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HIGHER ORDER RIESZ TRANSFORMS FOR THE DUNKL HARMONIC OSCILLATOR

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Abstract. The aim of this paper is to extend the study of Riesz transforms associated to the Dunkl harmonic oscillator considered by A.Nowak and K. Stempak to higher order. The methods used to establish the L^p-boundedness of these transforms call the Calderón-Zygmund theory.

1. Introduction and Statement of the Results

Let $T_j^\alpha,\ j=1,...,d,\ \alpha\in[-\frac{1}{2},\infty)^d$, be the Dunkl differential-difference operators defined by

$$T_j^\alpha=\partial_j f(x)+(\alpha_j+\frac{1}{2})\frac{f(x)-f(\sigma_j x)}{x_j},\quad f\in\mathcal{C}^1(\mathbb{R}^d),$$

here ∂_j is the jth partial derivative and σ_j denotes the reflection in the hyperplane orthogonal to e_j, the jth coordinate vector in $\mathbb{R}^d$.

In Dunkl's theory the operator :

$$\Delta_\alpha=\sum_{j=1}^d (T_j^\alpha)^2$$

plays the role of the Euclidean Laplacian.

We recall the definition of the Dunkl-Hermite oscillator, given in [8] by

$$L_\alpha=-\Delta_\alpha+\|x\|^2.$$

The corresponding weight ω_α has the form

$$\omega_\alpha(x)=\prod_{j=1}^d |x_j|^{2\alpha_j+1}.$$

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Given $\alpha \in [-\frac{1}{2}, \infty)^d$, the associated generalized Hermite functions are tensor products

$$h_n^\alpha(x) = h_{n_1}^{\alpha_1} \times ... \times h_{n_d}^{\alpha_d}, \quad x = (x_1, ..., x_d) \in \mathbb{R}^d, \quad n = (n_1, ..., n_d) \in \mathbb{N}^d,$$

where $h_{n_i}^{\alpha_i}$ are the one-dimensional generalized Hermite functions

$$h_{2n_i}^{\alpha_i}(x_i) = (-1)^{n_i}\Big(\frac{\Gamma(n_i+1)}{\Gamma(n_i+\alpha_i+1)}\Big)^{\frac{1}{2}} e^{-x_i^2/2} L_{n_i}^{\alpha_i}(x_i^2),$$

$$h_{2n_i+1}^{\alpha_i}(x_i) = (-1)^{n_i}\Big(\frac{\Gamma(n_i+1)}{\Gamma(n_i+\alpha_i+2)}\Big)^{\frac{1}{2}} e^{-x_i^2/2} x_i L_{n_i}^{\alpha_i+1}(x_i^2),$$

here $L_{n_i}^{\alpha_i}$ denotes the Laguerre polynomial of degree n_i and order α_i.

The system $\{h_n^\alpha : \ n \in \mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}^d, \ \omega_\alpha)$ consisting of eigenfunctions of L_α, (see [8])

$$L_\alpha h_n^\alpha = (2|n| + 2|\alpha| + 2d) h_n^\alpha,$$

where we denote $|\alpha| = \alpha_1 + ... + \alpha_d$.

We define the jth partial "derivative" δ_j^α, for $1 \le j \le d$, related to L_α by

$$\delta_j^\alpha = T_j^\alpha + x_j.$$

The formal adjoint of δ_j^α in $L^2(\mathbb{R}^d, \omega_\alpha)$ is

$$\delta_{-j}^\alpha = -T_j^\alpha + x_j.$$

This precisely means that

$$\langle \delta_j^\alpha f, g\rangle_\alpha = \langle f, \delta_{-j}^\alpha g\rangle_\alpha, \quad f, \ g \in \mathcal{C}_c^1(\mathbb{R}^d),$$

where $\langle ., .\rangle_\alpha$ is the canonical inner product in $L^2(\mathbb{R}^d, \omega_\alpha)$.

As a direct computation shows, we have

$$L_\alpha = \frac{1}{2}\sum_{j=1}^{d}(\delta_{-j}^\alpha \delta_j^\alpha + \delta_j^\alpha \delta_{-j}^\alpha).$$

We recall that for $1 \le j \le d$ (see [6])

$$\delta_{+j}^\alpha h_n^\alpha = m(n_j, \alpha_j) h_{n-e_j}^\alpha$$

$$\delta_{-j}^\alpha h_n^\alpha = m(n_j+1, \alpha_j) h_{n+e_j}^\alpha$$

where

$$m(n_j, \alpha_j) = \begin{cases} \sqrt{2n_j} & if \ n_j \ is \ even, \\ \sqrt{2n_j + 4\alpha_j + 2} & if \ n_j \ is \ odd. \end{cases}$$

