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AUTOMORPHISMS OF NEIGHBORHOOD SEQUENCE OF A GRAPH

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Abstract. Let G be a graph, u be a vertex of G, and $B(u)$(or $B_G(u)$) be the set of u with all its neighbors in G. A sequence $(B_1, B_2, ..., B_n)$ of subsets of an n-set S is a *neighborhood sequence* if there exists a graph G with a vertex set S and a permutation $(v_1, v_2, ..., v_n)$ of S such that $B(v_i) = B_i$ for $i = 1, 2, ..., n$. Define $Aut(B_1, B_2, ..., B_n)$ as the set $\{f : f$ is a permutation of $V(G)$ and $(f(B_1), f(B_2), ..., f(B_n))$ is a permutation of B_1, B_2, ..., $B_n\}$. In this paper, we first prove that, for every finite group Γ, there exists a neighborhood sequence $(B_1, B_2, ..., B_n)$ such that Γ is group isomorphic to $Aut(B_1, B_2, ..., B_n)$. Second, we show that, for each finite group Γ, there exists a neighborhood sequence $(B_1, B_2, ..., B_n)$ such that, for each subgroup H of Γ, H is group isomorphic to $Aut(E_1, E_2, ..., E_t)$ for some neighborhood sequence $(E_1, E_2, ..., E_t)$ where $E_i \subseteq B_{j_i}$ and $j_1 < j_2 < \cdots < j_t$. Finally, we give some classes of graphs G with neighborhood sequence $(B_1, B_2, ..., B_n)$ satisfying that $Aut(G)$ and $Aut(B_1, B_2, ..., B_n)$ are different.

1. INTRODUCTION

The identifying codes were first introduced by Karpovsky, Chakrabarty, and Levitin in [4]. Furthermore, they have formed a fundamental basis for a wide variety of theoretical work and practical applications. If we settle that every vertex v of a graph G only exhibits the messages from some neighbors of v in G, then we can get a code with size $\leq M(G)$. We call such code an *identifying set* of a graph G. If two graphs have the same neighborhood sequence, then they have the same minimum cardinality of an identifying code [4] and the choice identification number [1].

Here we introduce some definitions used in the paper. Let G be a graph, u be a vertex of G, and $B(u)$(or $B_G(u)$) be the set of u with all its neighbors in G. And $N(u)$(or $N_G(u)$) is the set of all neighbors of u in G. Then $B(u) = N(u) \cup \{u\}$. A

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sequence $(B_1, B_2, ..., B_n)$ of subsets of an n-set S is a *neighborhood sequence* if there exist a graph G with a vertex set S and a permutation $(v_1, v_2, ..., v_n)$ of S such that $B(v_i) = B_i$ for $i = 1, 2, ..., n$.

An automorphism f of a graph G is a permutation of vertex set $V(G)$ such that $xy \in E(G)$ if and only if $f(x)f(y) \in E(G)$. The collection $Aut(G)$ of all automorphisms of G is a group by a composition operator. Define $Aut(B_1, B_2, ..., B_n)$ as the set $\{f : f$ is a permutation of $V(G)$ and $(f(B_1), f(B_2), ..., f(B_n))$ is a permutation of B_1, B_2, ..., $B_n\}$ where $f(S) = \{f(x) : x \in S\}$ for $S \subseteq V(G)$. Such permutation is called a $(B_1, B_2, ..., B_n)$*-automorphism*. It is immediate that every automorphism f of G is also an element of $Aut(B_1, B_2, ..., B_n)$ where $(B_1, B_2, ..., B_n)$ is the neighborhood sequence of a graph G.

In this paper, we first prove that, for every finite group Γ, there exists a neighborhood sequence $(B_1, B_2, ..., B_n)$ such that Γ is isomorphic to $Aut(B_1, B_2, ..., B_n)$. We also get that, for each finite group Γ, there exists a neighborhood sequence $(B_1, B_2, ..., B_n)$ such that, for each subgroup H of Γ, H is group isomorphic to $Aut(E_1, E_2, ..., E_t)$ for some neighborhood sequence $(E_1, E_2, ..., E_t)$ where $E_i \subseteq B_{j_i}$ and $j_1 < j_2 < \cdots < j_t$. In the last section, we give some classes of graphs G with neighborhood sequence $(B_1, B_2, ..., B_n)$ satisfying $Aut(G)$ and $Aut(B_1, B_2, ..., B_n)$ are different, and construct non-isomorphic graphs with the same neighborhood sequence having different automorphism groups.

2. $Aut(B_1, B_2, ..., B_n)$

Let $[n]$ be the set $\{1, 2, ..., n\}$ and $B_1, B_2, ..., B_n$ be subsets of an n-set S. Then we say $(B_1, B_2, ..., B_n)$ has an *adjacent SDR* if there exist $v_i \in B_i$ for $i = 1, 2, ..., n$ such that $v_1, v_2, ..., v_n$ are distinct and $v_j \in B_i$ if and only if $v_i \in B_j$.

Theorem 1. *Let $B_1, B_2, ..., B_n$ be subsets of an n-set S. Then $(B_1, B_2, ..., B_n)$ is a neighborhood sequence if and only if $(B_1, B_2, ..., B_n)$ has an adjacent SDR.*

Proof. Let $(B_1, B_2, ..., B_n)$ be a neighborhood sequence of a graph G. Then there exists a permutation $(v_1, v_2, ..., v_n)$ of $V(G)$ such that $B(v_i) = B_i$ for all i; that is, $v_j \in B_i\{v_i\}$ if and only if $v_iv_j \in E(G)$. Thus we have an adjacent SDR.

Conversely, let $(v_1, v_2, ..., v_n)$ be an an adjacent SDR of $(B_1, B_2, ..., B_n)$. Define G as a graph with a vertex set $\{v_1, v_2, ..., v_n\}$ and an edge set $\{v_iv_j : v_i \in B_j\}$. It is easy to check that G is a graph. ∎

The girth of a graph G is the length of a shortest cycle in G.

