TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 2, pp. 519-533, April 2015
DOI: 10.11650/tjm.19.2015.4847
This paper is available online at http://journal.taiwanmathsoc.org.tw

NULL 2-TYPE HYPERSURFACES WITH AT MOST THREE DISTINCT PRINCIPAL CURVATURES IN EUCLIDEAN SPACE

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Abstract. The goal of this paper is to prove null 2-type hypersurfaces with at most three distinct principal curvatures in a Euclidean space have constant mean curvature.

1. Introduction

Let $x : M^n \to \mathbb{E}^m$ be an isometric immersion of an n-dimensional connected submanifold M^n into a Euclidean space $\mathbb{E}^m$. Denote by Δ the Laplace operator with respect to the induced Riemannian metric. A submanifold of $\mathbb{E}^m$ is said to be of *finite type* [1, 2, 7, 9] if the position vector x of M^n in $\mathbb{E}^m$ can be decomposed in the following form:

$$x = x_0 + x_1 + \cdots + x_k, \tag{1.1}$$

where x_0 is a constant vector and $x_1, \ldots, x_k$ are non-constant maps satisfying $\Delta x_i = \lambda_i x_i$, $i = 1, \ldots, k$. In particular, if all eigenvalues $\lambda_1, \ldots, \lambda_k$ are mutually different, then the submanifold M^n is said to be of k-type and if one of $\lambda_1, \ldots, \lambda_k$ is zero, M^n is said to be of null k-type.

We now focus on null 2-type submanifolds M^n in $\mathbb{E}^m$. By choosing a coordinate system on $\mathbb{E}^m$ with x_0 as its origin, we have the following simple spectral decomposition of x for a null 2-type submanifold M^n:

$$x = x_1 + x_2, \quad \Delta x_1 = 0, \quad \Delta x_2 = a x_2, \tag{1.2}$$

where a is non-zero constant. After applying Beltrami's formula $\Delta x = -n\overrightarrow{H}$, where $\overrightarrow{H}$ is the mean curvature vector, (1.2) implies the following equation

$$\Delta \overrightarrow{H} = a \overrightarrow{H}. \tag{1.3}$$

Chen proposed in 1991 the following interesting problem [2, Problem 12]:

Received May 21, 2014, accepted July 8, 2014.
Communicated by Bang-Yen Chen.
2010 *Mathematics Subject Classification*: 53D12, 53C40, 53C42.
Key words and phrases: Null 2-type submanifolds, 3 Distinct principal curvatures, Mean curvature vector.

"Determine all submanifolds of Euclidean spaces which are of null 2-type. In particular, classify null 2-type hypersurfaces in Euclidean spaces."

In 1988, Chen [3] firstly proved that a null 2-type surface in $\mathbb{E}^3$ is an open portion of a circular cylinder $S^1 \times \mathbb{R}$. Later on, Ferrândez and Lucas [14] generalized Chen's results by showing that a null 2-type Euclidean hypersurface in $\mathbb{E}^{n+1}$ with at most two distinct principal curvatures is a spherical cylinder $S^p \times \mathbb{R}^{n-p}$. In 1995, Hasanis and Vlachos [15] proved that null 2-type hypersurfaces in $\mathbb{E}^4$ have constant mean curvature (see also Defever's proof in [11]). Recently, Chen and Garray in [8] characterized $\delta(2)$-ideal null 2-type hypersurfaces in Euclidean space as spherical cylinders, where $\delta(2)$-ideal hypersurfaces are a class of hypersurfaces whose principal curvatures take three special values: η, μ and $\eta + \mu$. There are also some study on null 2-type submanifolds with codimension greater one due to U. Dursun ([12, 13]). For more work in this field, see Chen's recent excellent survey [10].

A remarkable property obtained by Chen [4] says that a submanifold M^n of Euclidean space satisfies (1.3) if and only if M^n is 1) Biharmonic (in this case, $a = 0$); 2) 1-type; 3) null 2-type.

As pointed out by Chen et al., for example, in [8], a 1-type submanifold of a Euclidean space $\mathbb{E}^m$ is either a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of a hypersphere in $\mathbb{E}^m$. Biharmonic submanifolds in $\mathbb{E}^m$ are defined by the equation $\Delta\overrightarrow{H} = 0$, which is equivalent to $\Delta^2 x = 0$. Chen [2] in 1991 stated a well-known conjecture: *The only biharmonic submanifolds of Euclidean spaces are the minimal ones*. This conjecture is still open so far and the study of biharmonic submanifolds is a very active field [10].

In this paper, we investigate null 2-type hypersurfaces with at most three distinct principal curvatures in Euclidean space. Precisely, we will prove that

Theorem 1.1. *Every null 2-type hypersurface with at most three distinct principal curvatures in a Euclidean space must have constant mean curvature.*

Remark that our result generalizes the results given in [3, 8, 14, 15].