The self-adjoint extension of L_α initially considered on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is given by the operator

$$\mathcal{L}_\alpha f = \sum_{\nu\in\mathbb{N}^d} (2|\nu| + 2|\alpha| + 2d)\langle f, h_\nu^\alpha\rangle_\alpha h_\nu^\alpha,$$

and defined on the domain

$$Dom(\mathcal{L}_\alpha) = \Big\{ f \in L^2(\mathbb{R}^d,\ \omega_\alpha) : \sum_{\nu\in\mathbb{N}^d} |(2|\nu| + 2|\alpha| + 2d)\langle f, h_\nu^\alpha\rangle_\alpha|^2 < \infty \Big\}.$$

The semigroup $e^{-t\mathcal{L}_\alpha},\ t \geq 0$, (in the one dimensional one can see [1]) generated by $\mathcal{L}_\alpha$ is a strongly continuous semigroup of contractions on $L^2(\mathbb{R}^d,\ \omega_\alpha)$. By the spectral theorem,

$$e^{-t\mathcal{L}_\alpha} f = \sum_{m=0}^\infty e^{-t(2m+2|\alpha|+2d)}\mathcal{P}_m^\alpha f, \quad f \in L^2(\mathbb{R}^d,\ \omega_\alpha),$$

where the spectral projections are

$$\mathcal{P}_m^\alpha f = \sum_{|n|=m} \langle f, h_n^\alpha\rangle_\alpha h_n^\alpha.$$

The integral representation of $e^{-t\mathcal{L}_\alpha}$ on $L^2(\mathbb{R}^d,\ \omega_\alpha)$ is

$$e^{-t\mathcal{L}_\alpha} f(x) = \int_{\mathbb{R}^d} G_t^\alpha(x,y) f(y)\, d\omega_\alpha(y), \quad x \in \mathbb{R}^d,$$

where the heat kernel is given by

$$G_t^\alpha(x,y) = \sum_{m=0}^\infty e^{-t(2m+2|\alpha|+2d)} \sum_{|n|=m} h_n^\alpha(x) h_n^\alpha(y).$$

It is shown in [6] that

$$G_t^\alpha(x,y) = \sum_{\varepsilon\in\{0,1\}^d} G_t^{\alpha,\varepsilon}(x,y),$$

where

$$G_t^{\alpha,\varepsilon}(x,y) = \frac{1}{(2\sinh 2t)^d} \exp\big(-\frac{1}{2}\coth(2t)(\|x\|^2 + \|y\|^2)\big) \prod_{i=1}^d (x_i y_i)^{\varepsilon_i} \frac{I_{\alpha_i+\varepsilon_i}(\frac{x_i y_i}{\sinh 2t})}{(x_i y_i)^{\alpha_i+\varepsilon_i}},$$

with I_β being the modified Bessel function of the first kind and order β,

$$I_\beta(z) = \sum_{\kappa=0}^\infty \frac{(z/2)^{\beta+2\kappa}}{\Gamma(\kappa+1)\Gamma(\kappa+\beta+1)}.$$

The Schlafli's integral representation of Poisson's type for the modified Bessel function is (see [9])

$$I_\mu(z) = z^\mu \int_{-1}^{1} \exp(-zs)\Pi_\mu(ds), \quad z > 0, \;\; \mu \geq -\frac{1}{2},$$

where Π_μ is the measure given by

$$\Pi_\mu(ds) = \frac{(1-s^2)^{\mu-\frac{1}{2}}}{\sqrt{\pi}2^\mu\Gamma(\mu+\frac{1}{2})}ds, \quad s \in (-1,1),$$

when $\mu > -\frac{1}{2}$, and in the limiting case of $\mu = -\frac{1}{2}$,

$$\Pi_{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}(\eta_{-1} + \eta_1),$$

where η_{-1} and η_1 denote point masses at -1 and 1, respectively.

In [6], Nowak and Stempak deduced the following expression for the kernel $G_t^{\alpha,\varepsilon}(x,y)$, for $\alpha \in [-\frac{1}{2},\infty)^d$

$$G_t^{\alpha,\varepsilon}(x,y) = \frac{(xy)^\varepsilon}{2^d(\sinh 2t)^{d+|\alpha|+|\varepsilon|}}\int_{[-1,1]^d} \exp\Big(-\frac{1}{2}\coth(2t)(\|x\|^2+\|y\|^2) - \sum_{i=1}^{d}\frac{x_iy_is_i}{\sinh(2t)}\Big)\Pi_{\alpha+\varepsilon}(ds),$$

where Π_α denotes the product measure $\bigotimes_{i=1}^d \Pi_{\alpha_i}$. And by the change of variable

$$t(\xi) = \frac{1}{2}\log\frac{1+\xi}{1-\xi}, \quad \xi \in (0,1), \tag{1}$$

they obtained the following symmetric formula

$$G_t^{\alpha,\varepsilon}(x,y) = \frac{1}{2^d}\Big(\frac{1-\xi^2}{2\xi}\Big)^{d+|\alpha|+|\varepsilon|}(xy)^\epsilon\int_{[-1,1]^d} \exp\Big(-\frac{1}{4\xi}q_+(x,y,s) - \frac{\xi}{4}q_-(x,y,s)\Big)\Pi_{\alpha+\varepsilon}(ds),$$

with

$$q_\pm(x,y,s) = \|x\|^2 + \|y\|^2 \pm 2\sum_{i=1}^{d} x_iy_is_i.$$

In [6] Nowak and Stempak introduced the Riesz transforms of order one related to the Dunkl harmonic oscillator L_α and they proved that these transforms are L^p bounded with $1 < p < \infty$. The aim of this paper is to present an extension of this result to the Riesz transforms of order k with $1 \leq k \leq d$.