Proposition 2. *If the girth of a connected graph G is greater than or equal to 5, then the neighborhood sequence of G has a uniquely adjacent SDR.*

Proof. Since the girth of G is greater than or equal to 5, we have that $n \geq 5$. Let $(B_1, B_2, ..., B_n)$ be the neighbor sequence of a graph G and $(v_1, v_2, ..., v_n)$ be an adjacent SDR of $(B_1, B_2, ..., B_n)$. If $|B_i| = 2$, then it is easy to see that $B_i = B_H(v_i)$ for each graph H with the neighbor sequence $(B_1, B_2, ..., B_n)$. If $|B_j| \geq 3$, then take any pair of two distinct vertices x and y in $B_j - \{v_j\}$. Since the girth of G is greater than or equal to 5, $\{x, y\}$ is not contained in B_k for all $k \neq j$. This implies that $B_j = B_H(v_j)$ for each graph H with the neighbor sequence $(B_1, B_2, ..., B_n)$. Therefore, $(B_1, B_2, ..., B_n)$ has a uniquely adjacent SDR. ∎

Corollary 3. *Let G be a connected graph with a neighborhood sequence $(B_1, B_2, ..., B_n)$. If the girth of G is greater than or equal to 5, then G is the unique graph with neighborhood sequence $(B_1, B_2, ..., B_n)$.*

Proposition 4. *Let G be a connected graph with a neighborhood sequence $(B_1, B_2, ..., B_n)$. Then $Aut(G)$ is a subset of $Aut(B_1, B_2, ..., B_n)$.*

Proof. It is immediate by definition. ∎

Proposition 5. *Let G be a graph with a neighborhood sequence $(B_1, B_2, ..., B_n)$ and $(v_1, v_2, ..., v_n)$ be an adjacent SDR of $(B_1, B_2, ..., B_n)$. If $f \in Aut(B_1, B_2, ..., B_n)$, then $(f(v_1), f(v_2), ..., f(v_n))$ is an adjacent SDR of $(f(B_1), f(B_2), ..., f(B_n))$.*

Proof. If $f(v_j)f(v_i) \in f(B_i)$, then $v_jv_i \in B_i$. By the definition of neighborhood sequences, $v_iv_j \in B_j$. This implies that $f(v_i)f(v_j) \in f(B_j)$. That is, $(f(v_1), f(v_2), ..., f(v_n))$ is an adjacent SDR of $(f(B_1), f(B_2), ..., f(B_n))$. ∎

Proposition 6. *Let G be a graph with neighborhood sequence $(B_1, B_2, ..., B_n)$. If the sequence $(B_1, B_2, ..., B_n)$ has the unique adjacent SDR, then $Aut(G) = Aut(B_1, B_2, ..., B_n)$.*

Proof. By Proposition 4, we have that $Aut(G) \subseteq Aut(B_1, B_2, ..., B_n)$. Let $f \in Aut(B_1, B_2, ..., B_n)$. Since $(f(B_1), f(B_2), ..., f(B_n))$ is a permutation of B_1, B_2, ..., B_n and Proposition 5, if $f(B_i) = B_j$ then $f(v_i) = v_j$; that is, f is an automorphism of G. ∎

A graph G is called *vertex transitive* if for each pair (x, y) of vertices in G there exists $f \in Aut(G)$ such that $f(x) = y$. Similarly, we say that a neighborhood sequence $(B_1, B_2, ..., B_n)$ is *vertex transitive* if for each pair (x, y) of elements in $\cup_{i=1}^{n} B_i$ there exists $f \in Aut(B_1, B_2, ..., B_n)$ such that $f(x) = y$. In the following proposition, some graphs are not vertex transitive, but their neighborhood sequences are vertex transitive. Let both G and H be graphs and the join of G and H be the graph $G+H$ with a vertex set $V(G) \cup V(H)$ and an edge set $E(G) \cup E(H) \cup \{uv : u \in V(H) \text{ and } v \in V(H)\}$.

Proposition 7. *Let n be a positive integer with $n \geq 3$, G be $K_{n,n}$ with two partite sets $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_n\}$, H_1 be $K_{n,n}$ with two partite sets $\{u_1, u_2, ..., u_n\}$ and $\{v_1, v_2, ..., v_n\}$, and H_2 be a graph with a vertex set $\{u_1, u_2, ..., u_n\}$ $\cup \{v_1, v_2, ..., v_n\}$ and an edge set $\{v_iv_j, u_iu_j : i \neq j\} \cup \{u_iv_i : i = 1, 2, ..., n\}$, $G_1 = G + H_1$, and $G_2 = G + H_2$. Then G_1 and G_2 have the same vertex transitive neighborhood sequence; but G_2 is not vertex transitive.*

Proof. Let $(B_1, B_2, ..., B_{2n})$ be the neighborhood sequence of G_1. By Proposition 4, we have that $Aut(G_1) \subseteq Aut(B_1, B_2, ..., B_{2n})$. Since G_1 is vertex transitive, $Aut(B_1, B_2, ..., B_{2n})$ is vertex transitive. If G_2 is vertex transitive, then there is an automorphism f on G_2 such that $f(a_1) = u_1$. Since $f(a_1) = u_1$ and $dist_{G_2}(a_1, a_i) = 2$ for $i = 2, 3, .., n$, we have that $f(\{a_1, a_2, ..., a_n\}) = \{u_1, v_2, ..., v_n\}$. Since $dist_{G_2}(a_i, a_j)$ $= 2$ for $i \neq j$ and v_2v_3 is an edge of G_2, we have that f is not an automorphism, a contradiction. Thus G_2 is not vertex transitive. ■

Frucht [2] proved that, for every finite group Γ, there exists a simple graph G such that Γ is group isomorphic to $Aut(G)$. In the following, we show that, for every finite group Γ, there exists a neighborhood sequence $(B_1, B_2, ..., B_n)$ such that Γ is isomorphic to $Aut(B_1, B_2, ..., B_n)$. Frucht defined a useful colored digraph from a group in [2]. We can use a subdigraph of the colored digraph in [2] to get the same results. Let Γ be a group and $S = \{g_1, g_2, ..., g_t\}$ be a generator of Γ. Define $D_S(\Gamma)$ as a digraph with a vertex set Γ and (x, y) with $x \neq y$ is an arc with color k if and only if $xy^{-1} = g_k$ for some k. Let $Aut^*(D_S(\Gamma)) = \{f : f$ is a permutation of Γ, and both (u, v) and $(f(u), f(v))$ have the same color$\}$. Then it is easy to check that $Aut^*(D_S(\Gamma))$ is a group with a composition operator. We have the following theorem.