2. Preliminaries

Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometric immersion of a hypersurface M^n into $\mathbb{E}^{n+1}$. Denote the Levi-Civita connections of M^n and $\mathbb{E}^{n+1}$ by ∇ and $\tilde{\nabla}$, respectively. Let X and Y denote vector fields tangent to M^n and let ξ be a unite normal vector field. Then the Gauss and Weingarten formulas are given, respectively, by (cf. [5, 6])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -AX, \tag{2.2}$$

where h is the second fundamental form, and A is the shape operator (or Weingarten operator). It is well known that the second fundamental form h and the shape operator A are related by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle. \tag{2.3}$$

The mean curvature vector $\overrightarrow{H}$ is given by

$$\overrightarrow{H} = \frac{1}{n}\mathrm{trace}\ h. \tag{2.4}$$

The Gauss and Codazzi equations are given respectively by

$$R(X,Y)Z = \langle AY, Z\rangle AX - \langle AX, Z\rangle AY,$$

$$(\nabla_X A)Y = (\nabla_Y A)X,$$

where R is the curvature tensor and $(\nabla_X A)Y$ is defined by

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y) \tag{2.5}$$

for all X, Y, Z tangent to M.

Assume that $\overrightarrow{H} = H\xi$. Note that H denotes the mean curvature. By identifying the tangent and the normal parts of the condition $\Delta\overrightarrow{H} = a\overrightarrow{H}$ ($a \neq 0$), we obtain necessary and sufficient conditions for M^n to be of null 2-type in $\mathbb{E}^{n+1}$.

Proposition 2.1. *Assume M^n is not 1-type. A hypersurface M^n in an $n+1$-dimensional Euclidean space $\mathbb{E}^{n+1}$ is null 2-type if and only if*

$$\begin{cases} \Delta H + H\mathrm{trace}\, A^2 = aH, \\ 2A\,\mathrm{grad}H + n\,H\mathrm{grad}H = 0, \end{cases} \tag{2.6}$$

where the Laplace operator Δ acting on scalar-valued function f is given by (e.g., [8])

$$\Delta f = -\sum_{i=1}^{n}(e_i e_i f - \nabla_{e_i} e_i f). \tag{2.7}$$

Here, $\{e_1, \ldots, e_n\}$ is an orthonormal local tangent frame on M^n.

3. Proof of Theorem 1.1

In what follows, we work on null 2-type hypersurfaces M^n with three distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$ with $n \geq 4$.

Suppose that the mean curvature H is not constant. We will derive a contradiction.

By the second equation of (2.6), it is easy to see that $\operatorname{grad} H$ is an eigenvector of the Weingarten operator A with the corresponding principal curvature $-\frac{n}{2}H$. Without loss of generality, we choose e_1 such that e_1 is parallel to $\operatorname{grad} H$, and therefore the Weingarten operator A of M^n takes the following form with respect to a suitable orthonormal frame $\{e_1, \ldots, e_n\}$.

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \tag{3.1}$$

where λ_i are the principal curvatures and $\lambda_1 = -\frac{n}{2}H$. Since e_1 is parallel to $\operatorname{grad} H$, we compute

$$\operatorname{grad} H = \sum_{i=1}^{n} e_i(H) e_i$$

and hence

$$e_1(H) \neq 0, \quad e_i(H) = 0, \quad i = 2, 3, \ldots, n. \tag{3.2}$$

We write

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k, \quad i, j = 1, 2, \ldots, n. \tag{3.3}$$

We compute $\nabla_{e_k}\langle e_i, e_i\rangle = 0$ and $\nabla_{e_k}\langle e_i, e_j\rangle = 0$, which imply respectively that

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.4}$$

for $i \neq j$ and $i, j, k = 1, 2, \ldots, n$. Furthermore, we deduce from (3.1) and (3.3) and the Codazzi equation that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{3.5}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \tag{3.6}$$

for distinct $i, j, k = 1, 2, \ldots, n$.

It follows from (3.2) and (3.3) that

$$[e_i, e_j](H) = 0, \quad i, j = 2, 3, \ldots, n, \quad i \neq j,$$

which yields

$$\omega_{ij}^1 = \omega_{ji}^1, \tag{3.7}$$

for distinct $i, j = 2, 3, \ldots, n$.

We claim that $\lambda_j \neq \lambda_1$ for $j = 2, 3, \ldots, n$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, by putting $i = 1$ in (3.5) we have that

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1), \tag{3.8}$$

which contradicts to the first expression of (3.2).

By the assumption, M^n is a nondegenerate hypersurface with three distinct principal curvatures. Without loss of generality, we assume that

$$\lambda_2 = \lambda_3 = \cdots = \lambda_p = \alpha,$$

$$\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_n = \beta$$

for $\frac{n+1}{2} \leq p < n$. The multiplicities of principal curvatures α and β are $p-1$ and $n-p$, respectively.