According to a general principle we now define higher Dunkl-Riesz transforms in the following way: given $k,\ 1 \leq k \leq d$ and $\tau_k = (i_1, ..., i_k) \in \{1, ..., d\}^k$ such that $i_j \neq i_m$ if $j \neq m$, the family of the Dunkl-Riesz transforms $\{\mathcal{R}_{\pm}^{\alpha,\tau_k}\}$ of order k is given by

$$\mathcal{R}_{\pm}^{\alpha,\tau_k} = \delta_{\pm}^{\alpha,\tau_k} \mathcal{L}_{\alpha}^{-\frac{k}{2}}$$

where $\delta_{\pm}^{\alpha,\tau_k} = \delta_{\pm i_1}^{\alpha} .....\delta_{\pm i_k}^{\alpha}$ and $\delta_{\pm j}^{\alpha} = \pm T_j^{\alpha} + x_j$.

We note that for technical reasons, we have considered the vector $\tau_k = (i_1, ..., i_k) \in \{1, ..., d\}^k$ such that $i_j \neq i_m$ if $j \neq m$.

In section 2 we define the kernel $R_{\pm,\varepsilon}^{\alpha,\tau_k}(x, y)$ and we give a Calderón-Zygmund type estimations for this kernel which will be used to study the operators $\mathcal{R}_{\pm}^{\alpha,\tau_k}$.

The aim of the next section is to prove that the operators $\mathcal{R}_{\pm}^{\alpha,\tau_k}$ have associated kernels satisfying the Calderón-Zygmund standard conditions and, in consequence, boundedness properties of these higher Dunkl-Riesz transforms are analogous to those of the Dunkl-Riesz transforms of order one studied in [6].

We point out that the methods used to establish our results, are borrowed from [5] and [6]. Consequently, these results extend naturally those established in [6] by A.Nowak and K.Stempak.

2. Kernel Estimates

In this section we define and study the kernel $R_{\pm,\varepsilon}^{\alpha,\tau_k}(x, y)$ and we give a Calderón-Zygmund type estimations for this kernel.

Notation 1. We denoted by :

- $\beta_{d,\alpha}^k(\xi)$ for $\alpha \in [-\frac{1}{2}, \infty)^d,\ 1 \leq k \leq d$, the function on ξ given by

$$\beta_{d,\alpha}^k(\xi) = \frac{2^{1-d-\frac{k}{2}}}{\Gamma(\frac{k}{2})} \big(\frac{1-\xi^2}{2\xi}\big)^{d+|\alpha|} \frac{1}{1-\xi^2} \big(\log \frac{1+\xi}{1-\xi}\big)^{\frac{k}{2}-1}.$$

  We point out that the definition of $\beta_{d,\alpha}^k$ agrees for $k = 1$ with $\beta_{d,\alpha}$ introduced in $[6], p.550$.

- $\psi_{\xi}^{\varepsilon}(x, y, s)$ for $\varepsilon \in \{0, 1\}^d$, the function on $x,\ y,\ s$ given by

$$\psi_{\xi}^{\varepsilon}(x, y, s) = (xy)^{\varepsilon} \exp\big(-\frac{1}{4\xi} q_+(x, y, s) - \frac{\xi}{4} q_-(x, y, s)\big)$$

Definition 1. Let $\alpha \in [-\frac{1}{2}, \infty)^d,\ 1 \leq k \leq d,\ \varepsilon \in \{0, 1\}^d$ and $\tau_k = (i_1, ..., i_k) \in \{1, ..., d\}^k$ such that $i_j \neq i_m$ if $j \neq m$. We define the kernel $R_{\pm,\varepsilon}^{\alpha,\tau_k}(x, y)$ by

$$R_{\pm,\varepsilon}^{\alpha,\tau_k}(x, y) = \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \beta_{d,\alpha+\varepsilon}^k(\xi) \delta_{\pm,x}^{\alpha,\tau_k} \psi_{\xi}^{\varepsilon}(x, y, s) d\xi$$

where

$$\delta_{\pm}^{\alpha,\tau_k} = \delta_{\pm i_1}^{\alpha} \cdots \delta_{\pm i_k}^{\alpha},$$

$$\delta_{\pm j}^{\alpha} = \pm T_j^{\alpha} + x_j.$$

The following lemmas are just Lemma 5.2 and Lemma 5.3 in [6], p. 551, correspondingly for $k \geq 1$.