Theorem 8. *For each finite group Γ, Γ and $Aut^*(D_S(\Gamma))$ are group isomorphic.*

Proof. Let $g \in \Gamma$ and p_g be a permutation on Γ by $p_g(x) = xg$ for all $x \in \Gamma$. If $xy^{-1} = g_k$, then $p_g(x)p_g(y)^{-1} = xg(yg)^{-1} = xy^{-1} = g_k$; that is, p_g is in $Aut^*(D_S(\Gamma))$. Let $f \in Aut^*(D_S(\Gamma))$ and $f(1) = g$ where 1 is the identity of Γ. Claim that $f = p_g$. Take $g_i \in S$. We have that $(1, g_i^{-1})$ and $(g, g_i^{-1}g)$ are arcs in $D_S(\Gamma)$ with the same color i. Since $f(1) = g$, $f(g_i^{-1}) = g_i^{-1}g$. Take $g_j \in S$, Since $(g_i^{-1}, g_j^{-1}g_i^{-1})$ and $(g_i^{-1}g, g_j^{-1}g_i^{-1}g)$ are arcs in $D_S(\Gamma)$ with color j and $f(g_i^{-1}) = g_i^{-1}g$, we have that $f(g_j^{-1}g_i^{-1}) = g_j^{-1}g_i^{-1}g$. By S being a generator of Γ, we have that $f(x) = p_g(x)$ for all $x \in \Gamma$. We can conclude that there is a group isomorphism from Γ to $Aut^*(D_S(\Gamma))$. ■

Define $P(x, y, k)$ as a graph with a vertex set $\{x = v_0(x, y), v_1(x, y), v_2(x, y), ...,$ $v_{k+2}(x, y),\ y = v_{k+3}(x, y), u(x, y)\}$ and an edge set $\{v_i(x, y), v_{i+1}(x, y) : i = 0, 1, ..., k + 2\} \cup \{v_k(x, y)u(x, y),\ u(x, y)v_{k+3}(x, y)\}$.

Theorem 9. *For each finite group Γ, there exists a 2-connected graph with its neighborhood sequence $(B_1, B_2, ..., B_n)$ such that Γ is group isomorphic $Aut(B_1, B_2, ..., B_n)$.*

Proof. For the order of Γ being 2, we have the graph K_2 with $Aut(\{1,2\},\{1,2\})$ being group isomorphic to Γ. Assume that the order of $\Gamma \geq 3$. Let 1 be the identity element of Γ, S be a generator of Γ with $|S| \geq 2$ and $1 \notin S$, and $G_S(\Gamma)$ be a graph obtained from $D_S(\Gamma)$ by $P(x, y, k)$ instead of (x, y) with color k in $D_S(\Gamma)$. It is easy to see that $G_S(\Gamma)$ is a 2-connected graph with girth 5. Since Corollary 3 and Proposition 6, we have that $Aut(G_S(\Gamma)) = Aut(B_1, B_2, ..., B_n)$ where $B_1, B_2, ..., B_n$ is a neighborhood sequence of $G_S(\Gamma)$.

Claim that $Aut^*(D_S(\Gamma))$ is group isomorphic to $Aut(G_S(\Gamma))$. Define a function g from $Aut^*(D_S(\Gamma))$ to $Aut(G_S(\Gamma))$ by $g(f(x)) = f(x)$ for all $x \in \Gamma$. For each $f \in Aut^*(D_S(\Gamma))$, define an automorphism h of $G_S(\Gamma)$ by $h(x) = f(x)$ for all $x \in \Gamma$ and if (x, y) is color k in $D_S(\Gamma)$ then $h(u(x, y)) = u(f(x), f(y))$ and $h(v_i(x, y)) = v_i(f(x), f(y))$ for $k = 0, 1, 2..., k+3$. Then h is the unique automorphism of $G_S(\Gamma)$ with $h(x) = f(x)$ for all $x \in \Gamma$. Then g is well defined and one to one.

On the other hand, let h be an automorphism of $G_S(\Gamma)$. By $|S| = t \geq 2$, the degree of v in $G_S(\Gamma)$ is $3t$ for $v \in \Gamma$ and the degree of u is less than or equals to 3. Then for each vertex $x \in \Gamma$, $h(x)$ must be in Γ. Then we have that there exists $P(x, y, k)$ between x and y in $G_S(\Gamma)$ if and only there exists $P(h(x), h(x), k)$ between $h(x)$ and $h(y)$ in $G(\Gamma)$. This implies that there exists $f \in Aut^*(D_S(\Gamma))$ with $f(x) = h(x)$ such that $g \circ f = h$. By this relation, we have a group isomorphism from $Aut^*(D_S(\Gamma))$ to $Aut(G_S(\Gamma))$. The proof is complete. ■

If S is a minimum generator of Γ and $|S| \geq 2$, then we can reduce the number of edges of graphs in Theorem 9. For cyclic groups, every minimum generator has only one element. Define $P^*(x, y)$ as a graph with a vertex set $\{x = v_0(x, y), v_1(x, y), v_2(x, y), v_3(x, y), y = v_4(x, y), u_1, u_2, u_3\}$ and an edge set $\{v_i(x, y), v_{i+1}(x, y) : i = 0, 1, 2, 3\} \cup \{v_1(x, y)u_1, v_4(x, y)u_1, v_2(x, y)u_2, u_2u_3, v_4(x, y)u_3\}$. In the proof of Theorem 9, $P^*(x, y)$ is instead of $P(x, y, 1)$ for $|S| = 1$. Then we can get a 2-connected graph with its neighborhood sequence $(B_1, B_2, ..., B_n)$ for a cyclic group Γ such that Γ is group isomorphic $Aut(B_1, B_2, ..., B_n)$.