By the definition (2.4) of $\overrightarrow{H}$, we have $nH = \sum_{i=1}^n \lambda_i$. Hence

$$\beta = \frac{\frac{3}{2}nH - (p-1)\alpha}{n-p}. \tag{3.9}$$

Hence, by $\lambda_1 = -\frac{n}{2}H$ and (3.9), $\alpha \neq \lambda_1, \beta$ and $\beta \neq \lambda_1$ yield directly that

$$\alpha \neq -\frac{n}{2}H,\ \frac{3n}{2(n-1)}H,\ \frac{n^2-(p-3)n}{2(p-1)}H. \tag{3.10}$$

Since $n \geq 4$, it follows from (3.9) that $p-1 \geq 2$. For $i, j = 2, 3, \ldots, p$ and $i \neq j$ in (3.5), one has

$$e_i(\alpha) = 0, \quad i = 2, 3, \ldots, p. \tag{3.11}$$

Depending on the multiplicity $n-p$ of the principal curvature β, we consider two cases:

Case A. $n-p \geq 2$. In this case, for $i, j = p+1, \ldots, n$ and $i \neq j$ in (3.5) we have

$$e_i(\beta) = 0, \quad i = p+1, \ldots, n. \tag{3.12}$$

Hence, it follows directly from (3.2), (3.9), (3.11) and (3.12) that

$$e_i(\alpha) = 0, \quad i = 2, \ldots, n. \tag{3.13}$$

Case B. $n-p = 1$. Then (3.11) reduces to

$$e_i(\alpha) = 0, \quad i = 2, \ldots, n-1. \tag{3.14}$$

In this case, we will show that $e_n(\alpha)=0$ in the following.

Let us compute $[e_1,e_i](H)=\big(\nabla_{e_1}e_i-\nabla_{e_i}e_1\big)(H)$ for $i=2,\dots,n$. From the first expression of (3.4), we have $\omega_{i1}^1=0$. For $j=1$ and $i\neq 1$ in (3.5), by (3.2) we have $\omega_{1i}^1=0$ $(i\neq 1)$. Hence we have

$$e_ie_1(H)=0,\quad i=2,\dots,n. \tag{3.15}$$

By (3.14), with a similar way we can show that

$$e_ie_1(\alpha)=0,\quad i=2,\dots,n-1. \tag{3.16}$$

For $j=1$, $k,i\neq 1$ in (3.6) we have

$$(\lambda_i-\lambda_1)\omega_{ki}^1=(\lambda_k-\lambda_1)\omega_{ik}^1,$$

which together with (3.7) yields

$$\omega_{ij}^1=0,\quad i\neq j,\quad i,j=2,\dots n. \tag{3.17}$$

Combining (3.17) with the second equation of (3.4) gives

$$\omega_{i1}^j=0,\quad i\neq j,\quad i,j=2,\dots n. \tag{3.18}$$

It follows from (3.5) that

$$\omega_{i1}^i=\frac{e_1(\lambda_i)}{\lambda_1-\lambda_i},\quad i=2,\dots n. \tag{3.19}$$

For $k=2$ and $i=n$ in (3.6), we have

$$(\lambda_n-\lambda_j)\omega_{2n}^j=(\lambda_2-\lambda_j)\omega_{n2}^j,$$

which yields

$$\omega_{2n}^j=0,\quad j=3,\dots n-1.$$

Hence, from the first expression of (3.4) and (3.17) we get

$$\omega_{2n}^j=0,\quad j=1,3,\dots n. \tag{3.20}$$

Also, (3.5) yields

$$\omega_{2n}^2=\frac{e_n(\alpha)}{\lambda_n-\alpha}. \tag{3.21}$$

In the following we will derive a useful equation.

From the Gauss equation and (3.1) we have $R(e_2, e_n)e_1 = 0$. Recall the definition of Gauss curvature tensor

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

It follows from (3.16), (3.18-21) and (3.4) that

$$\begin{aligned}
\nabla_{e_2}\nabla_{e_n}e_1 &= \frac{e_1(\lambda_n)e_n(\alpha)}{(\lambda_1-\lambda_n)(\lambda_n-\alpha)}e_2,\\
\nabla_{e_n}\nabla_{e_2}e_1 &= e_n(\frac{e_1(\alpha)}{\lambda_1-\alpha})e_2 + \frac{e_1(\alpha)}{\lambda_1-\alpha}\sum_{k=3}^{n}\omega_{n2}^k e_k,\\
\nabla_{[e_2,e_n]}e_1 &= \frac{e_n(\alpha)e_1(\alpha)}{(\lambda_n-\alpha)(\lambda_1-\alpha)}e_2 - \frac{e_1(\alpha)}{\lambda_1-\alpha}\sum_{k=3}^{n}\omega_{n2}^k e_k.
\end{aligned}$$

Hence

$$e_n(\frac{e_1(\alpha)}{\lambda_1-\alpha}) = \Big(\frac{e_1(\lambda_n)}{\lambda_1-\lambda_n} - \frac{e_1(\alpha)}{\lambda_1-\alpha}\Big)\frac{e_n(\alpha)}{\lambda_n-\alpha}. \tag{3.22}$$

Note that $\lambda_1 = -\frac{n}{2}H$ and $\lambda_n = \beta = \frac{3}{2}nH - (n-2)\alpha$.