Lemma 1. *Assume that* $\alpha \in [-\frac{1}{2}, \infty)^d$. *Let* $b \geq 0$ *and* $c > 0$ *and* $k \geq 1$ *be fixed. Then, for any* $j = 1, ..., d$, *we have*

$$\big(|x_j + y_j s_j| + |y_j + x_j s_j|\big)^b \exp\big(-c\frac{1}{\xi} q_+(x,y,s)\big) \lesssim \xi^{b/2}, \tag{2}$$

$$\big(|x_j - y_j s_j| + |y_j - x_j s_j|\big)^b \exp\big(-c\xi q_-(x,y,s)\big) \lesssim \xi^{-b/2}, \tag{3}$$

$$(x_j)^b \exp\big(-c\frac{1}{\xi} q_+(x,y,s) - c\xi q_-(x,y,s)\big) \lesssim \xi^{-b/2}, \tag{4}$$

$$\int_0^1 \beta_{d,\alpha}^k(\xi)\xi^{-b-\frac{k}{2}} \exp\big(-c\frac{1}{\xi} q_+(x,y,s)\big)d\xi \lesssim (q_+(x,y,s))^{-d-|\alpha|-b}, \tag{5}$$

uniformly in $x, y \in \mathbb{R}_+^d$, $s \in [-1,1]^d$ *and except of* (5), *in* $\xi \in (0,1)$.

Lemma 2. *Assume that* $\alpha \in [-\frac{1}{2}, \infty)^d$ *and let* $\gamma, \kappa \in [0,\infty)^d$ *be fixed. If a complex-valued kernel* $K(x,y)$ *defined on* $\mathbb{R}_+^d \times \mathbb{R}_+^d \backslash \{(x,y) : x = y\}$ *satisfies*

(a)

$$|K(x,y)| \lesssim (x+y)^{2\gamma} \int_{[-1,1]^d} \Pi_{\alpha+\gamma+\kappa}(ds)(q_+(x,y,s))^{-d-|\alpha|-|\gamma|},$$

then also

$$|K(x,y)| \lesssim \frac{1}{\omega_\alpha^+(B^+(x, \|y-x\|))}, \quad x, y \in \mathbb{R}_+^d \text{ and } x \neq y$$

with ω_α^+ *being the restriction of* ω_α *to* $\mathbb{R}_+^d$ *and* $B^+(x, \|y-x\|)$ *the intersection of the The Euclidean balls* $B(x, \|y-x\|)$ *with* $\mathbb{R}_+^d$.

(b) *Similarly, the estimate*

$$\|\nabla_{x,y} K(x,y)\| \lesssim (x+y)^{2\gamma} \int_{[-1,1]^d} \Pi_{\alpha+\gamma+\kappa}(ds)(q_+(x,y,s))^{-d-|\alpha|-|\gamma|-1/2},$$

implies

$$\|\nabla_{x,y} K(x,y)\| \lesssim \frac{1}{\|x-y\|} \frac{1}{\omega_\alpha^+(B^+(x, \|y-x\|))}, \quad x, y \in \mathbb{R}_+^d \text{ and } x \neq y.$$

Remark 1. Notice that for $\lambda \in [-1/2, \infty)^d$ and $\mu_i \geq 0,\ i \in \{1, ..., d\}$,

(6) $$\beta^k_{d,\lambda+\sum_{i=1}^d \mu_i e_i}(\xi) \leq \xi^{-\sum_{i=1}^d \mu_i} \beta^k_{d,\lambda}(\xi), \quad \xi \in (0,1).$$

This property follows directly from the exact formula for $\beta^k_{d,\alpha}$.

Theorem 1. *Let* $\alpha \in [-\frac{1}{2}, \infty)^d,\ 1 \leq k \leq d,\ \varepsilon \in \{0,1\}^d$ *and* $\tau_k = (i_1, ..., i_k) \in \{1, ..., d\}^k$ *such that* $i_j \neq i_m$ *if* $j \neq m$.

Then the kernel $R^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y)$*, satisfy*

$i)$

$$|R^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y)| \lesssim \frac{1}{\omega^+_\alpha(B^+(x, \|y-x\|))}, \quad x, y \in \mathbb{R}^d_+ \text{ and } x \neq y$$

$ii)$

$$\|\nabla_{x,y} R^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y)\| \lesssim \frac{1}{\|x-y\|} \frac{1}{\omega^+_\alpha(B^+(x, \|y-x\|))}, \quad x, y \in \mathbb{R}^d_+ \text{ and } x \neq y.$$

Proof.

