L_k-BIHARMONIC HYPERSURFACES IN THE EUCLIDEAN SPACE

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Abstract. Chen conjecture states that every Euclidean biharmonic submanifold is minimal. In this paper we consider the Chen conjecture for L_k-operators. The new conjecture (L_k-conjecture) is formulated as follows: If $L_k^2 x = 0$ then $H_{k+1} = 0$ where $x : M^n \to \mathbb{R}^{n+1}$ is an isometric immersion of a Riemannian manifold M^n into the Euclidean space $\mathbb{R}^{n+1}$, H_{k+1} is the $(k+1)$-th mean curvature of M, and L_k is the linearized operator of the $(k+1)$-th mean curvature of the Euclidean hypersurface M. We prove the L_k-conjecture for the hypersurface M with at most two principal curvatures.

1. INTRODUCTION

Let $x : M^n \to \mathbb{R}^m$ be an isometric immersion from a Riemannian manifold M into the Euclidean space $\mathbb{R}^m$, by the Beltrami formula $\Delta x = n\vec{H}$, x is harmonic if and only if M is minimal, i.e., $\vec{H} = 0$, where Δ is the Laplace operator on M, and $\vec{H}$ is the mean curvature vector field of M. Inspired by this nice result, B.Y. Chen in [7, 8] made a conjecture, known as Chen conjecture, saying that every biharmonic Riemannian submanifold $M^n \subset \mathbb{R}^m$ (i.e., any isometric immersion $x : M^n \to \mathbb{R}^m$ satisfying the condition $\Delta^2 x = 0$) is minimal. That is the mean curvature vector field of the submanifold M is zero.

Chen himself proved his conjecture for Euclidean surfaces, (cf. [7, 8, 11]). Following him, I. Dimitrić in [14, 15] generalized Chen result and proved the Chen conjecture for the following submanifolds: (a) curves; in this case a biharmonic curve is an open part of a straight line; (b) submanifolds with constant mean curvature; (c) Euclidean hypersurfaces with at most two principal curvatures. As it is known, Euclidean conformally flat submanifolds with dimension $n \neq 3$ have at most two principal curvatures. Thus an immediate consequence of this result is that every Euclidean biharmonic conformally flat submanifold of dimension $n \neq 3$ is minimal; (d) finite type submanifolds,

see [9] for a general good reference on finite type submanifolds; (e) pseudo-umblical submanifolds of dimension $n \neq 4$. After that T. Hasanis and T. Vlachos in [16] classified H-hypersurfaces in $\mathbb{R}^4$ which are hypersurfaces having the gradient of the mean curvature function as a principal vector field and a constant multiple of the mean curvature function as its corresponding principal curvature. So every biharmonic Euclidean hypersurface is an H-hypersurface. Afterwards, via computer calculation they have shown that the Chen conjecture is true for every hypersurface of $\mathbb{R}^4$. Also F. Defever in [13] proves the same result by a different and purely analytical proof. K. Akutagawa and S. Maeta investigate the Chen conjecture and prove that every complete biharmonic properly immersed Euclidean submanifold is minimal, [1]. Recently B.Y. Chen and M.I. Munteanu have proved that every $\delta(2)$-ideal and $\delta(3)$-ideal biharmonic hypersurface of a Euclidean space is minimal, [12].

Also some authors have investigated the Chen conjecture for indefinite metrics. Chen and Ishikawa in [10, 11] proved that every biharmonic isometric immersion of a pseudo-Riemannian surface M into $\mathbb{R}_s^3(s = 1, 2)$ is minimal. In [11] they have constructed many examples of non-minimal space-like biharmonic surfaces in $\mathbb{R}_s^4(s = 1, 2)$. A. Arvanitoyeorgos et al. in [6] have shown the conjecture is true for Lorentzian hypersurfaces of $\mathbb{R}_1^4$.

At the same time, Chen conjecture has been studied when $x : (M, g) \to (N, h)$ is an isometric immersion of a Riemannian manifold (M, g) into a Riemannian manifold (N, h) of non positive sectional curvature. Several authors have considered the conjecture in this case. Recently, N. Nakauchi and H. Urakawa have shown that when M is complete and $\int_M |\vec{H}|^2 \mathrm{d}M < \infty$, M is minimal, [19]. In [3], Al´as et al. have considered the same problem when $\lambda_1^L(M) \geq 0$ (spectral radius of the operator $L = \Delta + Ric^N$) and they got some interesting results. In general this version of Chen conjecture turned out to be false. Ou, Tang in [21] got counter examples to Chen conjecture in this case. But the original Chen conjecture is still open.