If we take $S = \Gamma - \{1\}$ with 1 being the identity element of Γ, then for every subgroup H of Γ, there exists an induced subgraph F of $G_S(\Gamma)$ such that H and $Aut(F)$ are group isomorphic.

Proposition 10. *Let Γ be a finite group, 1 be the identity element of Γ, and $S = \Gamma - \{1\}$. Then there exists a 2-connected graph G with its girth ≥ 5 such that for each subgroup H of Γ, H is group isomorphic to $Aut(F)$ for some induced subgraph F of G.*

Proof. Let G be $G_\Gamma(\Gamma)$ in the proof of Theorem 9, $H = \{h_1, h_2, ..., h_t\}$, and

$U = \cup_{i \neq j} V(P(h_i, h_j, k)) \cup H$ where $V(P(h_i, h_j, k))$ is the set of all vertices in $P(h_i, h_j, k)$. It is immediate that H is group isomorphic to the automorphism group of the induced subgraph of U in G. ■

Corollary 11. *Let Γ be a finite group, 1 be the identity element of Γ, and $S = \Gamma - \{1\}$. Then there exists a neighborhood sequence $(B_1, B_2, ..., B_n)$ such that, for each subgroup H of Γ, H is group isomorphic to $Aut(E_1, E_2, ..., E_t)$ for some neighborhood sequence $(E_1, E_2, ..., E_t)$ where $E_i \subseteq B_{j_i}$ and $j_1 < j_2 < \cdots < j_t$.*

Let $n_1, n_2, ..., n_t$ be positive integers. Define $T_{n_1,n_2,...,n_t}$ as a tree with a vertex set $\{v\} \cup \{(i, j) : 1 \leq i \leq t$ and $1 \leq j \leq n_i\}$ and an edge set $\{v(i, 1) : i = 1, 2, ..., t\} \cup \{(i, j)(i.j + 1) : 1 \leq i \leq t$ and $1 \leq j \leq n_i - 1\}$.

Proposition 12. *Let $t \geq 2$, $n_1, n_2, ..., n_{2t}$ be positive integers with $n_i = n_{i+t}$ for $i = 1, 2, ..., t$ and $n_1 < n_2 < \cdots < n_t$, and $\Gamma = Aut(T_{n_1,n_2,...,n_{2t}})$. If Γ' is a subgroup of Γ, then there exists a subgraph H of $T_{n_1,n_2,...,n_{2t}}$ such that $Aut(H)$ is group isomorphic to Γ'.*

Proof. Let p be an automorphism of $T_{n_1,n_2,...,n_{2t}}$. Since deg$(v) \geq 4$, p satisfies $p(v) = v$. We can observe that $p(i, n_i), p(i + t, n_{i+t}) \in \{(i, n_i), (i + t, n_{i+t})\}$ for $i = 1, 2, ..., t$. This implies that $Aut(T_{n_1,n_2,...,n_{2t}})$ is group isomorphic to the cartesian product of t Z_2s' where Z_2 is a group of order 2. Since every subgroup Γ' of $Aut(T_{n_1,n_2,...,n_{2t}})$ is isomorphic to the cartesian product of s Z_2s' for some $s \leq t$, we have that $Aut(T_{n_1,n_2,...,n_{2s}})$ is isomorphic to Γ'. The proof is complete. ■

But not all graphs G (neighborhood sequences $(B_1, B_2, ..., B_n)$) satisfy that, for each subgroup Γ of $Aut(G)$($Aut(B_1, B_2, ..., B_n)$, resp.), there exists a subgraph H (a neighborhood sequence $(E_1, E_2, ..., E_t)$, resp.) of G ($(B_1, B_2, ..., B_n)$ with $E_i \subseteq B_{j_i}$ for some $j_1 < j_2 < \cdots < j_t$, resp.) such that Γ is group isomorphic $Aut(H)$ ($Aut(E_1, E_2, ..., E_t)$, resp.).

For example, let G be a graph of order n with its edge set being empty, the automorphism group is the symmetric group of order n and each automorphism group of a subgraph of G is also a symmetric group. If $n \geq 3$, then G has no subgraph H such that $Aut(H)$ is group isomorphic to a cyclic group of order n. For the automorphism groups of neighborhood sequences, we give an example C_4. Let C_4 be a cycle of order 4 and (B_1, B_2, B_3, B_4) be the neighborhood sequence of C_4. Then $Aut(B_1, B_2, B_3, B_4)$ is group isomorphic to the symmetric group of order 4 and has a cyclic subgroup with 3 elements. But there is no subgraph H or neighborhood sequence $(E_1, E_2, ..., E_t)$ with $t \leq 4$ and $E_i \subseteq B_{j_i}$ for some $j_1 < j_2 < \cdots < j_t$ such that $Aut(H)$ or $Aut(E_1, E_2, ..., E_t)$ is isomorphic a cyclic group with 3 elements.