Equation (3.22) can be rewritten as

$$e_n e_1(\alpha) = \Big\{-\frac{e_1(\alpha)}{\lambda_1-\alpha} + \Big(\frac{e_1(\lambda_n)}{\lambda_1-\lambda_n} - \frac{e_1(\alpha)}{\lambda_1-\alpha}\Big)\frac{\lambda_1-\alpha}{\lambda_n-\alpha}\Big\}e_n(\alpha),$$

and hence

$$\begin{aligned}
e_n(\frac{e_1(\lambda_n)}{\lambda_1-\lambda_n}) &= -(n-2)\Big(\frac{e_n e_1(\alpha)}{\lambda_1-\lambda_n} + \frac{e_1(\lambda_n)e_n(\alpha)}{(\lambda_1-\lambda_n)^2}\Big)\\
&= -(n-2)\frac{e_n(\alpha)}{\lambda_1-\lambda_n}\Big(\frac{e_1(\lambda_n)}{\lambda_1-\lambda_n} - \frac{e_1(\alpha)}{\lambda_1-\alpha}\Big)\frac{\lambda_1+\lambda_n-2\alpha}{\lambda_n-\alpha}.
\end{aligned} \tag{3.23}$$

Consider the first equation of (2.6). It follows from (3.1) and (3.19) that

$$\begin{aligned}
&e_1e_1(H) + \Big(\frac{(n-2)e_1(\alpha)}{\lambda_1-\alpha} + \frac{e_1(\lambda_n)}{\lambda_1-\lambda_n}\Big)e_1(H)\\
&-H\big({\lambda_1}^2 + (n-2)\alpha^2 + {\lambda_n}^2\big) = -aH.
\end{aligned} \tag{3.24}$$

From (3.15) and $\omega_{1n}^1 = \omega_{n1}^1 = 0$, by computing $[e_1, e_n](e_1(H)) = (\nabla_{e_1}e_n - \nabla_{e_n}e_1)(e_1(H)) = 0$, we could deduce that $e_n(e_1e_1(H)) = 0$.

Now differentiating (3.24) along e_n, by (3.2), (3.15), (3.22) and (3.23) we get

$$\frac{2}{\lambda_1-\lambda_n}\Big(\frac{e_1(\lambda_n)}{\lambda_1-\lambda_n} - \frac{\alpha}{\lambda_1-\alpha}\Big)e_1(H)e_n(\alpha) + H\big(-3nH + 2(n-1)\alpha\big)e_n(\alpha) = 0.$$

If $e_n(\alpha)\neq 0$, then the above equation becomes

$$\frac{2}{\lambda_1-\lambda_n}\Big(\frac{e_1(\lambda_n)}{\lambda_1-\lambda_n}-\frac{\alpha}{\lambda_1-\alpha}\Big)e_1(H)+H\big(-3nH+2(n-1)\alpha\big)=0. \tag{3.25}$$

Differentiating (3.25) along e_n, using (3.22) and (3.23) one has

$$\begin{aligned}&\frac{2n(4-n)H+2(n-2)(n-1)\alpha}{(\lambda_1-\lambda_n)(\lambda_n-\alpha)}\Big(\frac{e_1(\lambda_n)}{\lambda_1-\lambda_n}-\frac{\alpha}{\lambda_1-\alpha}\Big)e_1(H)\\&+H\big((-7n+10)nH+4(n-1)(n-2)\alpha\big)=0.\end{aligned} \tag{3.26}$$

Therefore, combining (3.26) with (3.25) gives

$$3(n-2)H\big(3nH-2(n-1)\alpha\big)^2=0,$$

which implies that

$$\alpha=\frac{3n}{2(n-1)}H.$$

This contradicts to (3.10). Hence, we have that $e_n(\alpha)=0$.

Now we are ready to express the connection coefficients of hypersurfaces.