As it is known the natural generalization of the Laplace operator is the L_k-operator, [22, 23], which is the linearized operator of $(k+1)$-th mean curvature of a hypersurface for $k = 0, \ldots, n - 1$. Recently Al´as, Gürbüz and following him, Kashani, et al. [4, 5, 17], have used the L_k-operators to study some hypersurfaces such as hypersurfaces satisfying $L_k x = Ax + b$ and L_k-finite type hypersurfaces, and got nice results. Hence it is interesting to consider the Chen conjecture for Euclidean hypersurfaces, replacing Δ by L_k. Here we restate the Chen conjecture for operators L_k. Let $x : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion from a connected orientable Riemannian hypersurface into the Euclidean space $\mathbb{R}^{n+1}$ with N as the unit normal direction. In [4] it's proved that

$$(1) \qquad L_k x = (k + 1) \binom{n}{k + 1} H_{k+1} N,$$

where $0 \leq k \leq n - 1$ and H_{k+1} is $(k+1)$-th mean curvature of M. When $k = 0$, (1) reduces to the equation $\Delta x = n H_1 N = n\vec{H}$ which is the Beltrami equation. So we

state the following conjecture.

L_k-**Conjecture.** Every Euclidean hypersurface $x : M^n \to \mathbb{R}^{n+1}$ satisfying the condition $L_k^2 x = 0$ for some k, $0 \le k \le n-1$, has zero $(k+1)$-th mean curvature, namely it is k-minimal.

When the Euclidean hypersurface M satisfies the equation $L_k^2 x = 0$, we call it, L_k-biharmonic hypersurface. In this paper we prove that the L_k-conjecture is true for Euclidean hypersufaces with at most two principal curvatures. Also we prove the L_k-conjecture for L_k-finite type hypersurfaces. Our main results are Theorems 5, 7, 8 and Corollary 6. We should mention that although in the proofs we follow the papers [11, 14, 15] on Chen conjecture, but our computations and somehow methods are totally different from the ones' used in those papers. This is because of the definition of the operators L_k in which one replaces the identity of $\mathcal{X}(M)$ by the much more complicated Newton transformations P_k $(0 \le k \le n-1)$, see page 4.

2. Preliminaries

We recall the prerequisites from [4, 17, 20]. Throughout the paper we denote by $x : M^n \to \mathbb{R}^{n+1}(n \ge 2)$ an isometric immersion from a connected orientable Riemannian manifold M into the Euclidean space $\mathbb{R}^{n+1}$ with N as a unit normal vector field, $\overline{\nabla}$ and ∇ as the Levi-Civita connections on $\mathbb{R}^{n+1}$ and M, respectively. For every tangent vector fields X and Y on M, the Gauss formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \langle SX, Y \rangle N,$$

where the shape operator S is defined by

$$SX = -\overline{\nabla}_X N.$$

The covariant derivative of the shape operator is symmetric, by the Codazzi equation, i.e.,

$$(2) \qquad\qquad (\nabla_X S)Y = (\nabla_Y S)X.$$

From the Gauss formula it can be seen that,

$$(3) \qquad R(X,Y)Z = \langle SY, Z \rangle SX - \langle SX, Z \rangle SY \quad \forall X, Y, Z \in \mathcal{X}(M).$$

As it is known, the shape operator $S : \mathcal{X}(M) \to \mathcal{X}(M)$ is a self-adjoint linear operator. Let $k_1, \ldots, k_n$ be its eigenvalues which are called principal curvatures of M. Define $s_0 = 1$ and

$$s_k = \sum_{1 \le i_1 < \cdots < i_k \le n} k_{i_1} \cdots k_{i_k}.$$

Now the k-th mean curvature of M is defined by

$$\binom{n}{k} H_k = s_k \,.$$

For $k = 1$, $H_1 = \frac{1}{n} tr(S) = H$ is the mean curvature of M . For $k = 2$, the scalar curvature of M is $s = n(n-1)H_2$. In general, when k is odd, the sign of H_k depends on the chosen orientation and when k is even, H_k is an intrinsic geometric quantity.

