

## INEQUALITIES FOR GENERALIZED NORMALIZED $\delta$ -CASORATI CURVATURES OF SLANT SUBMANIFOLDS IN QUATERNIONIC SPACE FORMS

Jaewon Lee and Gabriel-Eduard Vîlcu

**Abstract.** In this paper we prove two sharp inequalities involving the normalized scalar curvature and the generalized normalized  $\delta$ -Casorati curvatures for slant submanifolds in quaternionic space forms. We also characterize those submanifolds for which the equality cases hold. These results are a generalization of some recent results concerning the Casorati curvature for a slant submanifold in a quaternionic space form obtained by Slesar et al., *J. Inequal. Appl.*, 2014, 2014:123.

### 1. INTRODUCTION

In order to provide answers to an open question raised by S. S. Chern [8] concerning the existence of minimal immersions into Euclidean spaces of arbitrary dimension, Prof. B.-Y. Chen [3] introduced in the early 1990's new types of Riemannian invariants, known in the literature as Chen invariants or  $\delta$ -invariants and established general optimal inequalities involving the new intrinsic invariants and the main extrinsic invariants for arbitrary Riemannian submanifolds. Thus was born the theory of Chen invariants, one of the most interesting research topic in differential geometry of submanifolds.

After  $\delta$ -invariants were invented and first inequalities were proved, such invariants and Chen-like inequalities were considered in different ambient spaces for many classes of submanifolds. For example, new optimal inequalities involving Chen invariants were recently proved in [1, 5, 6, 7, 11, 14, 15, 19].

On the other hand, it is well-known that the Casorati curvature of a submanifold in a Riemannian manifold is an extrinsic invariant defined as the normalized square of the length of the second fundamental form and it was preferred by Casorati over the traditional Gauss curvature because corresponds better with the common intuition

---

Received May 18, 2014, accepted August 12, 2014.

Communicated by Bang-Yen Chen.

2010 *Mathematics Subject Classification*: 53A07, 53B42, 53C42.

*Key words and phrases*: Scalar curvature, Mean curvature,  $\delta$ -Casorati curvature, Shape operator, Quaternionic space form, Slant submanifold, Optimal inequality.

of curvature [2]. Some optimal inequalities involving Casorati curvatures were proved in [9, 10, 12, 16] for several submanifolds in real, complex and quaternionic space forms. In this paper, we establish two sharp inequalities involving the normalized scalar curvature and the generalized normalized  $\delta$ -Casorati curvatures for slant submanifolds in quaternionic space forms and also completely characterize those submanifolds for which the equality cases hold, generalizing some recent results from [16].

## 2. PRELIMINARIES

This section gives several basic definitions and notations for our framework based mainly on [4, 13].

Let  $M^n$  be an  $n$ -dimensional Riemannian submanifold of an  $m$ -dimensional Riemannian manifold  $(\bar{M}^m, \bar{g})$ . Then we denote by  $g$  the metric tensor induced on  $M$ . Let  $K(\pi)$  be the sectional curvature of  $M$  associated with a plane section  $\pi \subset T_p M$ ,  $p \in M$ . If  $\{e_1, \dots, e_n\}$  is an orthonormal basis of the tangent space  $T_p M$  and  $\{e_{n+1}, \dots, e_m\}$  is an orthonormal basis of the normal space  $T_p^\perp M$ , then the scalar curvature  $\tau$  at  $p$  is given by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$$

and the normalized scalar curvature  $\rho$  of  $M$  is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

If  $\bar{\nabla}$  is the Levi-Civita connection on  $\bar{M}$  and  $\nabla$  is the covariant differentiation induced on  $M$ , then the Gauss and Weingarten formulas are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM)$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^\perp)$$

where  $h$  is the second fundamental form of  $M$ ,  $\nabla^\perp$  is the connection on the normal bundle and  $A_N$  is the shape operator of  $M$  with respect to  $N$ . If we denote by  $\bar{R}$  and  $R$  the curvature tensor fields of  $\bar{\nabla}$  and  $\nabla$ , then we have the Gauss equation:

$$(1) \quad \begin{aligned} \bar{R}(X, Y, Z, W) &= R(X, Y, Z, W) + \bar{g}(h(X, W), h(Y, Z)) \\ &\quad - \bar{g}(h(X, Z), h(Y, W)), \end{aligned}$$

for all  $X, Y, Z, W \in \Gamma(TM)$ .