To compute $\delta^{\alpha,\tau_k}_{\pm,x} \psi^\varepsilon_\xi(x,y,s)$, observe that the derivative δ^α_j may be replaced either by $\delta^{e,\alpha}_j = \frac{\partial}{\partial x_j} + x_j$ or by $\delta^{o,\alpha}_j = \frac{\partial}{\partial x_j} + x_j + \frac{2\alpha_j+1}{x_j}$, depending on whether $\varepsilon_j = 0$ or $\varepsilon_j = 1$, respectively. Then we prove that

$$\begin{aligned}
\delta^{\alpha,\tau_k}_{\pm,x} \psi^\varepsilon_\xi(x,y,s) &= \delta^\alpha_{i_1} ... \delta^\alpha_{i_k} \psi^\varepsilon_\xi(x,y,s) \\
&= \Big\{ (xy)^\varepsilon \prod_{j=1}^k \big(x_{i_j} - \frac{1}{2\xi}(x_{i_j} + y_{i_j} s_{i_j}) - \frac{\xi}{2}(x_{i_j} - y_{i_j} s_{i_j}) \big) \\
&+ \sum_{j=1}^k \chi_{\{\varepsilon_{i_1} = \varepsilon_{i_2} = ... = \varepsilon_{i_j} = 1\}} \prod_{r=1}^j (2\alpha_{i_r} + 2) y_{i_r} (xy)^{\varepsilon - \sum_{r=1}^j e_{i_r}} \\
&\times \prod_{p=j+1}^k \big(x_{i_p} - \frac{1}{2\xi}(x_{i_p} + y_{i_p} s_{i_p}) - \frac{\xi}{2}(x_{i_p} - y_{i_p} s_{i_p}) \big) \Big\} \\
&\quad \exp\big(-\frac{1}{4\xi} q_+(x,y,s) - \frac{\xi}{4} q_-(x,y,s) \big)
\end{aligned}$$

i) The growth estimate

$$\begin{aligned}
|R^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y)| &\lesssim \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \beta^k_{d,\alpha+\varepsilon}(\xi) (xy)^\varepsilon \\
&\qquad \Big[\prod_{j=1}^k \big(x_{i_j} + \frac{1}{\xi} |x_{i_j} + y_{i_j} s_{i_j}| + \xi |x_{i_j} - y_{i_j} s_{i_j}| \big) \Big]
\end{aligned}$$

$$\times \exp\big(-\frac{1}{4\xi}q_+(x,y,s)-\frac{\xi}{4}q_-(x,y,s)\big)d\xi.$$

$$+\sum_{j=1}^{k}\chi_{\{\varepsilon_{i_1}=\varepsilon_{i_2}=...=\varepsilon_{i_j}=1\}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)$$

$$\int_0^1 \beta^k_{d,\alpha+\varepsilon}(\xi)\prod_{r=1}^{j} y_{i_r}(xy)^{\varepsilon-\sum_{r=1}^{j}e_{i_r}}$$

$$\times\prod_{p=j+1}^{k}\big(x_{i_p}+\frac{1}{\xi}|x_{i_p}+y_{i_p}s_{i_p}|+\xi|x_{i_p}-y_{i_p}s_{i_p}|\big)$$

$$\exp\big(-\frac{1}{4\xi}q_+(x,y,s)-\frac{\xi}{4}q_-(x,y,s)\big)d\xi$$

$$\equiv I_1+\sum_{j=1}^{k}I_2^j$$

we treat the integrals I_1 and I_2^j, $1\le j\le k$ separately. we treat the integrals I_1 and I_2^j, $1\le j\le k$ separately.

Applying Lemma (1), inequalities $(2)-(3)-(4)$, we get

$$\big(x_{i_j}+\frac{1}{\xi}|x_{i_j}+y_{i_j}s_{i_j}|+\xi|x_{i_j}-y_{i_j}s_{i_j}|\big)\exp\big(-\frac{1}{8k\xi}q_+(x,y,s)-\frac{\xi}{8k}q_-(x,y,s)\big)\lesssim\xi^{-\frac{1}{2}}.$$

So

$$I_1\lesssim(xy)^{\varepsilon}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\xi^{-\frac{k}{2}}\exp\big(-\frac{1}{8\xi}q_+(x,y,s)-\frac{\xi}{8}q_-(x,y,s)\big)d\xi.$$

Next, using Lemma (1), inequality (5) and observing that $(xy)^{\varepsilon}\lesssim(x+y)^{2\varepsilon}$ gives

$$I_1\lesssim(x+y)^{2\varepsilon}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon|}.$$

Now Lemma $(2)-(a)$, taken with $\gamma=\varepsilon$ and $\kappa=(0,...,0)$ provides the required growth bound for I_1.