The Newton transformations $P_k : \mathcal{X}(M) \to \mathcal{X}(M)$ are defined inductively by $P_0 = I$ and

$$P_k = s_k I - S \circ P_{k-1} \,, \ 1 \le k \le n.$$

Therefore

(4)
$$P_k = \sum_{l=0}^{k} (-1)^l s_{k-l} S^l \,.$$

So from the Cayley-Hamilton theorem, one gets that $P_n = 0$. Each P_k is a self adjoint linear operator which commutes with S and the eigenvalues of P_k are given by

(5)
$$\mu_{k,i} = \sum_{1 \le i_1 < \cdots < i_k \le n, \, i_j \ne i} k_{i_1} \cdots k_{i_k} \,.$$

For $0 \le k \le n - 1$, the second order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ as the natural generalization of the Laplace operator for Euclidean hypersurfaces M , is defined by

$$L_k f = tr(P_k \circ \nabla^2 f) \,,$$

where $\nabla^2 f$ is metrically equivalent to the Hessian of f and is defined by $\langle (\nabla^2 f)X, Y \rangle = \langle \nabla_X (\nabla f), Y \rangle$ for all vector fields $X, Y \in \mathcal{X}(M)$, and ∇f is the gradient vector field of f. When $k = 0$, $L_0 = \Delta$.

Now we assume that M has two principal curvatures and denote them by

$$k_1 = \cdots = k_m = f, \ k_{m+1} = \cdots = k_n = g \,.$$

Notation. We let $\binom{N}{L} = 0$ if $L > N$ or $L < 0$. Therefore we get

$$s_k = \sum_{i=0}^{n} \binom{m}{i} \binom{n-m}{k-i} f^i g^{k-i} \,,$$

which we write it as

(6)
$$s_k = \binom{m}{i} \binom{n-m}{k-i} f^i g^{k-i} \,.$$

By formulae in [4] page 122, we have

(7)
$$L_k^2 x = - (k+1)(s_{k+1}\nabla s_{k+1} + 2(S \circ P_k)\nabla s_{k+1})$$
$$- (k+1)(s_{k+1}(s_1 s_{k+1} - (k+2)s_{k+2}) - L_k s_{k+1})N\,.$$

Here we consider the following standard examples in this context to be used later.

Example 1. Let $x : S^n(r) \to \mathbb{R}^{n+1}$ be the standard isometric embedding with $r > 0$. Its shape operator is $S = (1/r)I$ and its $(k+1)$-th mean curvature is $H_{k+1} = (1/r)^{k+1}$. Then formulas (1) and (7) gives that for $0 \le k \le n-1$,

$$L_k x \ne 0,\ L_k^2 x \ne 0\,.$$

Example 2. Let $x : S^m(r) \times \mathbb{R}^{n-m} \to \mathbb{R}^{n+1}$ be the standard isometric embedding with $r > 0$ and $1 \le m < n$. The shape operator of $M = S^m(r) \times \mathbb{R}^{n-m}$ is $S = \begin{bmatrix} (1/r)I_m & 0 \\ 0 & 0 \end{bmatrix}$, with respect to a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, where $\{e_i\}_{i=1}^m$ and $\{e_i\}_{i=m+1}^n$ are tangent to $S^m(r)$ and $\mathbb{R}^{n-m}$, respectively. So

$$s_{k+1} = \begin{cases} 0 & k+1 > m \\ \dbinom{m}{k+1}\dfrac{1}{r^{k+1}} & k+1 \le m \end{cases}$$

Therefore from (1) and (7),

$$\begin{cases} L_k x = L_k^2 x = 0 & k+1 > m \\ L_k x \ne 0,\ L_k^2 x \ne 0 & k+1 \le m \end{cases}$$

We recall the definition of L_k-finite type hypersurfaces from [17].

Definition 3. A Euclidean hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is called an L_k-finite type hypersurface if x satisfies a finite decomposition $x = x_0 + \cdots + x_p$ for some integer $p > 0$ such that x_0 is a constant vector and for $1 \le i \le p$, x_i's are smooth maps and $L_k x_i = \lambda_i x_i$ where λ_i's are real numbers. If all λ_i's are mutually distinct, M^n is said to be of p-type.

An L_k-finite type hypersurface is called null if for some $i > 0$, $\lambda_i = 0$.

3. Main Results

In this section we prove our main theorems. We consider Euclidean hypersurfaces with at most two principal curvatures. When the hypersurface has just one principal curvature the case reduces to the following remark.