We denote by  $H$  the mean curvature vector, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$$

and we also set

$$h_{ij}^\alpha = g(h(e_i, e_j), e_\alpha), \quad i, j \in \{1, \dots, n\}, \quad \alpha \in \{n + 1, \dots, m\}.$$

Then it is well-known that the squared mean curvature of the submanifold  $M$  in  $\bar{M}$  is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^m \left( \sum_{i=1}^n h_{ii}^\alpha \right)^2$$

and the squared norm of  $h$  over dimension  $n$  is denoted by  $\mathcal{C}$  and is called the Casorati curvature of the submanifold  $M$ . Therefore we have

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^m \sum_{i,j=1}^n (h_{ij}^\alpha)^2.$$

The submanifold  $M$  is called *invariantly quasi-umbilical* if there exists  $m - n$  mutually orthogonal unit normal vectors  $\xi_{n+1}, \dots, \xi_m$  such that the shape operators with respect to all directions  $\xi_\alpha$  have an eigenvalue of multiplicity  $n - 1$  and that for each  $\xi_\alpha$  the distinguished eigendirection is the same.

Suppose now that  $L$  is an  $s$ -dimensional subspace of  $T_pM$ ,  $s \geq 2$  and let  $\{e_1, \dots, e_s\}$  be an orthonormal basis of  $L$ . Then the scalar curvature  $\tau(L)$  of the  $s$ -plane section  $L$  is given by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq s} K(e_\alpha \wedge e_\beta)$$

and the Casorati curvature  $\mathcal{C}(L)$  of the subspace  $L$  is defined as

$$\mathcal{C}(L) = \frac{1}{s} \sum_{\alpha=n+1}^m \sum_{i,j=1}^s (h_{ij}^\alpha)^2.$$

The normalized  $\delta$ -Casorati curvatures  $\delta_c(n - 1)$  and  $\widehat{\delta}_c(n - 1)$  of the submanifold  $M^n$  are given by

$$[\delta_c(n - 1)]_p = \frac{1}{2} \mathcal{C}_p + \frac{n + 1}{2n} \inf\{\mathcal{C}(L) | L \text{ a hyperplane of } T_pM\}$$

and

$$[\widehat{\delta}_c(n - 1)]_p = 2\mathcal{C}_p - \frac{2n - 1}{2n} \sup\{\mathcal{C}(L) | L \text{ a hyperplane of } T_pM\}.$$

The generalized normalized  $\delta$ -Casorati curvatures  $\delta_C(r; n - 1)$  and  $\widehat{\delta}_C(r; n - 1)$  of the submanifold  $M^n$  are defined for any positive real number  $r \neq n(n - 1)$  as

$$[\delta_C(r; n - 1)]_p = r\mathcal{C}_p + \frac{(n - 1)(n + r)(n^2 - n - r)}{rn} \inf\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM\},$$

if  $0 < r < n^2 - n$ , and

$$[\widehat{\delta}_C(r; n - 1)]_p = r\mathcal{C}_p - \frac{(n - 1)(n + r)(r - n^2 + n)}{rn} \sup\{\mathcal{C}(L) \mid L \text{ a hyperplane of } T_pM\},$$

if  $r > n^2 - n$ .