Proposition 13. *Let C_n be a cycle of order n. If $n \geq 3$, then $Aut(C_n)$ has a cyclic subgroup C' of order n; but, C_n has no subgraph H with $Aut(H)$ group isomorphic to C'.*

Proof. It is easy to see that a cyclic subgroup C' of order n in $Aut(C_n)$ is a proper subgroup. If there is a subgraph H of C_n such that $Aut(H)$ is group isomorphic to C', then H is a proper subgraph of C_n. Since automorphism groups of a path or a null graph is not group isomorphic C', H is neither a path nor a null graph. This implies that H contains a component which is a path of order greater than or equal to 2. If H has at least two disjoint paths of order greater than or equal to 2, then $Aut(H)$ contains at least two elements of order 2; but every cycle contains at most one element of order 2. Thus H has only one component which is a path $(v_0, v_1, ..., v_{t-1})$ with $n-1 \geq t \geq 2$. If $t = n-1$, then $Aut(H)$ is group isomorphic to a cyclic subgroup of order 2, a contradiction. If $t \leq n-2$, then $Aut(H)$ is group isomorphic to $S_{n-t} \times Z_2$ where S_{n-t} is a symmetric group of order $n-t$ and Z_2 is a cyclic group of order 2. For each $n-1 \geq t \geq 2$, $S_{n-t} \times Z_2$ and C' are not group isomorphic. Therefore C_n has no subgraph H with $Aut(H)$ group isomorphic to C'. ∎

Corollary 14. *Let $n \geq 5$ and $(B_1, B_2, ..., B_n)$ be the neighborhood sequence of a cycle C_n. Then $Aut(B_1, B_2, ..., B_n)$ has a cyclic subgroup C' of order n; but, there is no neighborhood sequence $(E_1, E_2, ..., E_t)$ with $E_i \subseteq B_{j_i}$ for some $j_1 < j_2 < \cdots < j_t$ such that $Aut(E_1, E_2, ..., E_t)$ group isomorphic to C'.*

In the following we prove that for $n \geq 3$, there exists a subgroup G' of $Aut(K_n)$ such that G' is not group isomorphic to $Aut(H)$ for all subgraph H of K_n.

Theorem 15. (Bertrand's postulate). *For each integer n with $n \geq 2$, there exists a prime number p with $n < p < 2n-2$.*

Proposition 16. *Let K_n be a complete graph of order n. If $n \geq 3$, then there exists an integer p with $n < p < 2n$ such that $Aut(K_n)$ has a cyclic subgroup C' of order p; but there is no subgraph H of K_n such that $Aut(H)$ group isomorphic to C'.*

Proof. Let $V(K_n) = \{0, 1, 2, ..., n-1\}$. By Bertrand's postulate, we have a prime number p with $n/2 < p \leq n$ (if $n = 3$, then let $p = 3$). Then it is trivial that $Aut(K_n)$ has a cyclic subgroup C' of order p. If there is a subgraph H of K_n such that $Aut(H)$ group isomorphic to a cyclic group of order p; that is, $Aut(H)$ has a generator $\{f\}$ for some permutation f on $V(H)$. Since p is a prime number and $p \geq n/2$, without loss of generality, $f(i) = i+1$ for $i = 0, 1, 2, ..., p-2$, $f(p-1) = 0$, and $f(j) = j$ for $j \notin \{0, 1, 2, ..., p-1\}$. Then we have that if $ij \in E(H)$ with $i, j \in \{0, 1, 2, ..., p-1\}$ then $kl \in E(H)$ for each pair of $i, j \in \{1, 2, ..., p\}$ with $k - l \equiv i - j$ (mod p). (In fact, the induced subgraph of $\{0, 1, 2, ..., p-1\}$ in H is a circulant graph.) For each j with $j \notin \{0, 1, 2, ..., p-1\}$, if $jk \in E(H)$ for some $1 \leq k \leq p$ then $ji \in E(H)$ for all $1 \leq i \leq p$. Let g be a permutation on $V(H)$ with $g(0) = 0$, $g(1) = p-1$,$g(p-1) = 1$,...,$g((p-1)/2) = (p+1)/2$, $g((p+1)/2) = (p-1)/2$ and $g(j) = j$ for $j \notin \{0, 1, 2, ..., p-1\}$. Then g is an automorphism of H, but g can not be generated by f, a contradiction. Thus there is no subgraph H of K_n such that $Aut(H)$ group isomorphic to a cyclic group of order p. ■

Proposition 17. *Let n be an integer with $n \geq 3$ and $(B_1, B_2, ..., B_n)$ be the neighborhood sequence of a complete graph K_n. Then there exists an integer p with $p \leq n$ such that $Aut(B_1, B_2, ..., B_n)$ has a cyclic subgroup C' of order n, but there is no neighborhood sequence $(E_1, E_2, ..., E_t)$ with $E_i \subseteq B_{j_i}$ for some $j_1 < j_2 < \cdots < j_t$ such that $Aut(E_1, E_2, ..., E_t)$ group isomorphic to C'.*

Proof. Let $V(K_n) = \{0, 1, 2, ..., n-1\}$ and $(B_1, B_2, ..., B_n)$ be the neighborhood sequence of a complete graph K_n. By Bertrand's postulate, we have a prime number p with $n/2 < p \leq n$. Then it is trivial that $Aut(B_1, B_2, ..., B_n)$ has a cyclic subgroup C' of order p. If there is a neighborhood sequence $(E_1, E_2, ..., E_t)$ with $E_i \subseteq B_{j_i}$ for some $j_1 < j_2 < \cdots < j_t$ such that $Aut(E_1, E_2, ..., E_t)$ group isomorphic to C'; that is, $Aut(E_1, E_2, ..., E_t)$ has a generator $\{f\}$ for some permutation f on $\cup_{i=1}^{t} E_i$. Since p is a prime number, without loss of generality, $f(i) = i + 1$ for $i = 0, 1, 2, ..., p-2$ and $f(p-1) = 0$. If $f(E_i) = E_i$ for all i, then $Aut(E_1, E_2, ..., E_t)$ is not a cyclic group, a contradiction. If $f(E_i) \neq E_i$ for some i then $E_i, f(E_i), ..., f^{n-1}(E_i)$ are different. Since p is prime, without loss of generality, we say $f^{i-1}(E_1) = E_i$ for $i = 1, 2, ..., p$. If there exist $a \neq b$ with $f^i(E_1) = E_a = E_b$ for some i or $f(E_k) \notin \{E_k, E_1, E_2, ..., E_{p-1}\}$, then $t \geq 2p$, a contradiction. Thus $f(E_k) = E_k$ for $k \geq p$. We can conclude that

(a) $E_i \cap \{p, ..., n-1\} = E_j \cap \{p, ..., n-1\}$ for $0 \leq i, j \leq p-1$.