Lemma 3.1. *Under the assumptions above, we have*

$$\begin{aligned}&\nabla_{e_1}e_1=0;\ \nabla_{e_i}e_1=\frac{e_1(\lambda_i)}{\lambda_1-\lambda_i}e_i,\ i=2,\dots,n;\\&\nabla_{e_i}e_j=\sum_{k=2,k\neq j}^{p}\omega_{ij}^k e_k,\ i=1,\dots,n,\ j=2,\dots,p,\ i\neq j;\\&\nabla_{e_i}e_i=-\frac{e_1(\lambda_i)}{\lambda_1-\lambda_i}e_1+\sum_{k=2,k\neq i}^{p}\omega_{ii}^k e_k,\ i=2,\dots,p;\\&\nabla_{e_i}e_j=\sum_{k=p+1,k\neq j}^{n}\omega_{ij}^k e_k,\ i=1,\dots,n,\ j=p+1,\dots,n,\ i\neq j;\\&\nabla_{e_i}e_i=-\frac{e_1(\lambda_i)}{\lambda_1-\lambda_i}e_1+\sum_{k=p+1,k\neq i}^{n}\omega_{ii}^k e_k,\ i=p+1,\dots,n.\end{aligned}$$

Proof. For $j=1$ and $i=2,\dots,n$ in (3.5), by (3.2) we get $\omega_{1i}^1=0$. Moreover, by the first and second expressions of (3.4) we have

$$\omega_{1i}^1=\omega_{11}^i=0,\quad i=1,\dots,n. \tag{3.27}$$

For $i = 1$, $j = 2, \ldots, n$ in (3.5), we obtain

$$\omega_{j1}^{j} = -\omega_{jj}^{1} = \frac{e_1(\lambda_j)}{\lambda_1 - \lambda_j}, \quad j = 2, \ldots, n. \tag{3.28}$$

For $i = p+1, \ldots, n$, $j = 2, \ldots, p$ in (3.5), by (3.2) we have

$$\omega_{ji}^{j} = -\omega_{jj}^{i} = 0. \tag{3.29}$$

Similarly, for $i = 2, \ldots, p$, $j = p+1, \ldots, n$ in (3.5), we also have

$$\omega_{ji}^{j} = -\omega_{jj}^{i} = 0. \tag{3.30}$$

For $i = 1$, by choosing $j, k = 2, \ldots, p$ or $k, j = p+1, \ldots, n$ ($j \neq k$) in (3.6), we have

$$\omega_{k1}^{j} = \omega_{kj}^{1} = 0. \tag{3.31}$$

For $i = 2, \ldots, p$ and $j, k = p+1, \ldots, n$ ($j \neq k$) in (3.6), we get

$$\omega_{ki}^{j} = \omega_{kj}^{i} = 0. \tag{3.32}$$

For $i = 2, \ldots, p$, $j = 1$ and $k = p+1, \ldots, n$ in (3.6), one has

$$(\alpha - \lambda_1)\omega_{ki}^{1} = (\beta - \lambda_1)\omega_{ik}^{1},$$

which together with (3.7) and the second expression of (3.4) gives

$$\omega_{ki}^{1} = \omega_{ik}^{1} = \omega_{k1}^{i} = \omega_{i1}^{k} = 0. \tag{3.33}$$

For $i = 2, \ldots, p$, $k = 1$ and $j = p+1, \ldots, n$ in (3.6), we obtain

$$(\beta - \alpha)\omega_{1i}^{j} = (\lambda_1 - \alpha)\omega_{i1}^{j},$$

which together with (3.33) yields

$$\omega_{1i}^{j} = \omega_{1j}^{i} = 0. \tag{3.34}$$

Combining (3.27-3.34) with (3.4) completes the proof of the lemma. ∎

Define two smooth functions A and B as follows:

$$A = \frac{e_1(\alpha)}{\lambda_1 - \alpha}, \quad B = \frac{e_1(\beta)}{\lambda_1 - \beta}. \tag{3.35}$$

One can compute the curvature tensor R by Lemma 3.1, and apply the Gauss equation for different values of X, Y and Z. After comparing the coefficients with respect to the orthonormal basis $\{e_1, \ldots, e_n\}$ we get the following:

- $X = e_1, Y = e_2, Z = e_1$,

$$e_1(A) + A^2 = -\lambda_1\alpha; \tag{3.36}$$

- $X = e_1, Y = e_n, Z = e_1$,

$$e_1(B) + B^2 = -\lambda_1\beta; \tag{3.37}$$

- $X = e_n, Y = e_2, Z = e_n$,

$$AB = -\alpha\beta. \tag{3.38}$$

Note that equation (3.38) can be obtained by calculating $\langle R(e_n, e_2)e_n, e_2\rangle$.