Assume now that  $(\bar{M}, \bar{g})$  is a smooth manifold such that there is a rank 3-subbundle  $\sigma$  of  $End(T\bar{M})$  with local basis  $\{J_1, J_2, J_3\}$  satisfying for all  $\alpha \in \{1, 2, 3\}$ :

$$\bar{g}(J_\alpha \cdot, J_\alpha \cdot) = \bar{g}(\cdot, \cdot)$$

and

$$J_\alpha^2 = -Id, J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where  $Id$  denotes the identity tensor field of type  $(1, 1)$  on  $\bar{M}$  and the indices are taken from  $\{1, 2, 3\}$  modulo 3. Then  $(\bar{M}, \sigma, \bar{g})$  is said to be an almost quaternionic Hermitian manifold. It is easy to see that such manifold is of dimension  $4m$ ,  $m \geq 1$ . Moreover, if the bundle  $\sigma$  is parallel with respect to the Levi-Civita connection  $\bar{\nabla}$  of  $\bar{g}$ , then  $(\bar{M}, \sigma, \bar{g})$  is said to be a quaternionic Kähler manifold.

Let  $(\bar{M}, \sigma, \bar{g})$  be a quaternionic Kähler manifold and let  $X$  be a non-null vector field on  $\bar{M}$ . Then the 4-plane spanned by  $\{X, J_1X, J_2X, J_3X\}$ , denoted by  $Q(X)$ , is called a quaternionic 4-plane. Any 2-plane in  $Q(X)$  is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say  $c$ . It is well-known that a quaternionic Kähler manifold  $(\bar{M}, \sigma, \bar{g})$  is a quaternionic space form, denoted  $\bar{M}(c)$ , if and only if its curvature tensor is given by

$$(2) \quad \bar{R}(X, Y)Z = \frac{c}{4} \{ \bar{g}(Z, Y)X - \bar{g}(X, Z)Y + \sum_{\alpha=1}^3 [\bar{g}(Z, J_\alpha Y)J_\alpha X - \bar{g}(Z, J_\alpha X)J_\alpha Y + 2\bar{g}(X, J_\alpha Y)J_\alpha Z] \}$$

for all vector fields  $X, Y, Z$  on  $\bar{M}$  and any local basis  $\{J_1, J_2, J_3\}$  of  $\sigma$ .

A submanifold  $M$  of a quaternionic Kähler manifold  $(\bar{M}, \sigma, \bar{g})$  is said to be a slant submanifold [17] if for each non-zero vector  $X$  tangent to  $M$  at  $p$ , the angle  $\theta(X)$

between  $J_\alpha(X)$  and  $T_pM$ ,  $\alpha \in \{1, 2, 3\}$  is constant, i.e. it does not depend on the choice of  $p \in M$  and  $X \in T_pM$ . We can easily see that quaternionic submanifolds are slant submanifolds with  $\theta = 0$  and totally-real submanifolds are slant submanifolds with  $\theta = \frac{\pi}{2}$ . A slant submanifold of a quaternionic Kähler manifold is said to be proper (or  $\theta$ -slant proper) if it is neither quaternionic nor totally real. We recall that every proper slant submanifold of a quaternionic Kähler manifold is of even dimension  $n = 2s \geq 2$  and we can choose a canonical orthonormal local frame, called an adapted slant frame, as follows:

$$\{e_1, e_2 = \sec\theta P_\alpha e_1, \dots, e_{2s-1}, e_{2s} = \sec\theta P_\alpha e_{2s-1}\},$$

where  $P_\alpha e_{2k-1}$  denotes the tangential component of  $J_\alpha e_{2k-1}$ ,  $k \in \{1, \dots, s\}$  and  $\alpha$  is 1, 2 or 3 (see [18, 20]).

### 3. MAIN RESULTS

**Theorem 3.1.** *Let  $M^n$  be a  $\theta$ -slant proper submanifold of a quaternionic space form  $\bar{M}^{4m}(c)$ . Then:*

(i) *The generalized normalized  $\delta$ -Casorati curvature  $\delta_C(r; n - 1)$  satisfies*

$$(3) \quad \rho \leq \frac{\delta_C(r; n - 1)}{n(n - 1)} + \frac{c}{4} \left( 1 + \frac{9}{n - 1} \cos^2 \theta \right)$$

*for any real number  $r$  such that  $0 < r < n(n - 1)$ .*

(ii) *The generalized normalized  $\delta$ -Casorati curvature  $\widehat{\delta}_C(r; n - 1)$  satisfies*

$$(4) \quad \rho \leq \frac{\widehat{\delta}_C(r; n - 1)}{n(n - 1)} + \frac{c}{4} \left( 1 + \frac{9}{n - 1} \cos^2 \theta \right)$$

