -\frac{1}{2}e^{-\omega}\sinh(2k_1), \\
\partial\bar{\partial}\omega &= -e^{2\omega} + \frac{1}{2}e^{-\omega}\cosh(2k_1).
\end{aligned}$$

A similar set of PDE can be found in [3].
Now we return to the case where $-1 = z^2$ and $\tilde{\epsilon} = 1$. We assume first that ϵ is not specified. In this case $a_2 = 0$, $b_2 = \pm\sqrt{\epsilon + a_3^2}$ and $S(z) = 0$, so z is insufficient to construct the function g. Equations (7) are reduced to

$$(X_1 - iX_2)(a_6 + ib_6) = -i(b_1 + ia_1)(a_6 + ib_6), \tag{40}$$

$$X_3(a_6 + ib_6) = \pm i\frac{5}{3}\sqrt{\epsilon + a_3^2}(a_6 + ib_6), \tag{41}$$

$$X_4(a_6 + ib_6) = 2a_3(a_6 + ib_6), \tag{42}$$

$$X_1(b_1) - X_2(a_1) = 2r^2 - \frac{8}{3}(\epsilon + a_3^2) - a_1^2 - b_1^2, \tag{43}$$

$$X_1(a_1) + X_2(b_1) = 0, \tag{44}$$

$$X_3(a_1 + ib_1) = \frac{i}{3}\left(a_3(a_6 + ib_6) \pm 2\sqrt{\epsilon + a_3^2}(a_1 + ib_1)\right), \tag{45}$$

$$X_4(a_1 + ib_1) = a_3(a_1 + ib_1) \pm \sqrt{\epsilon + a_3^2}\frac{(a_6 + ib_6)}{3}. \tag{46}$$

The first equation is obtained by applying integrability on a_3. Now we define

$$w = \frac{a_6 + ib_6}{\epsilon + a_3^2},$$

which after derivation gives

$$X_4(w) = -2a_3\frac{(a_6 + ib_6)}{\epsilon + a_3^2} + 2a_3\frac{a_6 + ib_6}{\epsilon + a_3^2} = 0,$$
$$S(w) = \pm i\frac{5}{3}w.$$

The resulting differential equation for a function G of w will be

$$S(G) = \pm G' i\frac{5}{3}w = -1 \Leftrightarrow G' = \pm i\frac{3}{5w}.$$

The solution is that G is a logarithm of w. We find that H defined by

$$H = w^2(\epsilon + a_3^2)^2\rho^5 r$$

is a constant and hence can be used to express ρ. We can thus solve w as

$$w = e^{k_1 \pm i\frac{5}{3}s} = e^{k_1}\left(\cos(\frac{5}{3}s) \pm i\sin(\frac{5}{3}s)\right).$$

Applying (11), (12), (18), (41) and (42) when $\epsilon = 0$ yields

$$\begin{aligned}
a_3 &= -\frac{1}{t},\\
a_6 &= e^{k_1}\frac{\cos(\frac{5}{3}s)}{t^2},\\
b_6 &= \pm e^{k_1}\frac{\sin(\frac{5}{3}s)}{t^2},\\
r &= \frac{e^{k_2}}{t}.
\end{aligned}$$

The equation (10) now gives $a_1 + ib_1$ immediately without going through γ_1 because of (11). The final unknown, γ_1, can then be determined using (40). When we pick $b_2 = a_3$, we obtain

$$\begin{aligned}
\gamma_{1+} &= \frac{-5e^{k_1+\frac{5si}{3}} + e^{\frac{2k_1+k_2}{5}+\frac{2si}{3}}t\left((\frac{\partial k_2}{\partial v} - 3\frac{\partial k_1}{\partial v}) - i(\frac{\partial k_2}{\partial u} - 3\frac{\partial k_1}{\partial u})\right)}{5t^2},\\
a_{1+} &= \frac{e^{k_1}\cos(\frac{5s}{3}) + e^{\frac{2k_1+k_2}{5}}t\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3t^2},\\
b_{1+} &= \frac{e^{k_1}\sin(\frac{5s}{3}) - e^{\frac{2k_1+k_2}{5}}t\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3t^2},
\end{aligned}$$

and for $b_2 = -a_3$ we obtain

$$\begin{aligned}
\gamma_{1-} &= \frac{-5e^{k_1-\frac{5si}{3}} + e^{\frac{2k_1+k_2}{5}-\frac{2si}{3}}t\left((3\frac{\partial k_1}{\partial v} - \frac{\partial k_2}{\partial v}) - i(3\frac{\partial k_1}{\partial u} - \frac{\partial k_2}{\partial u})\right)}{5t^2},\\
a_{1-} &= \frac{-e^{k_1}\cos(\frac{5s}{3}) + e^{\frac{2k_1+k_2}{5}}t\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3t^2},\\
b_{1-} &= \frac{e^{k_1}\sin(\frac{5s}{3}) - e^{\frac{2k_1+k_2}{5}}t\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3t^2}.
\end{aligned}$$

Equations(33) and (43) result in restrictions on the functions k_1 and k_2 of (u, v) given by

(47)
$$\begin{aligned}
\Delta k_1 &= e^{-\frac{2}{5}(2k_1+k_2)}(6 - 2e^{2k_2}),\\
\Delta k_2 &= e^{-\frac{2}{5}(2k_1+k_2)}(8 - 6e^{2k_2}).
\end{aligned}$$

These equations are valid for both $b_2 = \pm a_3$. Defining $\omega_1 = \frac{6k_1-2k_2}{5}$ and $\omega_2 = \frac{2k_1-4k_2}{5}$, these equations become

$$\partial\bar{\partial}\omega_1 = e^{\omega_2-\omega_1},$$
$$\partial\bar{\partial}\omega_2 = e^{-2\omega_2} - e^{\omega_2-\omega_1}.$$

Using the constructed functions, the rest of the Gauss equations do not impose further conditions. We can summarize this result in the following theorem.

Theorem 3.4. *Each special Lagrangian submanifold of* $\mathbb{C}^4$ *with a pointwise* $SO(2) \rtimes S_3$*-symmetry where the only integral distribution containing* $\mathcal{N}_1$ *is the tangent bundle, can be constructed in the way above using either functions* $\{k_1, k_2, k_3, k_4\}$ *subject to* (39) *or functions* $\{k_1, k_2\}$ *subject to* (47)*. Conversely, each such a construction results in such a submanifold, unique up to local isometry.*

In the upcoming sections we will consider the construction for $\epsilon = \pm 1$.