(b) $E_k \cap \{0, 1, ..., p-1\}$ is either $\{0, 1, ..., p-1\}$ or empty set for $k \geq p$.

Let $(v_1, v_2, ..., v_t)$ be an adjacent SDR of $(E_1, E_2, ..., E_t)$.
If $\{v_1, v_2, ..., v_p\} = \{0, 1, ..., p-1\}$ then we have that $v_l \in E_k$ for each v_l with $|v_k - v_l| \equiv |v_i - v_j|$ (mod p). Let g be a permutation on $V(K_n)$ with $g(0) = 0$, $g(1) = p-1$,...,$g((n-1)/2) = (n+1)/2$. Then g is an automorphism of $Aut(E_1, E_2, ..., E_n)$, but g can not be generated by f, a contradiction.
If there exists $v \in \{0, 1, ..., p-1\} - \{v_1, v_2, ..., v_p\}$, then $v = v_k$ and $v_i \geq p$ for some $k \geq p$ and $1 \leq i \leq p-1$. By (b), we have that $v_k \in E_j$ for $j \leq p$ and $v_j \leq p-1$; that is, $v_j \in E_k$. By (a), $v_i \in E_r$ for $r \in \{1, 2, ..., p\}$ and $v_r \leq p-1$. By above, we have that B_j contains $0, 2, ..., p-1$ for $j \leq p$. Then $Aut(E_1, E_2, ..., E_t)$ is not a cyclic group. The proof is complete. ■

A permutation p on $\{1, 2, ..., n\}$ is a *transposition* if there exist $i \neq j \in \{1, 2, ..., n\}$ such that $p(i) = j$, $p(j) = i$, and $p(x) = x$ for $x \notin \{i, j\}$. A permutation p on $\{1, 2, ..., n\}$ is is even if it can be written as the composition of even number of transpositions. The set of all even permutations on $\{1, 2, ..., n\}$ is called the *alternating group* A_n of order n. Then A_n is a subgroup of the symmetric group on $\{1, 2, ..., n\}$ (the set of all permutations with the composition operator).

Proposition 18. *Let K_n be a complete graph of order n. If $n \geq 3$, then there is no subgraph H of K_n such that $Aut(H)$ group isomorphic to A_n.*

Proof. Assume that there exists a subgroup of K_n such that $Aut(H)$ group isomorphic to A_n. Then H is regular and nontrivial. If $xy \in E(G)$ and $z \in \{1, 2, ..., n\}$ then by $(x, y, z) \in A_n$, we have that $yz \in E(H)$; that is, H is K_n, a contradiction. ■

Proposition 19. *Let n be an integer with $n \geq 3$. Then there is no neighborhood sequence $(B_1, B_2, ..., B_n)$ such that $Aut(B_1, B_2, ..., B_n)$ group isomorphic to A_n.*

Proof. Assume that there exists a neighborhood sequence $(B_1, B_2, ..., B_n)$ such that $Aut(B_1, B_2, ..., B_n)$ group isomorphic to A_n. Then $|B_i|$ is a constant k. If k is either 1 or greater than or equal $n - 1$, then $Aut(B_1, B_2, ..., B_n)$ is symmetric group of order n, a contradiction. If $n \leq 4$, then $n = 4$ and $k = 2$. Without loss of generality, let $(B_1, B_2, B_3, B_4) = (\{1, 2\}, \{1, 2\}, \{3, 4\}, \{3, 4\})$. Then $Aut(\{1, 2\}, \{1, 2\}, \{3, 4\}, \{3, 4\}\}$ is not group isomorphic to A_4. Suppose that $n \geq 5$ and $2 \leq k \leq n - 2$. Let $B_1 = \{x_1, x_2, ..., x_k\}$. Let a, b, c be distinct in $\cup_{i=1}^{n} B_i$ and g be a permutation g by $g(a) = b$, $g(b) = c$, $g(c) = a$, and $g(x) = x$ for $x \notin \{a, b, c\}$. Since g is in A_n and $g(B_1) = B_j$ for some j, $k(n - k) \leq n$. It contradicts that $2 \leq k \leq n - 2$. ■

Corollary 20. *Let n be an integer with $n \geq 3$ and $(B_1, B_2, ..., B_n)$ be the neighborhood sequence of a complete graph K_n. Then A_n is group isomorphic to some subgroup of $Aut(B_1, B_2, ..., B_n)$, but there is no neighborhood sequence $(E_1, E_2, ..., E_t)$ with $E_i \subseteq B_{j_i}$ for some $j_1 < j_2 < \cdots < j_t$ such that $Aut(E_1, E_2, ..., E_t)$ group isomorphic to A_n.*

Proof. It is easy to see that A_n is group isomorphic to some subgroup of $Aut(B_1, B_2, ..., B_n)$. Since $|Aut(B_1, B_2, ..., B_n)|/|A_n| = 2$, $t = 2$. By Proposition 19, we have that there is no neighborhood sequence $(E_1, E_2, ..., E_t)$ with $E_i \subseteq B_{j_i}$ for some $j_1 < j_2 < \cdots < j_t$ such that $Aut(E_1, E_2, ..., E_t)$ group isomorphic to A_n. ■

3. $Aut(G)$ AND $Aut(B_1, B_2, ..., B_n)$

Sometimes, in a graph G with a neighborhood sequence $(B_1, B_2, ..., B_n)$, $Aut(G)$ and $Aut(B_1, B_2, ..., B_n)$ are different. Let $[n] = \{1, 2, ..., n\}$ and Q_n be the n-cube. Q_n is a graph with its vertex set being the power set of $[n]$ and AB being an edge of Q_n if and only if the symmetric difference of A and B having only one element.