*for any real number  $r > n(n - 1)$ .*

*Moreover, the equality sign holds in the inequalities (3) and (4) if and only if  $M^n$  is an invariantly quasi-umbilical submanifold with trivial normal connection in  $\bar{M}^{4m}(c)$ , such that with respect to suitable orthonormal tangent frame  $\{\xi_1, \dots, \xi_n\}$  and normal orthonormal frame  $\{\xi_{n+1}, \dots, \xi_{4m}\}$ , the shape operators  $A_r \equiv A_{\xi_r}$ ,  $r \in \{n + 1, \dots, 4m\}$ , take the following forms:*

$$(5) \quad A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{r}a \end{pmatrix}, \quad A_{n+2} = \dots = A_{4m} = 0.$$

*Proof.* Since  $M$  is  $\theta$ -slant, then it is known from [17] that

$$(6) \quad P_\beta P_\alpha X = -\cos^2 \theta X, \quad \forall X \in \Gamma(TM), \quad \alpha, \beta \in \{1, 2, 3\},$$

where  $P_\alpha X$  denotes the tangential component of  $J_\alpha X$ .

From (6) it follows immediately that

$$(7) \quad g(P_\alpha X, P_\beta Y) = \cos^2 \theta g(X, Y).$$

for  $X, Y \in \Gamma(TM)$  and  $\alpha, \beta \in \{1, 2, 3\}$ .

On the other hand, because  $M^{4m}(c)$  is a quaternionic space form, from (1) and (2) we derive

$$(8) \quad n^2 \|H\|^2 = 2\tau(p) + \|h\|^2 - \frac{n(n-1)c}{4} - \frac{3c}{4} \sum_{\beta=1}^3 \sum_{i,j=1}^n g^2(P_\beta e_i, e_j).$$

Choosing now an adapted slant basis  $\{e_1, e_2 = \sec \theta P_\alpha e_1, \dots, e_{2s-1}, e_{2s} = \sec \theta P_\alpha e_{2s-1}\}$  of  $T_p M$ ,  $p \in M$ , where  $2s = n$ , from (6) and (7), we derive

$$(9) \quad g^2(P_\beta e_i, e_{i+1}) = g^2(P_\beta e_{i+1}, e_i) = \cos^2 \theta, \quad \text{for } i = 1, 3, \dots, 2s - 1$$

and

$$(10) \quad g(P_\beta e_i, e_j) = 0, \quad \text{for } (i, j) \notin \{(2l-1, 2l), (2l, 2l-1) \mid l \in \{1, 2, \dots, s\}\}.$$

By using (9) and (10) in (8) we get

$$(11) \quad 2\tau(p) = n^2 \|H\|^2 - nC + \frac{c}{4} [n(n-1) + 9n \cos^2 \theta].$$

We consider now the following quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = rC + \frac{(n-1)(n+r)(n^2-n-r)}{rn} C(L) - 2\tau(p) + \frac{c}{4} [n(n-1) + 9n \cos^2 \theta],$$

where  $L$  is a hyperplane of  $T_p M$ . Without loss of generality we can assume that  $L$  is spanned by  $e_1, \dots, e_{n-1}$ . Then we derive

$$(12) \quad \mathcal{P} = \frac{r}{n} \sum_{\alpha=n+1}^{4m} \sum_{i,j=1}^n (h_{ij}^\alpha)^2 + \frac{(n+r)(n^2-n-r)}{rn} \sum_{\alpha=n+1}^{4m} \sum_{i,j=1}^{n-1} (h_{ij}^\alpha)^2 - 2\tau(p) + \frac{c}{4} [n(n-1) + 9n \cos^2 \theta].$$