4. Submanifolds in $\mathbb{C}P^4$

4.1. The case where $b_2 = 0$.

This means that both $\mathcal{N}_1$ and $\mathcal{N}_2$ are integrable distributions. We can assume $a_3 = 0$. However, the Gauss equation

$$X_3(a_2) = 1 + a_2^2 \tag{48}$$

no longer allows for a_2 being a constant. The following result is obtained:

Theorem 4.1. *Suppose* M *is a special Lagrangian submanifold in* $\mathbb{C}P^4$ *having a pointwise* $SO(2) \rtimes S_3$*-symmetry. Suppose* $\mathcal{N}_1$ *is integrable. Then* M *can be lifted horizontally to a submanifold in the unit sphere of* $\mathbb{C}^5$ *through* F *and this lift is congruent to*

$$F(t,s,u,v) = (\phi(u,v)\cos(s), \sin(s)\cos(t), \sin(s)\sin(t))\,, \tag{49}$$

where ϕ *is a minimal C-totally real submanifold in the unit sphere of* $\mathbb{C}^3$*.*

Remark 4.1. Thus, we can take a minimal C-totally real submanifold in an $S^5 \subset \mathbb{C}^3$ part and a unit circle in the complementary part. We then connect any random point of the submanifold in S^5 with any other point of the circle using a geodesic.

Proof. Equations (7) reduce ∇ to

$$\begin{array}{c|c}
\nabla_{X_1}X_1 = a_1X_2 + a_2X_3 & \nabla_{X_1}X_2 = -a_1X_1, \\
\nabla_{X_2}X_1 = b_1X_2 & \nabla_{X_2}X_2 = -b_1X_1 + a_2X_3, \\
\nabla_{X_3}X_1 = 0 & \nabla_{X_3}X_2 = 0, \\
\nabla_{X_4}X_1 = 0 & \nabla_{X_4}X_2 = 0, \\
\nabla_{X_1}X_3 = -a_2X_1 & \nabla_{X_1}X_4 = 0, \\
\nabla_{X_2}X_3 = -a_2X_2 & \nabla_{X_2}X_4 = 0, \\
\nabla_{X_3}X_3 = 0 & \nabla_{X_3}X_4 = 0, \\
\nabla_{X_4}X_3 = \frac{X_4}{a_2} & \nabla_{X_4}X_4 = -\frac{X_3}{a_2}.
\end{array}$$

The distributions $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy the conditions for a warped product $N_2 \times_{e^f} N_1$. Furthermore, the distributions $span\{X_3\}$ and $span\{X_4\}$ satisfy those of a warped product and we can write $M = \mathbb{R} \times_{e^g} \mathbb{R} \times_{e^f} N_1$. The functions f and g depend solely on the parameter corresponding to X_3 and are given by $X_3(f) = -a_2$ and $X_3(g) = \frac{1}{a_2}$. We can assume $X_3 = \frac{\partial}{\partial s}$ on the submanifold. We can also find a function $\mu(s)$ such that $\mu X_4 = \frac{\partial}{\partial t}$. To find a suitable μ, we set μ as a function of a_2 and solve

$$[X_3, \mu X_4] = \left(X_3(\mu) - \frac{\mu}{a_2} \right) X_4 = \left(\mu'(1 + a_2^2) - \frac{\mu}{a_2} \right) X_4 = 0.$$

The function $\mu = \frac{a_2}{\sqrt{1+a_2^2}}$ satisfies this equation. We can find $a_2(s)$ by solving

$$\frac{\partial a_2}{\partial s} = 1 + a_2^2 \Rightarrow a_2 = \tan(s).$$

Hence $\mu(s) = \sin(s)$ and we calculate for $i \in \{1, 2\}$ that

$$\begin{aligned}
D_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial s^2} = -F \\
&\Rightarrow F = A \cos(s) + B \sin(s), \\
D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial t \partial s} = -\frac{\partial A}{\partial t} \sin(s) + \frac{\partial B}{\partial t} \cos(s) \\
&= \cot(s) \frac{\partial F}{\partial t} = \frac{\cos(s)^2}{\sin(s)} \frac{\partial A}{\partial t} + \cos(s) \frac{\partial B}{\partial t} \\
&\Rightarrow \frac{\partial A}{\partial t} = 0, \\
D_{X_i} \frac{\partial}{\partial s} &= \frac{\partial F_* X_i}{\partial s} = -A_* X_i \sin(s) + B_* X_i \cos(s) \\
&= -\tan(s) X_i = -A_* X_i \sin(s) - \frac{\sin(s)^2}{\cos(s)} B_* X_i \\
&\Rightarrow B_* X_i = 0.
\end{aligned}$$

So A is the immersion of N_1 and B is a curve tangent to X_4. Because F lies in the unit sphere, one has

$$\langle F, F \rangle = \cos(s)^2 \langle A, A \rangle + \sin(s)^2 \langle B, B \rangle + \sin(2s) \langle A, B \rangle = 1,$$

which implies that A and B have both unit length and are orthogonal. We can also calculate

$$\begin{aligned}
D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= -\cos(s)\sin(s) \frac{\partial F}{\partial s} - \sin(s)^2 F = -\sin(s) B \\
&= \frac{\partial^2 F}{\partial t^2} = \sin(s) \frac{\partial^2 B}{\partial t^2} \\
&\Rightarrow B = B_1 \cos(t) + B_2 \sin(t).
\end{aligned}$$

Vector fields B_1 and B_2 are constant, normalized and orthogonal. This follows from the fact that $\langle B, B\rangle = 1$. The fact that $\frac{\partial F}{\partial s}$ is orthogonal to iF implies that A is orthogonal to both iB_1 and iB_2. Finally similar to (27), A can be shown to be any special Lagrangian submanifold in $\mathbb{C}P^2$ lifted to the unit sphere in $\mathbb{C}^3$ orthogonal to B_1 and B_2 directions. Fixing B_1 and B_2 by an isometry leads to (49). ■

4.2. The case where $b_2 \neq 0$.

When $\mathcal{N}_+$ is integrable, so when $a_6 = b_6 = 0$, the equations for ∇ are given by (28). We have:

Theorem 4.2. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}P^4$ having a pointwise $SO(2) \rtimes S_3$ symmetry on the cubic form. Suppose $\mathcal{N}_+$ is integrable. Then M can be lifted horizontally to a submanifold in the unit sphere of $\mathbb{C}^5$ through F and is locally isometric to*

(50) $$F(t, s, u, v) = (\phi(s, u, v)\cos(t), \sin(t)),$$

where ϕ is a minimal C-totally real submanifold with a pointwise S_3-symmetry of the unit sphere in $\mathbb{C}^4$.