Remark 1. Let $x : M^n \to \mathbb{R}^{n+1}$ be a totally umbilic hypersurface. The classification of totally umbilic hypersurfaces in $\mathbb{R}^{n+1}$, [20], states that M is an open piece of a hypersphere or a hyperplane. Example 1 shows that hyperspheres are not L_k-biharmonic, but every Euclidean hyperplane is totally geodesic, So all principal curvatures vanish. Hence both $L_k^2 x$ and H_{k+1} are zero. That is Euclidean hyperplanes are L_k-biharmonic and k-minimal, thus in the following we assume that the hypersurfaces have two principal curvatures.

Since $P_n = 0, (n = dim M)$; $S \circ P_{n-1} = s_n I$, by equation (7), hence one leads to consider the L_k-conjecture for $k = n - 1$, at first. Here we prove the following auxiliary proposition and lemma.

Proposition 2. *Let* $x : M^n \to \mathbb{R}^{n+1}$ *be an isometrically immersed connected hypersurface. If* x *satisfies* $L_{n-1}^2 x = 0$, *then* H_n *is constant.*
Proof.   By (7) we have

$$s_n \nabla s_n + 2(S \circ P_{n-1}) \nabla s_n = 0 \,.$$

We know that $P_n = 0$, hence $S \circ P_{n-1} = s_n I$. So

$$\frac{3}{2} \nabla s_n^2 = 0 \,.$$

Therefore s_n^2 is constant, as the result s_n is constant.   ∎

The proposition motivates us to consider the L_k-conjecture when the $(k+1)$-th mean curvature is constant. As we observed in the introduction, also the Chen conjecture has been proved when the mean curvature is constant. So we consider the analogous case on our context in the following lemma.

Lemma 3. *Let* $x : M^n \to \mathbb{R}^{n+1}$ *be an isometrically immersed connected hypersurface and let* M *has at most two principal curvatures. If* $L_k^2 x = 0$ *for some* k, $1 \le k \le n - 1$, *and* s_{k+1} *is constant then* $s_{k+1} = 0$.

Proof.   By using the hypothesis and equation (7), we show that the ratio of the principal curvatures of M should satisfy a polynomial equation with constant coefficients, from that we obtain that the principal curvatures are constant and by the classification of isoparametric Euclidean hypersurfaces we get the result. The detailed proof is as follows.
From (7) we have that either $s_{k+1} = 0$ or

$$s_1 s_{k+1} = (k + 2) s_{k+2} \,.$$

Let $k_1 = \cdots = k_m = f$, $k_{m+1} = \cdots = k_n = g$ be principal curvatures of M. By formulae (6), we get

$$
(mf + (n-m)g)\binom{m}{i}\binom{n-m}{k+1-i}f^i g^{k+1-i}
$$
(8)
$$
= (k+2)\binom{m}{i}\binom{n-m}{k+2-i}f^i g^{k+2-i}.
$$

If for some $p_0 \in M$, $g(p_0) = 0$, then equation (8) at p_0 becomes

(9)
$$
m\binom{m}{k+1}f^{k+2}(p_0) = (k+2)\binom{m}{k+2}f^{k+2}(p_0).
$$

So if $k+1 > m$,

$$
s_{k+1} = \binom{m}{k+1}f^{k+1}(p_0) = 0.
$$

If $k+1 \leq m$, then by (9) we have $f(p_0) = 0$. Thus $s_{k+1} = 0$. Now assume that $g \neq 0$ on M. Equation (8) can be written as

$$
m\binom{m}{i}\binom{n-m}{k+1-i}\left(\frac{f}{g}\right)^{i+1} + (n-m)\binom{m}{i}\binom{n-m}{k+1-i}\left(\frac{f}{g}\right)^i
$$
$$
= (k+2)\binom{m}{i}\binom{n-m}{k+2-i}\left(\frac{f}{g}\right)^i.
$$

Therefore f/g is a root of the above polynomial. So f/g is a constant α. Now formulae (6) gives that

$$
s_{k+1} = \binom{m}{i}\binom{n-m}{k+1-i}\alpha^i g^{k+1}.
$$

As the result we get that g, hence f is constant. Thus M is an isoparametric hypersurface in $\mathbb{R}^{n+1}$. So by the classification of such hypersurfaces, [18, 24], M is an open piece of a hyperplane or of a hypersphere or of a generalized right spherical cylinder. By Example 1, hyperspheres are not L_k-biharmonic and by Remark 1 and Example 2, hyperplanes and generalized right spherical cylinders have $s_{k+1} = 0$. ∎

Now we get the following corollary easily.