Compute the first equation of (2.6) again. It follows from (3.1) and Lemma 3.1 that

$$\begin{aligned} &-e_1e_1(H) - \big\{(p-1)A + (n-p)B\big\}e_1(H) \\ &+H\big(\lambda_1^2 + (p-1)\alpha^2 + (n-p)\beta^2\big) = aH. \end{aligned} \tag{3.39}$$

Lemma 3.2. *The functions A and B are related by*

$$\begin{aligned} &\big\{(4-p)A + (3+p-n)B\big\}e_1(H) + \frac{3n^2(n+6-p)}{4(n-p)}H^3 \\ &-\frac{3n(n-2+4p)}{2(n-p)}H^2\alpha + \frac{3n(p-1)}{n-p}H\alpha^2 - \frac{3}{2}aH = 0. \end{aligned} \tag{3.40}$$

Proof. From (3.35), (3.36) and (3.37) respectively reduce to

$$e_1e_1(\alpha) + 2Ae_1(\alpha) - Ae_1(\lambda_1) + \lambda_1\alpha(\lambda_1 - \alpha) = 0, \tag{3.41}$$

$$e_1e_1(\beta) + 2Be_1(\beta) - Be_1(\lambda_1) + \lambda_1\beta(\lambda_1 - \beta) = 0. \tag{3.42}$$

By (3.9), it follows from the second expression of (3.35) that

$$e_1(\alpha) = \frac{3n}{2(p-1)}e_1(H) - \frac{n-p}{p-1}B(\lambda_1 - \beta). \tag{3.43}$$

Similarly,

$$e_1(\beta) = \frac{3n}{2(n-p)}e_1(H) - \frac{p-1}{n-p}A(\lambda_1 - \alpha). \tag{3.44}$$

Substitute (3.9) into (3.42). Eliminating $e_1e_1(H)$ and $e_1e_1(\alpha)$, from (3.38), (3.39) and (3.41-44) we obtain the desired equation (3.40). ■

Now we are in a position to prove Theorem 1.1.

Proof. By the second expression of (3.35) and (3.9), equation (3.44) reduces to

$$e_1(H) = -\Big\{\frac{p-1}{3}H + \frac{2(p-1)}{3n}\alpha\Big\}A + \Big\{-\frac{n+3-p}{3}H + \frac{2(p-1)}{3n}\alpha\Big\}B. \tag{3.45}$$

Substituting (3.45) into (3.40), by (3.38) we have

$$\begin{aligned}&(4-p)(p-1)(nH+2\alpha)A^2\\ &+(3+p-n)\Big\{n(n+3-p)H - 2(p-1)\alpha\Big\}B^2 = f(H,\alpha),\end{aligned} \tag{3.46}$$

where

$$\begin{aligned} f(H,\alpha) = &\frac{9n^3(n+6-p)}{4(n-p)}H^3 + \frac{3n^2(p-1)(2p-2n-15)}{2(n-p)}H^2\alpha \\ &+\frac{n(p-1)(-2p^2+2pn+11p+n-12)}{n-p}H\alpha^2 \\ &-\frac{2(p-1)^2(2p-n-1)}{n-p}\alpha^3 - \frac{9}{2}naH. \end{aligned} \tag{3.47}$$

Multiplying A and B successively on the equation (3.40), using (3.38) one gets respectively

$$\begin{aligned}&(4-p)A^2e_1(H) - (3+p-n)\alpha\beta e_1(H)\\ &+\Big\{\frac{3n^2(n+6-p)}{4(n-p)}H^3 - \frac{3n(n-2+4p)}{2(n-p)}H^2\alpha + \frac{3n(p-1)}{n-p}H\alpha^2 - \frac{3}{2}aH\Big\}A = 0,\end{aligned} \tag{3.48}$$

$$\begin{aligned}&(3+p-n)B^2e_1(H) - (4-p)\alpha\beta e_1(H)\\ &+\Big\{\frac{3n^2(n+6-p)}{4(n-p)}H^3 - \frac{3n(n-2+4p)}{2(n-p)}H^2\alpha + \frac{3n(p-1)}{n-p}H\alpha^2 - \frac{3}{2}aH\Big\}B = 0.\end{aligned} \tag{3.49}$$

Differentiating (3.40) along e_1, and using (3.36-37) and (3.39) we get

$$\begin{aligned}&\Big\{(4-p)(\frac{n}{2}H\alpha - A^2) + (3+p-n)(\frac{n}{2}H\beta - B^2)\Big\}e_1(H)\\ &-\Big\{(4-p)A + (3+p-n)B\Big\}\Big\{(p-1)A + (n-p)B\Big\}e_1(H)\\ &+\Big\{(4-p)A + (3+p-n)B\Big\}\Big\{\frac{n^2}{4}H^3 + (p-1)H\alpha^2 + (n-p)H\beta^2 - aH\Big\}\\ &+\Big\{\frac{9n^2(n+6-p)}{4(n-p)}H^2 - \frac{3n(n-2+4p)}{n-p}H\alpha + \frac{3n(p-1)}{n-p}\alpha^2 - \frac{3}{2}a\Big\}e_1(H)\\ &-\frac{3n(n-2+4p)}{2(n-p)}H^2e_1(\alpha) + \frac{6n(p-1)}{n-p}H\alpha e_1(\alpha) = 0.\end{aligned} \tag{3.50}$$