Remark 4.2. The construction of this submanifold is similar to the construction of a Lagrangian cone in $\mathbb{C}^4$. You simply let a geodesic run through every point of the C-totally real submanifold and the fixed point $(0, 0, 0, 0, 1)$.

Proof. The manifold is a warped product $\mathbb{R} \times_{e^f} N^3$. Solving the Gauss equation

$$X_4(a_3) = \frac{\partial a_3}{\partial t} = 1 + a_3^2$$

yields $a_3 = \tan(t)$. For $i \in \{1, 2, 3\}$ this implies

$$\begin{aligned} D_{X_4}X_4 &= \frac{\partial^2 F}{\partial t^2} = -F \\ &\Rightarrow F = A\cos(t) + B\sin(t), \\ D_{X_i}X_4 &= -A_*X_i\sin(t) + B_*X_i\cos(t) \\ &= -\tan(t)X_i = -A_*X_i\sin(t) - B_*X_i\frac{\sin(t)^2}{\cos(t)} \\ &\Rightarrow B_* = 0. \end{aligned}$$

Thus B is a constant vector field along the submanifold and A is an immersion of a 3-fold N^3. Using the fact that F is of unit length, A and B are orthogonal and of unit length. Using calculations similar to (27) A is a minimal C-totally real submanifold in S^7 having pointwise S_3-symmetry on the cubic form, where S^7 lies in the subspace orthogonal to B and JB. Applying a suitable isometry results in (50). ■

The method to solve the case where the only integrable distribution containing $\mathcal{N}_1$ is the tangent bundle has been analyzed earlier for a non-specific complex space form. We can now fill in $\epsilon = 1$ and we find for $z^2 \neq -1$ that

$$\begin{aligned}
a_3 &= \tan(t),\\
a_2 &= \frac{\sin(2s)}{\cos(t)(\cos(2s)+\cosh(2k_1))},\\
b_2 &= \frac{\sinh(2k_1)}{\cos(t)(\cos(2s)+\cosh(2k_1))},\\
r &= \frac{e^{k_2}}{\cos(t)\sqrt{\cos(2s)+\cosh(2k_1)}},\\
a_6+ib_6 &= \frac{k_3+ik_4}{\rho}\sqrt{1+a_3^2}\left(1+\bar{z}^2\right)^{-\frac{1}{2}},
\end{aligned}$$

where the functions k_i depend only on (u, v). Solving (7), we obtain furthermore that

$$\begin{aligned}
\gamma_1 &= \frac{-\tan(t)(k_3+ik_4)\cos(s-ik_1)+\left(\frac{\partial k_1}{\partial v}-i\frac{\partial k_1}{\partial u}\right)}{\rho},\\
a_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\cos(t)^2\left(\cos(2s)+\cosh(2k_1)\right)^2}\big((\cos(2s)+\cosh(2k_1))(\rho_1\frac{\partial k_2}{\partial v}+\rho_2\frac{\partial k_2}{\partial u})\\
&\quad+\sin(2s)(\rho_1\frac{\partial k_1}{\partial u}-\rho_2\frac{\partial k_1}{\partial v})-\sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial v}+\rho_2\frac{\partial k_1}{\partial u})\\
&\quad-\tan(t)\sinh(2k_1)\left(\cos(s)\cosh(k_1)(k_4\rho_2-k_3\rho_1)\right.\\
&\quad\left.+\sin(s)\sinh(k_1)(k_4\rho_1+k_3\rho_2)\right)\big),\\
b_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\cos(t)^2\left(\cos(2s)+\cosh(2k_1)\right)^2}\big((\cos(2s)+\cosh(2k_1))(\rho_2\frac{\partial k_2}{\partial v}-\rho_1\frac{\partial k_2}{\partial u})\\
&\quad+\sin(2s)(\rho_2\frac{\partial k_1}{\partial u}+\rho_1\frac{\partial k_1}{\partial v})+\sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial u}-\rho_2\frac{\partial k_1}{\partial v})\\
&\quad-\sinh(2k_1)\tan(t)\left(\sin(s)\sinh(k_1)(k_4\rho_2-k_3\rho_1)\right.\\
&\quad\left.-\cos(s)\cosh(k_1)(k_4\rho_1+k_3\rho_2)\right)\big).
\end{aligned}$$

The other equations in (7) impose restrictions on $\{k_1, k_2, k_3, k_4\}$ given by

$$\begin{aligned}
\Delta k_1 &= \frac{\sinh(2k_1)}{2}\left(k_3^2+k_4^2-2e^{-\frac{2k_2-\ln(2)}{3}}\right),\\
\Delta k_2 &= 3e^{-\frac{2k_2-\ln(2)}{3}}\left(\cosh(2k_1)-e^{2k_2}\right),\\
\frac{\partial k_4}{\partial v}+\frac{\partial k_3}{\partial u} &= -2\coth(k_1)\left(k_3\frac{\partial k_1}{\partial u}+k_4\frac{\partial k_1}{\partial v}\right),\\
\frac{\partial k_4}{\partial u}-\frac{\partial k_3}{\partial v} &= 2\tanh(k_1)\left(k_3\frac{\partial k_1}{\partial v}-k_4\frac{\partial k_1}{\partial u}\right).
\end{aligned}\tag{51}$$