Corollary 4. *Let $x : M^2 \to \mathbb{R}^3$ be an isometrically immersed surface and let $L_1^2 x = 0$ then $H_2 = 0$. Proof.* Proposition 2 implies that s_2 is constant, then Lemma 3 gives that $s_2 = 0$. ∎

As we mentioned, Chen himself has proved his conjecture for Euclidean surfaces, [11], so the Corollary shows that the L_k-conjecture is also true for Euclidean surfaces.

In the following theorems, We prove the main results of the paper, that is we show that the L_k-conjecture is true when the Euclidean hypersurface has at most two

principal curvatures. In Theorem 5, we consider Euclidean hypersurfaces with two principal curvatures, such that both multiplicities are greater than one . The hypothesis on multiplicities is a key assumption in the proof of the theorem. For the case that one of the multiplicities is one, we use a different proof in Theorem 7.

Theorem 5. *Let* $x : M^n \to \mathbb{R}^{n+1}$ *be an isometrically immersed Euclidean hypersurface and let* $L_k^2 x = 0$ *for some* k, $1 \le k \le n-1$, *Suppose that* M *has two principle curvatures with both multiplicities greater than one. Then* $H_{k+1} = 0$.

Proof. By using equation (7), we get that either s_{k+1} is constant, from which by Lemma 3 we get that $s_{k+1} = 0$, or s_{k+1} is non constant. In this case, we consider the possibility that one of the principal curvatures be zero at some point(s) of M, this leads us to a contradiction. So, the principal curvatures are non zero on M, then we get that they have to be constant. In this case the restriction on multiplicities is essential. From the fact that principal curvatures are constant, we conclude the result. here is the detailed proof.

By using (7) one gets that

$$s_{k+1}\nabla s_{k+1} + 2(S \circ P_k)\nabla s_{k+1} = 0 \,.$$

Then the equation $P_{k+1} = s_{k+1}I - S \circ P_k$ gives that $P_{k+1}\nabla s_{k+1} = \frac{3}{2}s_{k+1}\nabla s_{k+1}$. Now let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame of eigenvectors of S and $k_1, \ldots, k_n$ as their eigenvalues, respectively. Then $P_{k+1}e_i = \mu_{k+1,i}e_i$ and

$$P_{k+1}\langle \nabla s_{k+1}, \, e_i\rangle e_i = \frac{3}{2}s_{k+1}\langle \nabla s_{k+1}, \, e_i\rangle e_i \,.$$

Therefore

$$\langle \nabla s_{k+1}, \, e_i\rangle \left(\mu_{k+1,i} - \frac{3}{2}s_{k+1}\right) = 0 \,.$$

If for every i, $\langle \nabla s_{k+1}, \, e_i\rangle = 0$, then s_{k+1} is constant. So Lemma 3 implies that $s_{k+1} = 0$. If for some j_0, $\langle \nabla s_{k+1}, \, e_{j_0}\rangle \neq 0$, then $\mu_{k+1,j_0} = \frac{3}{2}s_{k+1}$. We show that this case does not occur. We have $P_{k+1} = \sum_{l=0}^{k+1}(-1)^l s_{k+1-l}S^l$, so

$$\mu_{k+1,j_0} = \sum_{l=0}^{k+1}(-1)^l s_{k+1-l}k_{j_0}^l \,.$$

Let $k_1 = \cdots = k_m = f$, $k_{m+1} = \cdots = k_n = g$ with both $m, n-m$ greater than one. We consider the following two cases.

Case 1. If $j_0 \le m$, then $\mu_{k+1,j_0} = \sum_{l=0}^{k+1}(-1)^l s_{k+1-l}f^l = \frac{3}{2}s_{k+1}$. Thus

$$(10) \quad (-1)^l \binom{m}{i}\binom{n-m}{k+1-l-i}f^{i+l}g^{k+1-l-i} = \frac{3}{2}\binom{m}{i}\binom{n-m}{k+1-i}f^i g^{k+1-i} \,.$$

If for some $p_0 \in M$, $f(p_0) = 0$ then by (10) we have

$$\binom{n-m}{k+1} g^{k+1}(p_0) = \frac{3}{2} \binom{n-m}{k+1} g^{k+1}(p_0).$$

Now we look at two subcases.