Substituting (3.40), (3.47), (3.48) into (3.49), and using the first expression of (3.35) we obtain

$$\begin{aligned}&\Big\{\frac{3n^2(2n-2p+21)}{4(n-p)}H^2-\frac{3n(5p+1)}{n-p}H\alpha+\frac{(p-1)(2n+7)}{n-p}\alpha^2-\frac{3}{2}a\Big\}e_1(H)\\&+\Big\{\frac{n^2(2pn-2p^2+7n+17p+30)}{4(n-p)}H^3-\frac{3n(3np+2p^2+4p-3n-6)}{2(n-p)}H^2\alpha\\&+\frac{(p-1)(2np-2n+p-4)}{n-p}H\alpha^2+\frac{1}{2}(5p-8)aH\Big\}A\\&+\Big\{\frac{n^2\big(2(n-p)^2+15(n-p)+45\big)}{4(n-p)}H^3-\frac{3n(n^2+np-2p^2+10p+n-8)}{2(n-p)}H^2\alpha\\&+\frac{(p-1)(2n^2-2np+7n-p-3)}{n-p}H\alpha^2+\frac{1}{2}(5n-5p-3)aH\Big\}B=0.\end{aligned}$$

Moreover, it follows from (3.45) that the above equation further reduces to

$$L(H,\alpha)A+M(H,\alpha)B=0, \tag{3.51}$$

where

$$\begin{aligned}L(H,\alpha)=&\tfrac{9}{4}n^3(3n-2p+17)H^3-\tfrac{3}{2}n^2(-6p^2+11np+43p-11n-37)H^2\alpha\\&+n(p-1)(4np-4n+26p+1)H\alpha^2-2(p-1)^2(2n+7)\alpha^3\\&+\tfrac{9}{2}n(n-p)(2p-3)aH+3(n-p)(p-1)a\alpha,\end{aligned} \tag{3.52}$$

$$\begin{aligned}M(H,\alpha)=&-\tfrac{9}{2}(2n-2p+3)H^3-\tfrac{9}{2}n^2(2p^2+n^2-3np-7p+n-3)H^2\alpha\\&+2n(p-1)(2n^2-2np+4n-13p-18)H\alpha^2+2(p-1)^2(2n+7)\alpha^3\\&-9n(n-p)^2aH+3(n-p)(p-1)a\alpha.\end{aligned} \tag{3.53}$$

Multiplying LM on the equation (3.46), using (3.51-3.53) and (3.38) we can eliminate both A and B. Hence, we have

$$\begin{aligned}&(4-p)(p-1)(nH+2\alpha)M^2\frac{\frac{3}{2}nH\alpha-(p-1)\alpha^2}{n-p}\\&+(3+p-n)\big\{n(n+3-p)H-2(p-1)\alpha\big\}L^2\frac{\frac{3}{2}nH\alpha-(p-1)\alpha^2}{n-p}\\&+LMf=0.\end{aligned} \tag{3.54}$$

In view of (3.54), we notice that the equation should take the following form:

$$\begin{aligned}&c_{90}H^9+c_{81}H^8\alpha+c_{72}H^7\alpha^2+c_{63}H^6\alpha^3+c_{54}H^5\alpha^4+c_{45}H^4\alpha^5\\&+c_{36}H^3\alpha^6+c_{27}H^2\alpha^7+c_{18}H\alpha^8+c_{09}\alpha^9+a(c_{70}H^7+c_{61}H^6\alpha\\&+c_{52}H^5\alpha^2+c_{43}H^4\alpha^3+c_{34}H^3\alpha^4+c_{25}H^2\alpha^5+c_{16}H\alpha^6+c_{07}\alpha^7\\&+c_{50}H^5+c_{41}H^4\alpha+c_{32}H^3\alpha^2+c_{23}H^2\alpha^3+c_{14}H\alpha^5+c_{05}\alpha^5\\&+c_{30}H^3+c_{21}H^2\alpha+c_{12}H\alpha^2+c_{03}\alpha^3)=0,\end{aligned} \tag{3.55}$$

where the coefficients c_{ij} $(i, j = 0, \ldots, 9)$ are constants concerning n and p.

From (3.54), (3.52), (3.53) and (3.47), we compute a_{90} as follows

$$c_{90} = \frac{729n^6(n-p+6)(3n-2p+17)(2n-2p+3)}{32(n-p)}.$$

Since $n > p$, it is easy to see that $c_{90} \neq 0$.

Note that α is not constant in general. In fact, if α is a constant, then (3.55) becomes an algebraic equation of H with constant coefficients. Thus, the real function H satisfies a polynomial equation $q(H) = 0$ with constant coefficients, therefore it must be a constant. We obtain the conclusion immediately.