We can simplify these equations using $u+iv$ as a complex coordinate as we did in the $\mathbb{C}^4$ case. Defining $\alpha = \frac{\sinh(2k_1)(k_3-ik_4)}{2}$ and $\omega = \frac{2k_2-\ln(2)}{3}$, we obtain

$$\begin{aligned}
\bar{\partial}\alpha &= -2\bar{\alpha}\partial k_1 csch(2k_1),\\
2\partial\bar{\partial}k_1 &= |\alpha|^2 csch(2k_1) - \frac{1}{2}e^{-\omega}\sinh(2k_1),\\
\partial\bar{\partial}\omega &= -e^{2\omega} + \frac{1}{2}e^{-\omega}\cosh(2k_1).
\end{aligned}$$

These equations can also be compared to what is obtained in [3].
When $a_2 = 0$ and $b_2 = \pm\sqrt{1+a_3^2}$, we find

$$\begin{aligned}
a_6 &= \frac{e^{k_1}\cos(\frac{5}{3}s)}{\cos(t)^2};\\
b_6 &= \pm\frac{e^{k_1}\sin(\frac{5}{3}s)}{\cos(t)^2};\\
r &= \frac{e^{k_2}}{\cos(t)}.
\end{aligned}$$

Furthermore, we obtain for $b_2 = \sqrt{1+a_3^2}$ that

$$\begin{aligned}
\gamma_{1+} &= \frac{-5e^{k_1+\frac{5si}{3}}\tan(t) + e^{\frac{2k_1+k_2}{5}+\frac{2si}{3}}\left((\frac{\partial k_2}{\partial v} - 3\frac{\partial k_1}{\partial v}) - i(\frac{\partial k_2}{\partial u} - 3\frac{\partial k_1}{\partial u})\right)}{5\cos(t)},\\
a_{1+} &= \frac{e^{k_1}\cos(\frac{5s}{3})\tan(t) + e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3\cos(t)},\\
b_{1+} &= \frac{e^{k_1}\sin(\frac{5s}{3})\tan(t) - e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3\cos(t)},
\end{aligned}$$

and for $b_2 = -\sqrt{1+a_3^2}$ we obtain

$$\begin{aligned}
\gamma_{1-} &= \frac{-5e^{k_1-\frac{5si}{3}}\tan(t) + e^{\frac{2k_1+k_2}{5}-\frac{2si}{3}}\left((3\frac{\partial k_1}{\partial v} - \frac{\partial k_2}{\partial v}) - i(3\frac{\partial k_1}{\partial u} - \frac{\partial k_2}{\partial u})\right)}{5\cos(t)},\\
a_{1-} &= \frac{-e^{k_1}\cos(\frac{5s}{3})\tan(t) + e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3\cos(t)},\\
b_{1-} &= \frac{e^{k_1}\sin(\frac{5s}{3})\tan(t) - e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3\cos(t)}.
\end{aligned}$$

Solving the last equations in (7) implies restrictions on the functions $k_1(u,v)$ and $k_2(u,v)$ given by

$$\begin{aligned} \Delta k_1 &= 2e^{-\frac{2(2k_1+k_2)}{5}}\left(3 - e^{2k_1} - e^{2k_2}\right) \\ \Delta k_2 &= e^{-\frac{2(2k_1+k_2)}{5}}\left(8 - e^{2k_1} - 6e^{2k_2}\right). \end{aligned} \tag{52}$$

These equations are valid for both $b_2 = \pm\sqrt{1+a_3^2}$. Similarly to the case in $\mathbb{C}^4$, we can define $\omega_1 = \frac{6k_1-2k_2}{5}$ and $\omega_2 = \frac{2k_1-4k_2}{5}$ to obtain

$$\begin{aligned} \partial\bar{\partial}\omega_1 &= e^{\omega_2-\omega_1} - e^{\omega_1-\ln(2)}, \\ \partial\bar{\partial}\omega_2 &= e^{-2\omega_2} - e^{\omega_2-\omega_1}. \end{aligned}$$

We summarize this in the following theorem.

Theorem 4.3. *Each special Lagrangian submanifold of $\mathbb{C}P^4$ with a pointwise $SO(2) \rtimes S_3$-symmetry where the only integral distribution containing $\mathcal{N}_1$ is the tangent bundle, can be constructed in the way above using either functions $\{k_1,k_2,k_3,k_4\}$ subject to* (51) *or functions $\{k_1,k_2\}$ subject to* (52)*. Conversely, each such a construction results in such a submanifold, unique up to local isometry.*

5. Submanifolds in $\mathbb{C}H^4$

5.1. The case where $b_2 = 0$.

This is the case where $\mathcal{N}_1$ is an integrable distribution. We assume that $a_3 = 0$. Similar to the case in $\mathbb{C}P^4$ we have that M is a double warped product $\mathbb{R}\times_{e^g}\mathbb{R}\times_{e^f} N^2$. The function a_2 depends only on the coordinate s and is given by

$$\frac{\partial a_2}{\partial s} = a_2^2 - 1.$$

This equation has 4 possible solutions, depending on the initial conditions. For $a_2(0) = \pm 1$, it is a constant. For $|a_2(0)| > 1$ it is given as $a_2 = -\coth(s)$, up to translation of s. Finally for $|a_2(0)| < 1$, it is given as $a_2(s) = -\tanh(s)$, up to translation of s. The connection ∇ is given by

$$\begin{array}{c|c} \nabla_{X_1}X_1 = a_1X_2 + a_2X_3 & \nabla_{X_1}X_2 = -a_1X_1, \\ \nabla_{X_2}X_1 = b_1X_2 & \nabla_{X_2}X_2 = -b_1X_1 + a_2X_3, \\ \nabla_{X_3}X_1 = 0 & \nabla_{X_3}X_2 = 0, \\ \nabla_{X_4}X_1 = 0 & \nabla_{X_4}X_2 = 0, \\ \nabla_{X_1}X_3 = -a_2X_1 & \nabla_{X_1}X_4 = 0, \\ \nabla_{X_2}X_3 = -a_2X_2 & \nabla_{X_2}X_4 = 0, \\ \nabla_{X_3}X_3 = 0 & \nabla_{X_3}X_4 = 0, \\ \nabla_{X_4}X_3 = -\frac{X_4}{a_2} & \nabla_{X_4}X_4 = \frac{X_3}{a_2}. \end{array}$$

We have the following result.