From (11) and (12), we obtain

$$\mathcal{P} = \frac{n+r}{n} \sum_{\alpha=n+1}^{4m} \sum_{i,j=1}^n (h_{ij}^\alpha)^2 + \frac{(n+r)(n^2-n-r)}{rn} \sum_{\alpha=n+1}^{4m} \sum_{i,j=1}^{n-1} (h_{ij}^\alpha)^2 - \sum_{\alpha=n+1}^{4m} \left( \sum_{i=1}^n h_{ii}^\alpha \right)^2$$

which is equivalent to

$$\begin{aligned} \mathcal{P} &= \sum_{\alpha=n+1}^{4m} \sum_{i=1}^{n-1} \left[ \frac{n^2+n(r-1)-2r}{r} (h_{ii}^\alpha)^2 + \frac{2(n+r)}{n} (h_{in}^\alpha)^2 \right] \\ (13) \quad &+ \sum_{\alpha=n+1}^{4m} \left[ \frac{2(n+r)(n-1)}{r} \sum_{i<j=1}^{n-1} (h_{ij}^\alpha)^2 - 2 \sum_{i<j=1}^n h_{ii}^\alpha h_{jj}^\alpha + \frac{r}{n} (h_{nn}^\alpha)^2 \right]. \end{aligned}$$

From (13) it follows that critical points

$$h^c = (h_{11}^{n+1}, h_{12}^{n+1}, \dots, h_{nn}^{n+1}, \dots, h_{11}^{4m}, h_{12}^{4m}, \dots, h_{nn}^{4m})$$

of  $\mathcal{P}$  are the solutions of the following system of linear homogeneous equations:

$$(14) \quad \begin{cases} \frac{\partial \mathcal{P}}{\partial h_{ii}^\alpha} = \frac{2(n+r)(n-1)}{r} h_{ii}^\alpha - 2 \sum_{k=1}^n h_{kk}^\alpha = 0 \\ \frac{\partial \mathcal{P}}{\partial h_{nn}^\alpha} = \frac{2r}{n} h_{nn}^\alpha - 2 \sum_{k=1}^{n-1} h_{kk}^\alpha = 0 \\ \frac{\partial \mathcal{P}}{\partial h_{ij}^\alpha} = \frac{4(n+r)(n-1)}{r} h_{ij}^\alpha = 0 \\ \frac{\partial \mathcal{P}}{\partial h_{in}^\alpha} = \frac{4(n+r)}{n} h_{in}^\alpha = 0 \end{cases}$$

with  $i, j \in \{1, \dots, n-1\}$ ,  $i \neq j$ , and  $\alpha \in \{n+1, \dots, 4m\}$ .

From (14) it follows that every solutions  $h^c$  has  $h_{ij}^\alpha = 0$  for  $i \neq j$  and the determinant which corresponds to the first two sets of equations of the above system is zero (there exist solutions for non-totally geodesic submanifolds). Moreover it is easy to see that the Hessian matrix of  $\mathcal{P}$  has the form

$$\mathcal{H}(\mathcal{P}) = \begin{pmatrix} H_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} \frac{2(n+r)(n-1)}{r} - 2 & -2 & \dots & -2 & -2 \\ -2 & \frac{2(n+r)(n-1)}{r} - 2 & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & \frac{2(n+r)(n-1)}{r} - 2 & -2 \\ -2 & -2 & \dots & -2 & \frac{2r}{n} \end{pmatrix},$$

$\mathbf{0}$  denotes the null matrix of corresponding dimensions and  $H_2, H_3$  are the next diagonal matrices

$$H_2 = \text{diag} \left( \frac{4(n+r)(n-1)}{r}, \frac{4(n+r)(n-1)}{r}, \dots, \frac{4(n+r)(n-1)}{r} \right),$$

$$H_3 = \text{diag} \left( \frac{4(n+r)}{n}, \frac{4(n+r)}{n}, \dots, \frac{4(n+r)}{n} \right).$$

Therefore we find that  $\mathcal{H}(\mathcal{P})$  has the following eigenvalues:

$$\lambda_{11} = 0, \lambda_{22} = \frac{2(n^3 - n^2 + r^2)}{rn}, \lambda_{33} = \dots = \lambda_{nn} = \frac{2(n+r)(n-1)}{r},$$

$$\lambda_{ij} = \frac{4(n+r)(n-1)}{r}, \lambda_{in} = \frac{4(n+r)}{n}, \forall i, j \in \{1, \dots, n-1\}, i \neq j.$$