Now consider an integral curve of e_1 passing through $p = \gamma(t_0)$ as $\gamma(t), t \in I$. Since $e_i(H) = e_i(\alpha) = 0$ for $i = 2, \ldots, n$ and $e_1(H), e_1(\alpha) \neq 0$, we can assume $t = t(\alpha)$ and $H = H(\alpha)$ in some neighborhood of $\alpha_0 = \alpha(t_0)$.

From the first expression of (3.35), (3.45) and (3.51), we have

$$\begin{aligned} \frac{dH}{d\alpha} &= \frac{dH}{dt}\frac{dt}{d\alpha} = \frac{e_1(H)}{e_1(\alpha)} \\ &= \frac{-(\frac{p-1}{3}H + \frac{2(p-1)}{3n}\alpha)A + (-\frac{n+3-p}{3}H + \frac{2(p-1)}{3n}\alpha)B}{(-\frac{n}{2}H - \alpha)A} \\ &= \frac{2(p-1)}{3n} + \frac{(-\frac{n+3-p}{3}H + \frac{2(p-1)}{3n}\alpha)B}{(-\frac{n}{2}H - \alpha)A} \\ &= \frac{2(p-1)}{3n} + \frac{2\big((n+3-p)H - 2(p-1)\alpha\big)L}{3n(nH+2\alpha)M}. \end{aligned} \tag{3.56}$$

Differentiating (3.55) with respect to α and substituting $\frac{dH}{d\alpha}$ from (3.56), combining these with (3.51) we get another algebraic equation of twelfth degree concerning H and α

$$\begin{aligned} & b_{12,0}H^{12} + b_{11,1}H^{11}\alpha + b_{10,2}H^{10}\alpha^2 + b_{93}H^9\alpha^3 + b_{84}H^8\alpha^4 + b_{75}H^7\alpha^5 \\ & + b_{66}H^6\alpha^6 + b_{57}H^5\alpha^7 + b_{48}H^4\alpha^8 + b_{39}H^3\alpha^9 + b_{2,10}H^2\alpha^{10} + b_{1,11}H\alpha^{11} \\ & + b_{0,12}\alpha^{12} + c(b_{10,0}H^{10} + b_{91}H^9\alpha + b_{82}H^8\alpha^2 + b_{73}H^7\alpha^3 + b_{64}H^6\alpha^4 \\ & + b_{55}H^5\alpha^5 + b_{46}H^4\alpha^6 + b_{37}H^3\alpha^7 + b_{28}H^2\alpha^8 + b_{19}H\alpha^9 + b_{0,10}\alpha^{10} + b_{80}H^8 \\ & + b_{71}H^7\alpha + b_{62}H^6\alpha^2 + b_{53}H^5\alpha^3 + b_{44}H^4\alpha^4 + b_{35}H^3\alpha^5 + b_{26}H^2\alpha^6 + b_{17}H\alpha^7 \\ & + b_{08}\alpha^8 + b_{60}H^6 + b_{51}H^5\alpha + b_{42}H^4\alpha^2 + b_{33}H^3\alpha^3 + b_{24}H^2\alpha^4 + b_{15}H\alpha^5 \\ & + b_{06}\alpha^6 + b_{40}H^4 + b_{31}H^3\alpha + b_{22}H^2\alpha^2 + b_{13}H\alpha^3 + b_{04}\alpha^4) = 0, \end{aligned} \tag{3.57}$$

where the coefficients b_{ij} $(i, j = 0, \ldots, 12)$ are constants concerning n and p.

Note that equation (3.57) is non-trivial and different from (3.55).

We rewrite (3.55) and (3.57) respectively in the following forms

(3.58) $$\sum_{i=0}^{9} q_i(H)\alpha^i = 0, \qquad \sum_{j=0}^{12} \bar{q}_j(H)\alpha^j = 0,$$

where $q_i(H)$ and $\bar{q}_j(H)$ are polynomials concerning function H.

We may eliminate α between the two equations of (3.58). Multiplying $\bar{q}_{12}(H)\alpha^3$ and $q_8(H)$ respectively on the first and second equations of (3.58), we obtain a new polynomial equation of α with eleventh degree. Combining this equation with the first equation of (3.58), we successively obtain a polynomial equation of α with tenth degree. In a similar way, by using the first equation of (3.58) and its consequences we are able to gradually eliminate α.

At last, we obtain a non-trivial algebraic polynomial equation of H with constant coefficients. Therefore, we conclude that the real function H must be a constant, which contradicts our original assumption.

In conclusion, we complete the proof of Theorem 1.1. ∎

Acknowledgments

The author is supported by the Mathematical Tianyuan Youth Fund of China (No. 11326068), the Natural Science Foundation of China (No. 71271045), the Excellent Innovation talents Project of DUFE (No. DUFE2014R26), and the Youth Project of DUFE (No. DUFE2014Q25).

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