To estimate I_2^j we assume that $\varepsilon_{i_1}=\varepsilon_{i_2}=...=\varepsilon_{i_j}=1$ and use inequality (6) for $\lambda=\alpha+\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}$ and $\mu_j=\frac{1}{2}$, with Lemma (1), inequalities $(2)-(3)-(4)$, we obtain that

$$I_2^j=\chi_{\{\varepsilon_{i_1}=\varepsilon_{i_2}=...=\varepsilon_{i_j}=1\}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\prod_{r=1}^{j}y_{i_r}(xy)^{\varepsilon-\sum_{r=1}^{j}e_{i_r}}$$

$$\times\prod_{p=j+1}^{k}\big(x_{i_p}+\frac{1}{\xi}|x_{i_p}+y_{i_p}s_{i_p}|+\xi|x_{i_p}-y_{i_p}s_{i_p}|\big)$$

$$\exp\big(-\frac{1}{4\xi}q_+(x,y,s)-\frac{\xi}{4}q_-(x,y,s)\big)d\xi$$
$$\lesssim \int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1 \beta^k_{d,\alpha+\varepsilon-\sum_{r=1}^j\frac{e_{i_r}}{2}}(\xi)(x+y)^{2\varepsilon-\sum_{r=1}^j e_{i_r}}\xi^{-\frac{k}{2}}$$
$$\times\exp\big(-\frac{1}{8\xi}q_+(x,y,s)-\frac{\xi}{8}q_-(x,y,s)\big)d\xi.$$

Then Lemma (1), inequality (5) shows that

$$I_2^j\lesssim (x+y)^{2(\varepsilon-\sum_{r=1}^j\frac{e_{i_r}}{2})}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon-\sum_{r=1}^j\frac{e_{i_r}}{2}|}.$$

Combining the above with Lemma $(2)-(a)$, applied with $\gamma=\varepsilon-\sum_{r=1}^j\frac{e_{i_r}}{2}$ and $\kappa=\sum_{r=1}^j\frac{e_{i_r}}{2}$, produces the relevant bound for I_2^j , $1\le j\le k$.

This finishes proving the growth estimate for $R^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y)$.

ii) The smoothness estimate

Passing with $\nabla_{x,y}$ under the integral signs is legitimate, the justification being implicitely contained in the estimates that follow.

To simplify the writing we note

$$\exp(q\pm)=\exp\big(-\frac{1}{4\xi}q_+(x,y,s)-\frac{\xi}{4}q_-(x,y,s)\big)$$

and

$$\varphi_\xi^j(x,y,s)=x_j-\frac{1}{2\xi}(x_j+y_js_j)-\frac{\xi}{2}(x_j-y_js_j).$$

To proceed, we need to compute the relevant derivatives. Denoting $\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}=\delta^\alpha_{i_1}\dots\delta^\alpha_{i_k}\psi^\varepsilon_\xi(x,y,s)$, we get:

For $1\le l\le k$

$$\frac{\partial}{\partial x_{i_l}}\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}$$
$$=\Big\{\chi_{\{\varepsilon_{i_l}=1\}}y_{i_l}(xy)^{\varepsilon-e_{i_l}}\prod_{j=1}^k\varphi_\xi^{i_j}(x,y,s)+(xy)^\varepsilon(1-\frac{1}{2\xi}-\frac{\xi}{2})\prod_{j=1,j\neq l}^k\varphi_\xi^{i_j}(x,y,s)$$
$$+\sum_{j=1}^{l-1}\big\{\chi_{\{\varepsilon_{i_1}=\ldots=\varepsilon_{i_j}=\varepsilon_{i_l}=1\}}\prod_{r=1}^j(2\alpha_{i_r}+2)y_{i_r}y_{i_l}(xy)^{\varepsilon-\sum_{r=1}^j e_{i_r}-e_{i_l}}\prod_{p=j+1}^k\varphi_\xi^{i_p}(x,y,s)$$
$$+\chi_{\{\varepsilon_{i_1}=\ldots=\varepsilon_{i_j}=1\}}\prod_{r=1}^j(2\alpha_{i_r}+2)y_{i_r}(xy)^{\varepsilon-\sum_{r=1}^j e_{i_r}}(1-\frac{1}{2\xi}-\frac{\xi}{2})$$
$$\prod_{p=j+1,p\neq l}^k\varphi_\xi^{i_p}(x,y,s)\big\}\Big\}\exp(q\pm)$$
$$-\big(\frac{1}{2\xi}(x_{i_l}+y_{i_l}s_{i_l})+\frac{\xi}{2}(x_{i_l}-y_{i_l}s_{i_l})\big)\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}.$$