Theorem 5.1. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}H^4$ having a pointwise $SO(2) \rtimes S_3$-symmetry on the cubic form. Suppose $\mathcal{N}_1$ is integrable. Then M can be lifted horizontally to a submanifold in H^9 through F and is locally isometric to either*

$$F(t,s,u,v) = (\phi(u,v)\cosh(s), \sin(t)\sinh(s), \cos(t)\sinh(s)), \tag{53}$$

where ϕ is a minimal C-totally real submanifold of $H^5 \subset \mathbb{C}^3$ in case $a_2^2 < 1$, or

$$F(t,s,u,v) = (\cosh(t)\cosh(s), \sinh(t)\cosh(s), \phi(u,v)\sinh(s)), \tag{54}$$

where ϕ is a minimal C-totally real submanifold of $S^5 \subset \mathbb{C}^3$ in case $a_2^2 > 1$, or

$$F(t,s,u,v) = \Big((\phi(u,v),t)\, e^{-s}, -\frac{1}{2}e^{-s}, \big(\|(\phi(u,v),t)\|^2 + if(u,v)\big)\, e^{-s} + e^{s}\Big), \tag{55}$$

where ϕ is a holomorphic curve in $\mathbb{C}^2$ and f is an integral of the differential form

$$2\sum_{i=1}^{2}\left(x^i dy^i - y^i dx^i\right)$$

on $\mathbb{C}^2$ in case $a_2^2 = 1$.

Remark 5.1.

- In (55) the coordinates are taken slightly differently, namely such that

$$\langle \vec{z}, \vec{w}\rangle = \Re\left(\sum_{j=1}^{3} z_j\bar{w}_j + z_4\bar{w}_5 + z_5\bar{w}_4\right).$$

- The first 2 cases are similar to the $\mathbb{C}P^4$ case in that we connect a C-totally real submanifold in the first part and connect it with a complementary geodesic curve using a geodesic.

Proof. We can check similarly to the case in $\mathbb{C}P^4$ that $M = \mathbb{R}\times_{e^g}\mathbb{R}\times_{e^f}\times N^2$, where f and g are functions on the first factor, determined by $X_3(g) = -\frac{1}{a_2}$ and $X_3(f) = -a_2$. We can treat the cases separately for each solution to $a_2(s)$.

Assume $a_2 = -\tanh(s)$, then it is easy to see that $\frac{\partial}{\partial t} = -\sinh(s)X_4$ commutes with $\frac{\partial}{\partial s}$. The Gauss identity now implies for $i \in \{1,2\}$ that

$$\begin{aligned}
D_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial s^2} = F \\
&\Rightarrow F = A\sinh(s) + B\cosh(s), \\
D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial t\partial s} = \frac{\partial A}{\partial t}\cosh(s) + \frac{\partial B}{\partial t}\sinh(s)
\end{aligned}$$

$$
\begin{aligned}
&= \coth(s)\frac{\partial F}{\partial t} = \frac{\partial A}{\partial t}\cosh(s) + \frac{\partial B}{\partial t}\frac{\cosh(s)^2}{\sinh(s)} \\
&\Rightarrow \frac{\partial B}{\partial t} = 0, \\
D_{X_i}\frac{\partial}{\partial s} &= \frac{\partial F_* X_i}{\partial s} = A_* X_i \cosh(s) + B_* X_i \sinh(s) \\
&= \tanh(s) X_i = A_* X_i \frac{\sinh(s)^2}{\cosh(s)} + B_* X_i \sinh(s) \\
&\Rightarrow A_* X_i = 0.
\end{aligned}
$$

Using the fact that $\langle F, F\rangle_1 = -1$, we get that $\langle B, B\rangle_1 = -\langle A, A\rangle_1 = -1$ and $\langle A, B\rangle_1 = 0$. Furthermore, we find

$$
\begin{aligned}
D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} &= \frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 A}{\partial t^2}\sinh(s) \\
&= -\cosh(s)\sinh(s)\frac{\partial F}{\partial s} + \sinh(s)^2 F = -A\sinh(s) \\
&\Rightarrow A = A_1\cos(t) + A_2\sin(t).
\end{aligned}
$$

Because A has unit length, so do A_1 and A_2 and they are both orthogonal. Calculations similar to (27) show that B can be taken as the horizontal lift of any special Lagrangian submanifold in $\mathbb{C}H^2$ and applying a suitable isometry gives (53).
For $a_2 = -\coth(s)$ up to translation of s calculations similar to the previous case result in (54).
Finally we assume $a_2 = 1$. Then the vector field given by $\frac{\partial}{\partial t} = e^{-s}X_4$ commutes with $\frac{\partial}{\partial s}$. We can calculate for $i \in \{1, 2\}$ that

$$
\begin{aligned}
D_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial s^2} = F \Rightarrow F = Ae^{s} + Be^{-s}, \\
D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial t \partial s} = \frac{\partial A}{\partial t}e^{s} - \frac{\partial B}{\partial t}e^{-s} \\
&= -\frac{\partial F}{\partial t} = -\frac{\partial A}{\partial t}e^{s} - \frac{\partial B}{\partial t}e^{-s} \\
&\Rightarrow \frac{\partial A}{\partial t} = 0, \\
D_{X_i}\frac{\partial}{\partial s} &= \frac{\partial F_* X_i}{\partial s} = A_* X_i e^{s} - B_* X_i e^{-s} \\
&= -F_* X_i = -\left(A_* X_i e^{s} + B_* X_i e^{-s}\right) \\
&\Rightarrow A_* = 0.
\end{aligned}
$$

Using the fact that $\langle F, F\rangle_1 = -1$, we obtain that A and B are vector fields with 0

length and they satisfy $\langle A, B\rangle_1 = -\frac{1}{2}$. Further calculations show

$$\begin{aligned} D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= \frac{\partial^2 F}{\partial t^2} = e^{-s} \frac{\partial^2 B}{\partial t^2} \\ &= e^{-2s} \left(\frac{\partial F}{\partial s} + F \right) = 2Ae^{-s} \\ &\Rightarrow B = At^2 + B_1 t + B_2, \\ D_{X_i} \frac{\partial}{\partial t} &= \frac{\partial F_* X_i}{\partial t} = B_{1*} X_i e^{-s} = 0 \\ &\Rightarrow B_{1*} = 0. \end{aligned}$$