Hence we deduce that  $\mathcal{P}$  is parabolic and reaches a minimum  $\mathcal{P}(h^c)$  for each solution  $h^c$  of the system (14). Inserting now (14) in (13) we get that  $\mathcal{P}(h^c) = 0$ . So  $\mathcal{P} \geq 0$ , and this implies

$$2\tau(p) \leq r\mathcal{C} + \frac{(n-1)(n+r)(n^2 - n - r)}{rn}\mathcal{C}(L) + \frac{c}{4} [n(n-1) + 9n \cos^2 \theta]$$

Therefore we derive

$$(15) \quad \rho \leq \frac{r}{n(n-1)}\mathcal{C} + \frac{(n+r)(n^2 - n - r)}{rn^2}\mathcal{C}(L) + \frac{c}{4} \left( 1 + \frac{9}{n-1} \cos^2 \theta \right)$$

for every tangent hyperplane  $L$  of  $M$  and both inequalities (3) and (4) obviously follow from (15).

Moreover, we can easily see now that the equality sign holds in the inequalities (3) and (4) if and only if

$$(16) \quad h_{ij}^\alpha = 0, \forall i, j \in \{1, \dots, n\}, i \neq j$$

and

$$(17) \quad h_{nn}^\alpha = \frac{n(n-1)}{r}h_{11}^\alpha = \frac{n(n-1)}{r}h_{22}^\alpha = \dots = \frac{n(n-1)}{r}h_{n-1,n-1}^\alpha$$



for all  $\alpha \in \{n + 1, \dots, 4m\}$ .

Finally, from (16) and (17) we deduce that the equality sign holds in (3) and (4) if and only if the submanifold  $M$  is invariantly quasi-umbilical with trivial normal connection in  $\bar{M}$ , such that the shape operators take the forms (5) with respect to suitable tangent and normal orthonormal frames. ■

**Corollary 3.2.** *Let  $M^n$  be a  $\theta$ -slant proper submanifold of a quaternionic space form  $\bar{M}^{4m}(c)$ . Then:*

(i) *The normalized  $\delta$ -Casorati curvature  $\delta_c(n - 1)$  satisfies*

$$(18) \quad \rho \leq \delta_c(n - 1) + \frac{c}{4} \left( 1 + \frac{9}{n - 1} \cos^2 \theta \right).$$

*Moreover, the equality sign holds if and only if  $M^n$  is an invariantly quasi-umbilical submanifold with trivial normal connection in  $\bar{M}^{4m}(c)$ , such that with respect to suitable orthonormal tangent frame  $\{\xi_1, \dots, \xi_n\}$  and normal orthonormal frame  $\{\xi_{n+1}, \dots, \xi_{4m}\}$ , the shape operators  $A_r \equiv A_{\xi_r}$ ,  $r \in \{n + 1, \dots, 4m\}$ , take the following forms:*

$$(19) \quad A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 0 & 2a \end{pmatrix}, \quad A_{n+2} = \dots = A_{4m} = 0.$$

(ii) *The normalized  $\delta$ -Casorati curvature  $\widehat{\delta}_c(n - 1)$  satisfies*

$$(20) \quad \rho \leq \widehat{\delta}_c(n - 1) + \frac{c}{4} \left( 1 + \frac{9}{n - 1} \cos^2 \theta \right).$$

*Moreover, the equality sign holds if and only if  $M^n$  is an invariantly quasi-umbilical submanifold with trivial normal connection in  $\bar{M}^{4m}(c)$ , such that with respect to suitable orthonormal tangent frame  $\{\xi_1, \dots, \xi_n\}$  and normal orthonormal frame  $\{\xi_{n+1}, \dots, \xi_{4m}\}$ , the shape operators  $A_r \equiv A_{\xi_r}$ ,  $r \in \{n + 1, \dots, 4m\}$ , take the following forms:*

$$(21) \quad A_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \dots & 0 & 0 \\ 0 & 2a & 0 & \dots & 0 & 0 \\ 0 & 0 & 2a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 2a & 0 \\ 0 & 0 & 0 & \dots & 0 & a \end{pmatrix}, \quad A_{n+2} = \dots = A_{4m} = 0.$$