$$\begin{aligned}
&\frac{\partial}{\partial y_{i_l}}\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}\\
&=\Big\{\chi_{\{\varepsilon_{i_l}=1\}}x_{i_l}(xy)^{\varepsilon-e_{i_l}}\prod_{j=1}^{k}\varphi_\xi^{i_j}(x,y,s)-(xy)^{\varepsilon}\big(\frac{s_{i_l}}{2\xi}-\frac{\xi s_{i_l}}{2}\big)\prod_{j=1,j\neq l}^{k}\varphi_\xi^{i_j}(x,y,s)\\
&\quad+\sum_{j=1}^{l-1}\big\{\chi_{\{\varepsilon_{i_1}=\ldots=\varepsilon_{i_j}=\varepsilon_{i_l}=1\}}\prod_{r=1}^{j}(2\alpha_{i_r}+2)y_{i_r}x_{i_l}(xy)^{\varepsilon-\sum_{r=1}^{j}e_{i_r}-e_{i_l}}\prod_{p=j+1}^{k}\varphi_\xi^{i_p}(x,y,s)\\
&\quad+\chi_{\{\varepsilon_{i_1}=\ldots=\varepsilon_{i_j}=1\}}\prod_{r=1}^{j}(2\alpha_{i_r}+2)y_{i_r}(xy)^{\varepsilon-\sum_{r=1}^{j}e_{i_r}}\big(-\frac{s_{i_l}}{2\xi}+\frac{\xi s_{i_l}}{2}\big)\prod_{p=j+1,p\neq l}^{k}\varphi_\xi^{i_p}(x,y,s)\big\}\\
&\quad+\sum_{j=l}^{k}\big\{\chi_{\{\varepsilon_{i_1}=\ldots=\varepsilon_{i_l}=1\}}\prod_{r=1,r\neq l}^{j}(2\alpha_{i_r}+2)y_{i_r}(xy)^{\varepsilon-\sum_{r=1}^{j}e_{i_r}}\prod_{p=j+1}^{k}\varphi_\xi^{i_p}(x,y,s)\big\}\Big\}\exp(q\pm)\\
&\quad-\big(\frac{1}{2\xi}(y_{i_l}+x_{i_l}s_{i_l})+\frac{\xi}{2}(y_{i_l}-x_{i_l}s_{i_l})\big)\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}.
\end{aligned}$$

For $k+1\leq l\leq d$

$$\begin{aligned}
\frac{\partial}{\partial x_{i_l}}\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}&=\Big\{\chi_{\{\varepsilon_{i_l}=1\}}y_{i_l}(xy)^{\varepsilon-e_{i_l}}\prod_{j=1}^{k}\varphi_\xi^{i_j}(x,y,s)\\
&\quad+\sum_{j=1}^{k}\big\{\chi_{\{\varepsilon_{i_1}=\ldots=\varepsilon_{i_j}=\varepsilon_{i_l}=1\}}\\
&\qquad\prod_{r=1}^{j}(2\alpha_{i_r}+2)y_{i_r}y_{i_l}(xy)^{\varepsilon-\sum_{r=1}^{j}e_{i_r}-e_{i_l}}\prod_{p=j+1}^{k}\varphi_\xi^{i_p}(x,y,s)\big\}\Big\}\exp(q\pm)\\
&\quad-\big(\frac{1}{2\xi}(x_{i_l}+y_{i_l}s_{i_l})+\frac{\xi}{2}(x_{i_l}-y_{i_l}s_{i_l})\big)\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y_{i_l}}\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}&=\Big\{\chi_{\{\varepsilon_{i_l}=1\}}x_{i_l}(xy)^{\varepsilon-e_{i_l}}\prod_{j=1}^{k}\varphi_\xi^{i_j}(x,y,s)\\
&\quad+\sum_{j=1}^{k}\big\{\chi_{\{\varepsilon_{i_1}=\ldots=\varepsilon_{i_j}=\varepsilon_{i_l}=1\}}\prod_{r=1}^{j}(2\alpha_{i_r}+2)y_{i_r}x_{i_l}(xy)^{\varepsilon-\sum_{r=1}^{j}e_{i_r}-e_{i_l}}\\
&\qquad\prod_{p=j+1}^{k}\varphi_\xi^{i_p}(x,y,s)\big\}\Big\}\exp(q\pm)\\
&\quad-\big(\frac{1}{2\xi}(y_{i_l}+x_{i_l}s_{i_l})+\frac{\xi}{2}(y_{i_l}-x_{i_l}s_{i_l})\big)\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}.
\end{aligned}$$

We have

$$\begin{aligned}\|\nabla_{x,y}R^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y)\| &= \|\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\nabla_{x,y}(\delta^{\alpha,\tau_k}_{\pm,x}\psi^{\varepsilon}_{\xi}(x,y,s))d\xi\| \\ &\le \int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\|\nabla_{x,y}(\delta^{\alpha,\tau_k}_{\pm,x}\psi^{\varepsilon}_{\xi}(x,y,s))\|d\xi,\end{aligned}$$

we will estimate separately the four case of integrals resulting from replacing $\nabla_{x,y}$ by one of the above derivatives. Denote these integrals by $J_{x_{i_l}}$ and $J_{y_{i_l}}$ for $1\le l\le k$ or $k+1\le l\le d$ and assume that $\varepsilon_{i_1}=...=\varepsilon_{i_l}=1$.