We can conclude that F has the form

$$F = \left(At^2 + B_1 t + \phi \right) e^{-s} + Ae^s$$

Here, ϕ is an immersion of a 2-fold in $\mathbb{C}^5_1$ tangent to $\mathcal{N}_1$. Calculating the scalar products of B and A, we get

$$\begin{aligned} \langle A, B\rangle_1 &= t\langle A, B_1\rangle_1 + \langle A, \phi\rangle_1 = -\frac{1}{2} \\ &\Rightarrow \langle A, B_1\rangle_1 = 0 \text{ and } \langle A, \phi\rangle_1 = -\frac{1}{2}, \\ \langle B, B\rangle_1 &= t^2 \left(\langle B_1, B_1\rangle_1 - 1 \right) + 2t\langle B_1, \phi\rangle_1 + \langle \phi, \phi\rangle_1 = 0 \\ &\Rightarrow \langle B_1, B_1\rangle_1 = 1 \text{ and } \langle B_1, \phi\rangle_1 = 0 \text{ and } \langle \phi, \phi\rangle_1 = 0. \end{aligned} \tag{56}$$

We can shift to a different standard basis of $\mathbb{C}^5_1$ such that

$$\langle \vec{z}, \vec{w}\rangle = \Re \left(\sum_{j=1}^{3} z_j \bar{w}_j + z_4 \bar{w}_5 + z_5 \bar{w}_4 \right).$$

In this case the constant light-like vector A and time-like B_1, after applying a suitable isometry, can be chosen to be

$$\begin{aligned} A &= (0, 0, 0, 0, 1), \\ B_1 &= (0, 0, 1, 0, 0). \end{aligned}$$

We can write $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$ where $\phi_j = x_j + iy_j$. Then (56) implies

$$\begin{aligned} x_4 &= -\frac{1}{2}, \\ x_3 &= 0, \\ x_5 - 2y_4 y_5 &= \sum_{j=1}^{3} |\phi_j|^2. \end{aligned}$$

We can use the fact that both F and iF are orthogonal to the tangent space in $\mathbb{C}_1^5$ and this results in

$$\begin{aligned}\phi_4 &= -\frac{1}{2},\\ \phi_3 &= 0,\\ dy_5 &= 2\sum_{i=1}^{2}(x_i dy_i - y_i dx_i).\end{aligned}$$

This last equation is integrable if and only if its derivative vanishes on the submanifold. But this derivative is nothing more than a multiple of the Kähler form on $\mathbb{C}^2$ spanned by the first 2 complex coordinates. In other words, for such a submanifold to exist, the projection of ϕ onto the first 2 coordinates should be a Lagrangian submanifold in $\mathbb{C}^2$. Calculating the Gauss identity on $D_{X_i}X_j$ we find that the metric on this immersion is given by

$$\langle \phi_* X_i, \phi_* X_j \rangle = e^{2s}\delta_{ij},$$

where $\langle a, b\rangle$ is the standard scalar product on $\mathbb{C}^2$ and ϕ here is the restriction to the first 2 complex coordinates. Because $\langle F_* X_i, F_* X_j\rangle = \delta_{ij}$ and because $\phi_{3*} = 0$ and $\phi_{4*} = 0$ this condition is included in the warped product structure. Using calculations like (27) we conclude that (ϕ_1, ϕ_2) can be any special Lagrangian 2-fold in $\mathbb{C}^2$. The result is summarized in (55).
It can be remarked that the solution $a_2 = -1$ just switches e^s and e^{-s} in the immersion, hence corresponds to a coordinate change of $s \to -s$. ■

5.2 The case where $b_2 \neq 0$.

First we assume that $\mathcal{N}_+$ is an integrable distribution. This is equivalent to $a_6 + ib_6 = 0$. The connection is given by (28), resulting in a warped product structure $\mathbb{R} \times_{e^f} N^3$. The equation

$$X_4(a_3) = \frac{\partial a_3}{\partial t} = a_3^2 - 1$$

has a solution given as $|a_3| = 1$, $a_3 = -\tanh(t)$ or $a_3 = -\coth(t)$, depending on the initial value of a_3. Using an analysis similar to the case of $\mathbb{C}P^4$ and the cases above gives the following result.

Theorem 5.2. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}H^4$ having a pointwise $SO(2) \rtimes S_3$-symmetry on the cubic form. Suppose $\mathcal{N}_1$ is non-integrable, but is contained in the integrable $\mathcal{N}_+$ distribution. Then M can be lifted horizontally to a submanifold in H^9 through F and is locally isometric to either*

(57) $$F(t, s, u, v) = (\phi(s, u, v)\cosh(t), \sinh(t)),$$

where ϕ is a minimal C-totally real submanifold with a pointwise S_3-symmetry of $H^7 \subset \mathbb{C}^4$ in case $a_3^2 < 1$, or

$$F(t,s,u,v) = (\cosh(t), \phi(s,u,v)\sinh(t)), \tag{58}$$

where ϕ is a minimal C-totally real submanifold with a pointwise S_3-symmetry of $S^7 \subset \mathbb{C}^4$ in case $a_3^2 > 1$, or

$$F(t,s,u,v) = \left(\phi(s,u,v)e^{-t}, -e^{-t}/2, \left(\|\phi(s,u,v)\|^2 + if(s,u,v)\right)e^{-t} + e^t\right), \tag{59}$$

where ϕ is a special Lagrangian submanifold in $\mathbb{C}^3$ with a pointwise S_3-symmetry and f is an integral of the differential form

$$2\sum_{i=1}^{3}\left(x^i dy^i - y^i dx^i\right).$$

Remark 5.2. As in $\mathbb{C}P^4$ the first 2 immersions are constructed similarly to a Lagrangian cone in $\mathbb{C}^4$.