Subcase 1. If $k + 1 \leq n - m$, then $g(p_0) = 0$. By (6),

$$\nabla s_{k+1} = \binom{m}{i}\binom{n-m}{k+1-i} g^{k+1-i} \nabla f^i + \binom{m}{i}\binom{n-m}{k+1-i} f^i \nabla g^{k+1-i}.$$

Therefore $\nabla s_{k+1}(p_0) = 0$. Hence $\langle \nabla s_{k+1}, e_{j_0} \rangle (p_0) = 0$ which is a contradiction.

Subcase 2. If $k + 1 > n - m$, then

$$\nabla s_{k+1}(p_0) = \binom{m}{1}\binom{n-m}{k} g^k(p_0) \nabla f(p_0).$$

If $k > n - m$, then $\nabla s_{k+1}(p_0) = 0$. Therefore $\langle \nabla s_{k+1}, e_{j_0} \rangle (p_0) = 0$ that is a contradiction.

If $k = n - m$, then

$$\langle \nabla s_{k+1}, e_{j_0} \rangle (p_0) = m g^k(p_0) \langle \nabla f, e_{j_0} \rangle (p_0).$$

So $g(p_0) \neq 0$. Since g is continuous, on some neighborhood U_{p_0} of p_0, $g \neq 0$. Now dividing (10) by g^{k+1}, we have

$$(-1)^l \binom{m}{i}\binom{n-m}{k+1-l-i}\left(\frac{f}{g}\right)^{i+l} = \frac{3}{2}\binom{m}{i}\binom{n-m}{k+1-i}\left(\frac{f}{g}\right)^{i}.$$

Thus f/g is a root of the above polynomial. So f/g is a constant α, hence $f = \alpha g$. Since $f(p_0) = 0$, $f \equiv 0$ on U_{p_0}. Thus $\langle \nabla s_{k+1}, e_{j_0} \rangle (p_0) = 0$ which is a contradiction. Subcases 1 and 2 give that $f \neq 0$. By (10) we have

(11) $\quad (-1)^l \binom{m}{k+1-l-i}\binom{n-m}{i} f^{k+1-i} g^i = \frac{3}{2}\binom{m}{k+1-i}\binom{n-m}{i} f^{k+1-i} g^i.$

So by dividing (11) by f^{k+1}, we have

$$(-1)^l \binom{m}{k+1-l-i}\binom{n-m}{i}\left(\frac{g}{f}\right)^{i} = \frac{3}{2}\binom{m}{k+1-i}\binom{n-m}{i}\left(\frac{g}{f}\right)^{i}.$$

Thus g/f is a root of the above polynomial. Therefore g/f is a constant α. So $s_{k+1} = \beta f^{k+1}$, where β is some constant. Since both multiplicities are greater than

one, the Codazzi equation (2), $(\nabla_{e_i} S)e_j = (\nabla_{e_j} S)e_i$ implies that for every i, $\nabla_{e_i} f = 0$. Therefore we have

$$\langle \nabla s_{k+1}, e_{j_0} \rangle = \beta \left\langle (k+1) f^k \nabla f, e_{j_0} \right\rangle = 0 ,$$

which is not possible.

Case 2. If $j_0 > m$. By choosing the frame $e'_1 = e_{m+1}, \ldots, e'_{n-m} = e_n$, $e'_{n-m+1} = e_1, \ldots, e'_n = e_m$ one can prove this case exactly similar to the previous case. ∎

As a consequence of Theorem 5, we can give the following uniqueness result.

Corollary 6. *The* only L_k-biharmonic hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$ having two principal curvatures, both with multiplicities greater than one, are the standard product embeddings $\mathbb{S}^m(r) \times \mathbb{R}^{n-m} \subset \mathbb{R}^{n+1}$ with $r > 0$ and $m \le k$.

Proof.   Observe that since both principal curvatures have multiplicities greater than one, then the distributions of the space of principal vectors corresponding to each principal curvature are completely integrable and each principal curvature is constant on each of the integral leaves of the corresponding distribution. By Theorem 5, the $(k+1)$-th mean curvature H_{k+1} is constant and by a suitable variant of the proof of Lemma 6 in [2] one can see that the principal curvatures are constant. So the hypersurface is an isoparametric hypersurface with exactly two constant principal curvatures, with multiplicities m and $n - m$, and $1 < m < n - 1$. Then, by the classical results on isoparametric hypersurfaces in Euclidean space the hypersurface must be an open piece of standard product embeddings $\mathbb{S}^m(r) \times \mathbb{R}^{n-m} \subset \mathbb{R}^{n+1}$ with $r > 0$. Once we know that the hypersurface is an open piece of a standard product $\mathbb{S}^m(r) \times \mathbb{R}^{n-m}$, by Example 2 it follows that $m \le k$. ∎

As we already mentioned, In the following theorem we prove the L_k-conjecture, when one of the multiplicities of principal curvatures is one. To prove the theorem, by the use of equation (3) and formulas, (2, 4, 5, 6), we show that the principal curvatures should satisfy some equation and by using it, we get the result.