*Proof.* (i) It is easy to see that the following relation holds

$$(22) \quad \left[ \delta_C \left( \frac{n(n-1)}{2}; n-1 \right) \right]_p = n(n-1) [\delta_C(n-1)]_p$$

in any point  $p \in M$ . Therefore, taking  $r = \frac{n(n-1)}{2}$  in (3) and making use of (22) we obtain the conclusion.

(ii) The following relation can be easily verified:

$$(23) \quad \left[ \widehat{\delta}_C(2n(n-1); n-1) \right]_p = n(n-1) \left[ \widehat{\delta}_c(n-1) \right]_p, \quad \forall p \in M.$$

Replacing now  $r = 2n(n-1)$  in (4) and taking account of (23) we derive the conclusion.  $\blacksquare$

**Remark 3.3.** We note that the above Corollary was recently proved in [16], but with a slightly modified coefficient in the definition of  $\delta_C(n-1)$ ; in fact, in [16] it was used the coefficient  $\frac{n+1}{2n(n-1)}$ , as in [9, 12], instead of  $\frac{n+1}{2n}$ , like in the present paper. However, because the normalized  $\delta$ -Casorati curvature  $\delta_C(n-1)$  should be able to be recovered from the generalized normalized  $\delta$ -Casorati curvature  $\delta_C(r; n-1)$  for a positive real number  $r \neq n(n-1)$ , it would be more appropriate to define  $\delta_C(n-1)$  using the coefficient  $\frac{n+1}{2n}$  and therefore the proof of Theorem 1.1 (i) in [16] should be adapted to the amended coefficient. But we would like to point out that this can be done easily replacing by  $\frac{(n+1)(n-1)}{2}$  the coefficient  $\frac{n+1}{2n}$  of  $\mathcal{C}(L)$  in the definition of a quadratic polynomial  $\mathcal{P}$  between (12) and (13) from [16] and modifying the corresponding coefficients in (13), (14) and (15). Hence the eigenvalues of the new Hessian matrix of  $\mathcal{P}$  become

$$\begin{aligned} \lambda_{11} &= 0, \quad \lambda_{22} = n+3, \quad \lambda_{33} = \dots = \lambda_{nn} = 2(n+1), \\ \lambda_{ij} &= 4(n+1), \quad \lambda_{in} = 2(n+1), \quad \forall i, j \in \{1, \dots, n-1\}, \quad i \neq j. \end{aligned}$$

and the rest of the proof in [16] remains unchanged.

**Remark 3.4.** We note that the techniques used in this paper for prove optimal inequalities involving the generalized normalized  $\delta$ -Casorati curvatures for slant submanifolds in quaternionic space forms are based on an optimization procedure by showing that a quadratic polynomial in the components of the second fundamental form is parabolic. We should point out that this approach is different for that used in [18, 19] for establish some inequalities of Chen's type between certain intrinsic invariants and the squared mean curvature of  $\theta$ -slant submanifolds in quaternionic space forms, which is mainly based on algebraic lemmas.

## ACKNOWLEDGMENT

The authors would like to thank the referee for his valuable comments and suggestions which helped to improve the paper. The first author was supported by the Education Research Institute, Busan National University of Education in 2015. The second author was supported by CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0118.