Finally we assume that there is no integrable distribution that contains $\mathcal{N}_1$ except for the tangent bundle. We return to the analysis as done for $\mathbb{C}^4$, but set $\epsilon = -1$. The result will depend on the initial value of a_3. First assume that $a_3^2 < 1$, then $\tilde{\epsilon} = -1$. We find functions $\{k_1, k_2, k_3, k_4\}$ of (u,v) such that

$$\begin{aligned} a_3 &= -\tanh(t), \\ a_2 &= -\frac{\sinh(2s)}{\cosh(t)\left(\cosh(2s) + \cos(2k_1)\right)}, \\ b_2 &= -\frac{\sin(2k_1)}{\cosh(t)\left(\cosh(2s) + \cos(2k_1)\right)}, \\ r &= \frac{e^{k_2}}{\cosh(t)\sqrt{\cosh(2s) + \cos(2k_1)}}, \\ a_6 + ib_6 &= \frac{k_3 + ik_4}{\rho}\sqrt{1 - a_3^2}\left(1 - \bar{z}^2\right)^{-\frac{1}{2}}. \end{aligned}$$

Using (7) as earlier, we obtain a_1, b_1, γ_1 as

$$\begin{aligned} \gamma_1 &= \frac{-\tanh(t)(k_3 + ik_4)\cosh(s - ik_1) + \left(\frac{\partial k_1}{\partial v} - i\frac{\partial k_1}{\partial u}\right)}{\rho}, \\ a_1 &= \frac{2^{\frac{2}{3}} e^{\frac{2}{3}k_2}}{3\cosh(t)^2\left(\cosh(2s) + \cos(2k_1)\right)^2}\Big((\cosh(2s) + \cos(2k_1))(\rho_1\frac{\partial k_2}{\partial v} + \rho_2\frac{\partial k_2}{\partial u}) \\ &\quad + \sinh(2s)(\rho_2\frac{\partial k_1}{\partial v} - \rho_1\frac{\partial k_1}{\partial u}) + \sin(2k_1)(\rho_1\frac{\partial k_1}{\partial v} + \rho_2\frac{\partial k_1}{\partial u}) \end{aligned}$$

$$
\begin{aligned}
&+\sin(2k_1)\tanh(t)\,(\cosh(s)\cos(k_1)(k_4\rho_2-k_3\rho_1)\\
&-\sinh(s)\sin(k_1)(k_4\rho_1+k_3\rho_2))\big),\\
b_1=&\frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\cosh(t)^2\,(\cosh(2s)+\cos(2k_1))^2}\big((\cosh(2s)+\cos(2k_1))(\rho_2\frac{\partial k_2}{\partial v}-\rho_1\frac{\partial k_2}{\partial u})\\
&-\sinh(2s)(\rho_2\frac{\partial k_1}{\partial u}+\rho_1\frac{\partial k_1}{\partial v})-\sin(2k_1)(\rho_1\frac{\partial k_1}{\partial u}-\rho_2\frac{\partial k_1}{\partial v}))\\
&-\sin(2k_1)\tanh(t)\,(\sinh(s)\sin(k_1)(k_4\rho_2-k_3\rho_1)\\
&+\cosh(s)\cos(k_1)(k_4\rho_1+k_3\rho_2))\big).
\end{aligned}
$$

The other equations in (7) put restrictions on $\{k_1,k_2,k_3,k_4\}$ given by

$$
\begin{aligned}
\Delta k_1&=\frac{\sin(2k_1)}{2}\left(2e^{-\frac{2k_2-\ln(2)}{3}}+k_3^2+k_4^2\right),\\
\Delta k_2&=-3e^{-\frac{2k_2-\ln(2)}{3}}\left(e^{2k_2}+\cos(2k_1)\right),\\
\frac{\partial k_4}{\partial v}+\frac{\partial k_3}{\partial u}&=-2\cot(k_1)\left(k_3\frac{\partial k_1}{\partial u}+k_4\frac{\partial k_1}{\partial v}\right),\\
\frac{\partial k_4}{\partial u}-\frac{\partial k_3}{\partial v}&=2\tan(k_1)\left(k_4\frac{\partial k_1}{\partial u}-k_3\frac{\partial k_1}{\partial v}\right).
\end{aligned}
\tag{60}
$$

We can use $u+iv$ as a complex coordinate to simplify these equations. In this case, we define $\alpha=\frac{\sin(2k_1)(k_3-ik_4)}{2}$ and $\omega=\frac{2k_2-\ln(2)}{3}$ and obtain

$$
\begin{aligned}
\bar\partial\alpha&=-2\bar\alpha\partial k_1\csc(2k_1),\\
2\partial\bar\partial k_1&=|\alpha|^2\csc(2k_1)+\frac{1}{2}e^{-\omega}\sin(2k_1),\\
\partial\bar\partial\omega&=-e^{2\omega}-\frac{1}{2}e^{-\omega}\cos(2k_1).
\end{aligned}
$$

Here, the equations look slightly different than the other cases because of the presence of standard goniometric functions instead of hyperbolic ones.
Then we set $a_3^2>1$ and assume $z^2\neq -1$. We then find

$$
\begin{aligned}
a_3&=-\coth(t),\\
a_2&=\frac{\sin(2s)}{\sinh(t)\,(\cos(2s)+\cosh(2k_1))},\\
b_2&=\frac{\sinh(2k_1)}{\sinh(t)\,(\cos(2s)+\cosh(2k_1))},\\
r&=\frac{e^{k_2}}{\sinh(t)\sqrt{\cos(2s)+\cosh(2k_1)}},\\
a_6+ib_6&=\frac{k_3+ik_4}{\rho}\sqrt{a_3^2-1}\,\left(1+\bar z^2\right)^{-\frac{1}{2}}.
\end{aligned}
$$

We obtain

$$\gamma_1 = \frac{\coth(t)(k_3 + ik_4)\cos(s - ik_1) + \left(\frac{\partial k_1}{\partial v} - i\frac{\partial k_1}{\partial u}\right)}{\rho},$$

$$a_1 = \frac{2^{\frac{2}{3}} e^{\frac{2}{3}k_2}}{3\sinh(t)^2\left(\cos(2s) + \cosh(2k_1)\right)^2}\big((\cos(2s) + \cosh(2k_1))(\rho_1\frac{\partial k_2}{\partial v} + \rho_2\frac{\partial k_2}{\partial u})$$
$$+ \sin(2s)(\rho_1\frac{\partial k_1}{\partial u} - \rho_2\frac{\partial k_1}{\partial v}) - \sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial v} + \rho_2\frac{\partial k_1}{\partial u})$$
$$+ \sinh(2k_1)\coth(t)\left(\cos(s)\cosh(k_1)(k_4\rho_2 - k_3\rho_1)\right.$$
$$\left.+ \sin(s)\sinh(k_1)(k_4\rho_1 + k_3\rho_2)\right)\big),$$