Theorem 7. *Let* $x : M^n \to \mathbb{R}^{n+1}$ *be an isometric immersed Euclidean hypersurface. If* $L_k^2 x = 0$ *for some* k, $1 \le k \le n - 1$, *and* M *has two principle curvatures with multiplicities* 1 *and* $n - 1$, *then* $H_{k+1} = 0$.

Proof.   Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame of eigenvectors of S and $k_1, \ldots, k_n$ be their principal eigenvalues, respectively. Let $k_1 = f$, $k_2 = \cdots = k_n = g$. First we compute $\mu_{k,i}$. By using (4) and (5), we get that

(12)
$$\begin{cases} \mu_{k,2} = \cdots = \mu_{k,n} \\ \mu_{k,2} = \binom{n-2}{k-1} f g^{k-1} + \binom{n-2}{k} g^k \\ \mu_{k,1} = \binom{n-1}{k} g^k \end{cases}$$

If s_{k+1} is constant, Lemma 3 gives that $s_{k+1} = 0$. Otherwise, as in the proof of Theorem 5, for some j_0, $\langle \nabla s_{k+1}, e_{j_0} \rangle \neq 0$ and $\mu_{k+1,j_0} = \frac{3}{2} s_{k+1}$. If $j_0 = 1$, then

$$\binom{n-1}{k+1} g^{k+1} = \frac{3}{2} \binom{n-1}{k+1} g^{k+1} + \frac{3}{2} \binom{n-1}{k} f g^k .$$

So similar to the proof of Theorem 5, we get that $g = \frac{-3k-3}{n-k-1} f$. Since $f \neq 0$, we assume that $f > 0$. Let $\alpha = \frac{-3k-3}{n-k-1}$. The Codazzi equation (2), $(\nabla_{e_i} S) e_j = (\nabla_{e_j} S) e_i$ gives that

$$(13) \quad \begin{cases} (a) \ i > 1 \ \nabla_{e_i} f = 0, \ \omega_{1i}^1 = 0, \ \alpha \nabla_{e_1} f = (1-\alpha) f \, \omega_{i1}^i, \ \omega_{i1}^{k>1, \, k \neq i} = 0, \\ (b) \ i, j > 1, \ i \neq j \ \omega_{ij}^1 = \omega_{ji}^1 , \end{cases}$$

where $\nabla_{e_i} e_j = \sum_k \omega_{ij}^k e_k$ and the symbol $\omega_{i1}^{k>1, \, k \neq i}$ denotes ω_{i1}^k when $k > 1$ and $k \neq i$. Now by (13)-(a),for every $i > 1$, $\nabla_{e_i} f = 0$. So let f be a function of s where the integral curve of e_1 is parametrized by s. Thus $e_1 = \frac{\partial}{\partial s}$, $\nabla_{e_1} f = f'$. The equation (13) and (3) yield the following equation

$$(14) \quad \langle R(e_1, e_2) e_2, e_1 \rangle = \nabla_{e_1} \omega_{22}^1 - (\omega_{22}^1)^2 = \alpha f^2 .$$

From (13)-(a),(14) one gets that

$$\omega_{22}^1 = \frac{\alpha f'}{(\alpha - 1) f} , \ \left(\frac{\alpha f'}{(\alpha - 1) f} \right)' - \left(\frac{\alpha f'}{(\alpha - 1) f} \right)^2 = \alpha f^2 .$$

Therefore

$$(15) \quad f f'' = (\alpha - 1) f^4 + \left(\frac{2\alpha - 1}{\alpha - 1} \right) f'^2 .$$

Define $f' = P$. So

$$f'' = \left(\frac{dP}{df} \right) \left(\frac{df}{ds} \right) = \left(\frac{dP}{df} \right) P .$$

Replacing in (15), we get that

$$\frac{1}{2} f \frac{d}{df} (P^2) - \left(\frac{2\alpha - 1}{\alpha - 1} \right) P^2 = (\alpha - 1) f^4 .$$

Let $Q = P^2$, then the above equation yields

$$\frac{dQ}{df} - \frac{2}{f} \left(\frac{2\alpha - 1}{\alpha - 1} \right) Q = 2(\alpha - 1) f^3 .$$