Proposition 21. *Let n be an integer with $n \geq 2$ and $(B_1, B_2, ..., B_m)$ be the neighborhood sequence of Q_n. Then $Aut(Q_n)$ is not equals to $Aut(B_1, B_2, ..., B_m)$.*

Proof. Let $S = \{A : A \subseteq [n-1]$ and $|A|$ is even$\}$ and $S^* = \{A \cup \{n\} : A \in S\}$. Define a permutation f on $V(Q_n)$ by $f(A) = A \cup \{n\}$ for $A \in S$, $f(C) = C - \{n\}$ for $C \in S^*$, and $f(D) = D$ for $D \in V(Q_n) - (S \cup S^*)$. It is easy to see that f is not an automorphism of Q_n. Let A be a subset of $[n]$. If $A \in S \cup S^*$, then $f(B(A)) = B(A)$. If $A \in V(Q_n) - (S \cup S^*)$ with $n \notin A$, then $f(B(A)) = B(A \cup \{n\})$. These imply that $(f(B_1), f(B_2), ..., f(B_m))$ is a permutation of $(B_1, B_2, ..., B_m)$; that is, $f \in Aut(B_1, B_2, ..., B_m)$. ■

Proposition 22. *Let f be the permutation in the proof of Proposition 21. Then $Aut(B_1, B_2, ..., B_m)$ is generated by $Aut(Q_n) \cup \{f\}$.*

Proof. Let $p \in Aut(B_1, B_2, ..., B_m)$. If $p(B(x)) = B(y)$ with $p(x) = y$, then we have that $N(x) = \{x_1, x_2, ..., x_n\}$ and $N(y) = \{y_1, y_2, ..., y_n\}$ such that $p(x_i) = y_i$ for all i. Since every pair of adjacent vertices are contained in exactly two B_is', $p(B(x_i)) = B(y_i)$ for $i = 1, 2, ..., n$. By the similar arguments, we have that $p(B(u)) = B(p(u))$ for all $u \in V(Q_n)$. This implies that $p \in Aut(Q_n)$.

If $p(B(x)) = B(y)$ with $p(x) = w$ for some $w \in N(y)$, then there exists a vertex $z \in N(x)$ such that $p(z) = y$. Let a, b be different elements in $[n]$, and f_a and g_{ab} be permutations of $V(Q_n)$ with $f_a(A) = A - \{a\}$ for $a \in A$, $f_a(B) = B \cup \{a\}$ for $a \notin B$, $g_{ab}(A) = (A - \{a\}) \cup \{b\}$ for $a \in A$ and $b \notin A$, $g_{ab}(B) = (B - \{b\}) \cup \{a\}$ for $a \notin B$ and $b \in B$, and $g_{ab}(C) = C$ for $a, b \in C$ or $a, b \notin C$. It is easy to see that f_a and g_{ab} are in $Aut(B_1, B_2, ..., B_m)$. Since the composition of two automorphisms in $Aut(B_1, B_2, ..., B_m)$, f_a, and g_{ab} are in $Aut(B_1, B_2, ..., B_m)$, without loss of generality, we can assume that x and y are empty sets, and z and w are $\{n\}$. Since $p(B(x)) = B(y)$, let $N(x) = \{z_1, z_2, ..., z_n = z\}$ and $N(y) = \{w_1, w_2, ..., w_n = w\}$ with $p(z_i) = w_i$ for $i = 1, 2, ..., n-1$. Since every pair of adjacent vertices are contained in exactly two B_is', $p(B(z)) = B(w)$. We observe that $N(z) \cap N(z_i) = \{x, z'_i\}$ and $N(w) \cap N(w_i) = \{y, w'_i\}$ for $i = 1, 2, ..., n-1$. Since $p(z_i) = w_i$ for $i = 1, 2, ..., n-1$, we have that $p(B(z_i)) = B(w'_i)$ and $p(B(z'_i)) = B(w_i)$ for $i = 1, 2, ..., n-1$. By $B(z_i) \cap B(z) = \{x, z'_i\}$, $p(B(z_i)) \cap p(B(z)) = \{p(x), p(z'_i)\}$. Then $\{w, w'_i\} = \{w, p(z'_i)\}$. Thus we have that $p(z'_i) = w'_i$ for $i = 1, 2, ..., n-1$. Let $N(z_j) = \{u_1, u_2, ..., u_{n-1} = x, u_n = z'_j\}$ and $N(z'_j) = \{v_1, v_2, ..., v_{n-1} = z, v_n = z_j\}$ with $u_t v_t \in E(Q_n)$ for all t. Since $p(B(z_j)) = B(w'_j)$ and $p(B(z'_j)) = B(w_j)$, let $N(w_j) = \{r_1, r_2, ..., r_{n-1} = y, u_n = w'_j\}$ and $N(w'_j) = \{s_1, s_2, ..., s_{n-1} = w, v_n = w_j\}$ with $r_t s_t \in E(Q_n)$ for all t and $p(u_i) = s_i$ for $i = 1, 2, ..., n-1$. Since $p(v_i) \in \{r_1, r_2, ..., r_{n-2}\}$ and $s_i \in p(B(v_i))$, $p(v_i) = r_i$ for $i = 1, 2, ..., n-2$. By the similar discussion, we have that $p = f$ where f is the permutation in the proof of Proposition 21. Therefore, every element of $Aut(B_1, B_2, ..., B_m)$ can be generated by $Aut(Q_n) \cup \{f\}$. ■

Let K_{2n}^m be a graph obtained from the complete graph K_{2n} deleting a perfect matching.

Proposition 23. *For each positive integer* $n \geq 2$, $Aut(K_{2n}^m)$ *is not equals to* $Aut(B_1, B_2, ..., B_{2n})$ *where* $(B_1, B_2, ..., B_{2n})$ *is a neighborhood sequence of* K_{2n}^m.