$$b_1 = \frac{2^{\frac{2}{3}} e^{\frac{2}{3}k_2}}{3\sinh(t)^2\left(\cosh(2s) + \cos(2k_1)\right)^2}\big((\cosh(2s) + \cos(2k_1))(\rho_2\frac{\partial k_2}{\partial v} - \rho_1\frac{\partial k_2}{\partial u})$$
$$+ \sin(2s)(\rho_2\frac{\partial k_1}{\partial u} + \rho_1\frac{\partial k_1}{\partial v}) + \sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial u} - \rho_2\frac{\partial k_1}{\partial v})\big)$$
$$- \sinh(2k_1)\coth(t)\left(\sin(s)\sinh(k_1)(k_3\rho_1 - k_4\rho_2)\right.$$
$$\left.+ \cos(s)\cosh(k_1)(k_4\rho_1 + k_3\rho_2)\right)\big).$$

The functions $\{k_1, k_2, k_3, k_4\}$ have to satisfy

(61)
$$\begin{aligned}
\Delta k_1 &= -\frac{\sinh(2k_1)}{2}\left(2e^{-\frac{2k_2 - \ln(2)}{3}} + k_3^2 + k_4^2\right),\\
\Delta k_2 &= 3e^{-\frac{2k_2 - \ln(2)}{3}}\left(\cosh(2k_1) - e^{2k_2}\right),\\
\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} &= -2\coth(k_1)\left(k_3\frac{\partial k_1}{\partial u} + k_4\frac{\partial k_1}{\partial v}\right),\\
\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} &= 2\tanh(k_1)\left(k_3\frac{\partial k_1}{\partial v} - k_4\frac{\partial k_1}{\partial u}\right).
\end{aligned}$$

Using $u + iv$ as a complex coordinate and defining $\alpha = \frac{\sinh(2k_1)(k_3 - ik_4)}{2}$ and $\omega = \frac{2k_2 - \ln(2)}{3}$, we simplify these as

$$\begin{aligned}
\bar\partial\alpha &= -2\bar\alpha\partial k_1 csch(2k_1),\\
2\partial\bar\partial k_1 &= -|\alpha|^2 csch(2k_1) - \frac{1}{2}e^{-\omega}\sinh(2k_1),\\
\partial\bar\partial\omega &= -e^{2\omega} + \frac{1}{2}e^{-\omega}\cosh(2k_1).
\end{aligned}$$

Again, these equations compare rather well to those obtained in [3].
Finally, assume $a_3^2 > 1$ and $z^2 = -1$. Then we find

$$\begin{aligned} a_3 &= -\coth(t),\\ a_6 &= \frac{e^{k_1}\cos(\frac{5}{3}s)}{\sinh(t)^2},\\ b_6 &= \pm\frac{e^{k_1}\sin(\frac{5}{3}s)}{\sinh(t)^2},\\ r &= \frac{e^{k_2}}{\sinh(t)}. \end{aligned}$$

We obtain for $b_2 = \sqrt{a_3^2 - 1}$ that

$$\begin{aligned} \gamma_{1+} &= \frac{5e^{k_1+\frac{5si}{3}}\coth(t) + e^{\frac{2k_1+k_2}{5}+\frac{2si}{3}}\left((\frac{\partial k_2}{\partial v} - 3\frac{\partial k_1}{\partial v}) - i(\frac{\partial k_2}{\partial u} - 3\frac{\partial k_1}{\partial u})\right)}{5\sinh(t)},\\ a_{1+} &= \frac{-e^{k_1}\cos(\frac{5s}{3})\coth(t) + e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3\sinh(t)},\\ b_{1+} &= \frac{-e^{k_1}\sin(\frac{5s}{3})\coth(t) - e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3\sinh(t)}, \end{aligned}$$

and for $b_2 = -\sqrt{a_3^2 - 1}$ we obtain

$$\begin{aligned} \gamma_{1-} &= \frac{5e^{k_1-\frac{5si}{3}}\coth(t) + e^{\frac{2k_1+k_2}{5}-\frac{2si}{3}}\left((3\frac{\partial k_1}{\partial v} - \frac{\partial k_2}{\partial v}) - i(3\frac{\partial k_1}{\partial u} - \frac{\partial k_2}{\partial u})\right)}{5\sinh(t)},\\ a_{1-} &= \frac{e^{k_1}\cos(\frac{5s}{3})\coth(t) + e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3\sinh(t)},\\ b_{1-} &= \frac{-e^{k_1}\sin(\frac{5s}{3})\coth(t) - e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3\sinh(t)}. \end{aligned}$$

Solving the other Gauss equations results in the relations

(62)
$$\begin{aligned} \Delta k_1 &= e^{-\frac{2}{5}(2k_1+k_2)}\left(6 + 2e^{2k_1} - 2e^{2k_2}\right),\\ \Delta k_2 &= e^{-\frac{2}{5}(2k_1+k_2)}\left(8 + e^{2k_1} - 6e^{2k_2}\right). \end{aligned}$$

These equations are valid for both $b_2 = \pm\sqrt{a_3^2 - 1}$. As in the previous cases, we can define $\omega_1 = \frac{6k_1-2k_2}{5}$ and $\omega_2 = \frac{2k_1-4k_2}{5}$ to obtain

$$\partial\bar{\partial}\omega_1 = e^{\omega_2-\omega_1} - e^{\omega_1-(\ln(2)+i\pi)},$$
$$\partial\bar{\partial}\omega_2 = e^{-2\omega_2} - e^{\omega_2-\omega_1}.$$

We can conclude with the following proposition.

Theorem 5.3. *Each special Lagrangian submanifold of $\mathbb{C}H^4$ with a pointwise $SO(2) \rtimes S_3$-symmetry where the only integral distribution containing $\mathcal{N}_1$ is the whole tangent bundle, can be constructed in the way above using functions $\{k_1, k_2, k_3, k_4\}$ subject to* (60) *in case $a_3^2(0) < 1$, subject to* (61) *in case $a_3^2(0) > 1$, or functions $\{k_1, k_2\}$ subject to* (62) *when $a_3^2(0) > 1$ and $z^2 = -1$. Conversely, each such a construction results in such a submanifold, unique up to local isometry.*

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