## REFERENCES

1. P. Alegre, B.-Y. Chen and M. Munteanu, Riemannian submersions,  $\delta$ -invariants, and optimal inequality, *Ann. Global Anal. Geom.*, **42(3)** (2012), 317-331.
2. F. Casorati, Mesure de la courbure des surfaces suivant l'idée commune, Ses rapports avec les mesures de courbure gaussienne et moyenne, *Acta Math.*, **14(1)** (1890), 95-110.
3. B.-Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Arch. Math.*, **60** (1993), 568-578.
4. B.-Y. Chen, *Pseudo-Riemannian Geometry,  $\delta$ -invariants and Applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
5. B.-Y. Chen, An optimal inequality for CR-warped products in complex space forms involving CR  $\delta$ -invariant, *Internat. J. Math.*, **23(3)** (2012), 1250045, 17 pp.
6. B.-Y. Chen, F. Dillen, J. Van der Veken and L. Vrancken, Curvature inequalities for Lagrangian submanifolds: The final solution, *Differential Geom. Appl.*, **31(6)** (2013), 808-819.
7. B.-Y. Chen, A. Prieto-Martín and X. Wang, Lagrangian submanifolds in complex space forms satisfying an improved equality involving  $\delta(2,2)$ , *Publ. Math. Debrecen*, **82(1)** (2013), 193-217.
8. S. S. Chern, *Minimal Submanifolds in a Riemannian Manifold*, University of Kansas Press, Lawrence, 1968.
9. S. Decu, S. Haesen and L. Verstraelen, Optimal inequalities involving Casorati curvatures, *Bull. Transilv. Univ. Braşov Ser. B (N.S.)*, **14(49)** (2007), Supp. 1, 85-93.
10. S. Decu, S. Haesen and L. Verstraelen, Optimal inequalities characterising quasi-umbilical submanifolds, *J. Inequal. Pure Appl. Math.*, **9(3)** (2008), Article 79, 1-7.
11. S. Deng, Improved Chen-Ricci inequality for Lagrangian submanifolds in quaternion space forms, *Int. Electron. J. Geom.*, **5(1)** (2012), 163-170.
12. V. Ghişoiu, *Inequalities for the Casorati curvatures of slant submanifolds in complex space forms*, Riemannian geometry and applications. Proceedings RIGA 2011, ed. Univ. Bucureşti, Bucharest, 2011, pp. 145-150.
13. S. Ishihara, Quaternion Kählerian manifolds, *J. Diff. Geometry*, **9** (1974), 483-500.

14. C. Özgür and A. Mihai, Chen inequalities for submanifolds of real space forms with a semi-symmetric non-metric connection, *Canad. Math. Bull.*, **55(3)** (2012), 611-622.
15. M. H. Shahid and F. R. Al-Solamy, Ricci tensor of slant submanifolds in a quaternion projective space, *C. R. Math. Acad. Sci. Paris*, **349(9)** (2011), 571-573.
16. V. Slesar, B. Şahin and G. E. Vîlcu, Inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms, *J. Inequal. Appl.*, 2014, 2014:123.
17. B. Şahin, Slant submanifolds of quaternion Kaehler manifolds, *Commun. Korean Math. Soc.*, **22(1)** (2007), 123-135.
18. G. E. Vîlcu, B.-Y. Chen inequalities for slant submanifolds in quaternionic space forms, *Turk. J. Math.*, **34(1)** (2010), 115-128.
19. G. E. Vîlcu, Slant submanifolds of quaternionic space forms, *Publ. Math. Debrecen*, **81(3-4)** (2012), 397-413.
20. G. E. Vîlcu, On Chen invariants and inequalities in quaternionic geometry, *J. Inequal. Appl.*, 2013, 2013:66.

Jaewon Lee  
Busan National University of Education  
Department of Mathematics Education  
Busan, 611-739  
South Korea  
E-mail: leejaew@bnue.ac.kr

Gabriel Eduard Vîlcu  
Petroleum-Gas University of Ploieşti  
Department of Mathematical Modelling  
Economic Analysis and Statistics  
Bulevardul Bucureşti  
Nr. 39, Ploieşti 100680, Romania  
E-mail: gvilcu@upg-ploiesti.ro  
and  
University of Bucharest  
Faculty of Mathematics and Computer Science  
Research Center in Geometry  
Topology and Algebra, Str. Academiei  
Nr. 14, Sector 1, Bucureşti 70109, Romania  
E-mail: gvilcu@gta.math.unibuc.ro