Proof. Let $V(K_{2n}^m)$ be $\{1, 2, ..., 2n\}$ and $i(i+1) \notin E(K_{2n}^m)$ for all odd number i. Since $B_1, B_2, ..., B_{2n}$ is all $(2n-1)$-subsets of $\{1, 2, ..., 2n\}$, every permutation of $\{1, 2, ..., 2n\}$ is in $Aut(B_1, B_2, ..., B_{2n})$; that is, $Aut(B_1, B_2, ..., B_{2n})$ is isomorphic to the symmetric group S_{2n}. But, let p be a permutation on $\{1, 2, ..., 2n\}$ with $p(1) = 3$, $p(3) = 1$, and $p(x) = x$ for $x \in \{2, 4, 5, ..., 2n\}$, it is not an automorphism of K_{2n}^m. ■

Remark. By the definitions of Proposition 23, $Aut(B_1, B_2, ..., B_{2n}) = S_{2n}$ is the symmetric group of order $2n$ and $Aut(K_{2n}^m) = \{p \in S_{2n} : p(\{i, i+1\}) = \{j, j+1\}$ for all odd integer $i\}$.

The following proposition shows that there are two non-isomorphic graph G and H with the same neighborhood sequence $(B_1, B_2, ..., B_n)$, but $Aut(G) = Aut(B_1, B_2, ..., B_n)$ is not group isomorphic to $Aut(H)$.

Proposition 24. *Let* G *be the graph with a vertex set* $\{1, 2, ..., 2n\}$ *and an edge set* $\{i(i+n) : i = 1, 2, ..., n\} \cup \{ij : \{i, j\} \subseteq \{1, 2, ..., n\}$ *or* $\{i, j\} \subseteq \{n+1, n+2, ..., 2n\}\}$ *and* $H = K_{n,n}$ *with two partite sets* $\{1, 2, ..., n\}$ *and* $\{n+1, n+2, ..., 2n\}$. *Then*

(1) G *and* H *have the same neighborhood sequence* $(B_1, B_2, ..., B_{2n})$,

(2) $Aut(G)$ *is not group isomorphic to* $Aut(H)$, *and*

(3) $Aut(H) = Aut(B_1, B_2, ..., B_{2n})$.

Proof. Let $B_i = \{i, n+1, n+2, ..., 2n\}$ for $i = 1, 2, ..., n$ and $B_j = \{j, 1, 2, ..., n\}$ for $j = n+1, n+2, ..., 2n$. Then it is easy to check that $(B_1, B_2, ..., B_{2n})$ is a neighborhood sequence of G and H.

We have that f is an automorphism of G if and only if $\{f(1), f(2), ..., f(n)\}$ is either $\{1, 2, ..., n\}$ or $\{n+1, n+2, ..., 2n\}$. Let g be a permutation on $\{1, 2, ..., 2n\}$ with $p(1) = 2$, $p(2) = 1$, and $p(x) = x$ for $x \notin \{1, 2\}$. Then $p \in Aut(H)$, but $p \notin Aut(G)$. (In fact, $Aut(G)$ is a proper subgroup of $Aut(H)$.)

It is trivial that $Aut(H) \subseteq Aut(B_1, B_2, ..., B_{2n})$. Take $f \in Aut(B_1, B_2, ..., B_{2n})$.

(a) Assume that $f(i) = j$ for some $i, j \in \{1, 2, ..., n\}$. If $f(\{n+1, ..., 2n\}) = \{n+1, ..., 2n\}$, then $f(\{1, ..., n\}) = \{1, ..., n\}$; that is, $f \in Aut(H)$. If $f(x) = y$ for some $x \in \{n+1, n+2, ..., 2n\}$ and $y \in \{1, 2, ..., n\}$, then $f(B_i) = B_t$ for some $t \in \{n+1, ..., 2n\}$. This implies that there is a $u \in \{n+1, ..., 2n\} - \{x\}$ and $v \in \{1, ..., n\} - \{y\}$ such that, $f(u) = v$. Then we have that, for each $a \in \{1, ..., n\}$, there exists $b \in \{n+1, ..., 2n\}$ such that $f(B_a) = B_b$. Since $f(B_i) = B_t$ and $f(i) = j$, there exists $w \in \{n+1, ..., 2n\}$ such that $f(w) = t$. By above, we conclude that $f(B_c) = B_t$ for all $c \in \{1, 2, ..., n\}$, a contradiction.

(b) Assume that $f(i) = j$ for some $i \in \{1, 2, ..., n\}$ and $j \in \{n+1, n+2, ..., 2n\}$. If $f(\{n+1, ..., 2n\}) = \{1, ..., n\}$, then $f(\{1, ..., n\}) = \{n+1, ..., 2n\}$; that is, $f \in Aut(H)$. If $f(x) = y$ for some $x \in \{n+1, n+2, ..., 2n\}$ and $y \in \{n+1, n+2, ..., 2n\}$, then $f(B_i) = B_t$ for some $t \in \{1, ..., n\}$. This implies that there is a $u \in \{n+1, ..., 2n\} - \{x\}$ and $v \in \{n+1, ..., 2n\} - \{y\}$ such that, $f(u) = v$. Then we have that, for each $a \in \{1, ..., n\}$, there exists $b \in \{1, ..., n\}$ such that $f(B_a) = B_b$. Since $f(B_i) = B_t$ and $f(i) = j$, there exists $w \in \{n+1, ..., 2n\}$ such that $f(w) = t$. By above, we conclude that $f(B_c) = B_t$ for all $c \in \{1, 2, ..., n\}$, it is also a contradiction. Therefore, every automorphism in $Aut(B_1, B_2, ..., B_{2n})$ must be an automorphism of G. Hence $Aut(H) = Aut(B_1, B_2, ..., B_{2n})$. ■

Remark. In the proposition 24, $Aut(G) = \{p : p$ is a permutation of $\{1, 2, ..., 2n\}$ and $p(\{1, 2, ..., n\})$ is either $\{1, 2, ..., n\}$ or $\{n+1, n+2, ..., 2n\}\}$ and $Aut(H) = \{p \in Aut(G) : p(\{i, n+i\}) = \{j, j+n\}$ for $i = 1, 2, ..., n\}$.

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