By multiplying both sides of the above equation by $f^{-\left(\frac{4\alpha-2}{\alpha-1}\right)}$ and then integrating we obtain that

$$(16) \qquad f'^2 = -(\alpha-1)^2 f^4 + c_0 f^{\left(\frac{4\alpha-2}{\alpha-1}\right)},$$

where c_0 is a constant. Since $f \neq 0$, then $c_0 \neq 0$. By equation (15) and (16), we get

$$(17) \qquad ff'' = -2(\alpha-1)^2 f^4 + c_0 \left(\frac{2\alpha-1}{\alpha-1}\right) f^{\left(\frac{4\alpha-2}{\alpha-1}\right)}.$$

Now we compute $L_k f^{k+1}$. We have

$$\nabla f = \langle \nabla f,\, e_i \rangle\, e_i = f' e_1.$$

Therefore we get

$$L_k f^{k+1} = tr\left(P_k \circ \nabla^2(f^{k+1})\right)$$

$$= (k+1)\left\langle (\nabla_{e_i} f^k) f' e_1 + f^k \left((\nabla_{e_i} f') e_1 + f' \nabla_{e_i} e_1\right),\, P_k e_i \right\rangle$$

$$= (k+1)\left(k(f')^2 f^{k-1}\mu_{k,1} + f^k f'' \mu_{k,1} + f^k f' \omega_{i1}^i \mu_{k,i}\right).$$

By replacing formulas (12) for $\mu_{k,i}$ in the above equation, it yields that

$$(18) \qquad \frac{1}{k+1} L_k f^{k+1} = k\binom{n-1}{k}\alpha^k f^{2k-1}(f')^2 + \binom{n-1}{k}\alpha^k f^{2k} f''$$

$$+ \frac{(n-1)(2k+3)}{k(\alpha-1)}\binom{n-2}{k-1}\alpha^k f^{2k-1}(f')^2.$$

We have $L_k^2 x = 0$. Thus from (7),

$$(19) \qquad s_{k+1}(s_1 s_{k+1} - (k+2)s_{k+2}) - L_k s_{k+1} = 0.$$

Since $g = \alpha f$, we have

$$(20) \qquad s_{r+1} = \left(\binom{n-1}{r+1}\alpha^{r+1} + \binom{n-1}{r}\alpha^r\right) f^{r+1}$$

$$= \binom{n-1}{r}\left(\left(\frac{n-r-1}{r+1}\right)\alpha + 1\right)\alpha^r f^{r+1}.$$

So using equations (16), (17), (18), (19) and (20) and substituting $\alpha = \frac{-3k-3}{n-k-1}$ gives that

$$(21) \qquad \begin{aligned} & 2\,\alpha\left(\alpha n + (\alpha^2-1)k^2 + (3\alpha^2 - 4\alpha + 1)k + 2\alpha^2 - 5\alpha + 3\right) f^{2k+3} \\ & -2\,c_0\,\alpha(k+1)(k+2)\left(\frac{\alpha+1}{\alpha-1}\right) f^{\left(\frac{2k\alpha+3\alpha-2k-1}{\alpha-1}\right)} = 0. \end{aligned}$$

Now equation (21) implies that f and thus s_{k+1} are constant which is a contradiction. If $j_0 \neq 1$, the proof is similar to the argument in Theorem 5. ∎

As we mentioned in the introduction, Chen conjecture has been proved for finite type hypersurfaces. With Definition 3 in the preliminaries, we can prove the L_k-conjecture for L_k-finite type hypersurfaces as well.

Theorem 8. *Let* $x : M^n \to \mathbb{R}^{n+1}$ *be an* L_k*-finite type hypersurface and* $L_k^2 x = 0$, *then* M *is of null one type and especially* $s_{k+1} = 0$.

Proof. $x = x_0 + \cdots + x_p$ with x_0 is a constant and for every $1 \leq i \leq p$, $L_k x_i = \lambda_i x_i$. So

$$L_k x = \lambda_1 x_1 + \cdots + \lambda_p x_p.$$

Thus for every $s \geq 2$,

$$\lambda_1^s x_1 + \cdots + \lambda_p^s x_p = 0.$$

So we get that for every i, $\lambda_i = 0$. Therefore M is a null one-type hypersurface. Hence $L_k^2 x = L_k x = 0$ and by (1), $s_{k+1} = 0$. ∎

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