For $1\le l\le k$, applying $(2)-(3)-(4)$ of Lemma (1) we get

$$\begin{aligned}J_{x_{i_l}} &\lesssim (x+y)^{2\varepsilon-e_{i_l}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\xi^{-\frac{k}{2}}\sqrt{\exp(q\pm)}d\xi \\ &+ (x+y)^{2\varepsilon}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\xi^{-\frac{k+1}{2}}\sqrt{\exp(q\pm)}d\xi \\ &+\sum_{j=1}^{l-1}\Big\{(x+y)^{2\varepsilon-\sum_{r=1}^{j}e_{i_r}-e_{i_l}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\xi^{-\frac{k-j}{2}}\sqrt{\exp(q\pm)}d\xi \\ &+ (x+y)^{2\varepsilon-\sum_{r=1}^{j}e_{i_r}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\xi^{-\frac{k-j+1}{2}}\sqrt{\exp(q\pm)}d\xi\Big\} \\ &+\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\xi^{-\frac12}|\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}|\frac{1}{\sqrt{\exp(q\pm)}}d\xi \\ &\equiv J^1_{x_{i_l}}+J^2_{x_{i_l}}+\sum_{j=1}^{l-1}J^{3,j}_{x_{i_l}}+J^4_{x_{i_l}}.\end{aligned}$$

To estimate $J^1_{x_{i_l}}$ we use (6) and inequality (5) of Lemma (1), obtaining

$$\begin{aligned}J^1_{x_{i_l}} &\lesssim (x+y)^{2\varepsilon-e_{i_l}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon-\frac{e_{i_l}}{2}}(\xi)\xi^{-\frac{k+1}{2}}\sqrt{\exp(q\pm)}d\xi \\ &\lesssim (x+y)^{2(\varepsilon-\frac{e_{i_l}}{2})}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon-\frac{e_{i_l}}{2}|-\frac12}.\end{aligned}$$

The expression of $J^2_{x_{i_l}}$ give that

$$J^2_{x_{i_l}}\lesssim (x+y)^{2\varepsilon}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon|-\frac12}.$$

For $1\le j\le l-1$ we give this inequality for $J^{3,j}_{x_{i_l}}$,

$$J^{3,j}_{x_{i_l}}\lesssim (x+y)^{2\varepsilon-\sum_{r=1}^{j}e_{i_r}-e_{i_l}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)$$

$$\int_0^1 \beta^k_{d,\alpha+\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}-\frac{e_{i_l}}{2}}(\xi)\xi^{-\frac{k+1}{2}}\sqrt{\exp(q\pm)}d\xi$$
$$+(x+y)^{2\varepsilon-\sum_{r=1}^{j}e_{i_r}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1 \beta^k_{d,\alpha+\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}}(\xi)\xi^{-\frac{k+1}{2}}\sqrt{\exp(q\pm)}d\xi$$
$$\lesssim (x+y)^{2(\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}-\frac{e_{i_l}}{2})}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}-\frac{e_{i_l}}{2}|-\frac{1}{2}}$$
$$+(x+y)^{2(\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2})}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}|-\frac{1}{2}}.$$

Finally, estimating $J^4_{x_{i_l}}$ is completely analogous to estimating the growth of $|R^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y)|$ performed earlier. The result is

$$J^4_{x_{i_l}}\lesssim (x+y)^{2\varepsilon}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon|-\frac{1}{2}}$$
$$+\sum_{j=1}^{k}(x+y)^{2(\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2})}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}|-\frac{1}{2}},$$

combining the above estimates of $J^1_{x_{i_l}}$, $J^2_{x_{i_l}}$, $J^{3,j}_{x_{i_l}}$ and $J^4_{x_{i_l}}$ with Lemma $(2)-(b)$, specified to either $\gamma=\varepsilon$ and $\kappa=(0,...,0)$ or $\gamma=\varepsilon-\frac{e_{i_l}}{2}$ and $\kappa=\frac{e_{i_l}}{2}$ or $\gamma=\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}$ and $\kappa=\sum_{r=1}^{j}\frac{e_{i_r}}{2}$ or $\gamma=\varepsilon-\sum_{r=1}^{j}\frac{e_{i_r}}{2}-\frac{e_{i_l}}{2}$ and $\kappa=\sum_{r=1}^{j}\frac{e_{i_r}}{2}+\frac{e_{i_l}}{2}$ provides the required smoothness bound for $J_{x_{i_l}}$.

Considering $J_{y_{i_l}}$, items $(2)-(3)-(4)$ of Lemma (1) lead to

$$J_{y_{i_l}}\lesssim J^1_{x_{i_l}}+J^2_{x_{i_l}}+\sum_{j=1}^{l-1}J^{3,j}_{x_{i_l}}+J^4_{x_{i_l}}$$
$$+\sum_{j=l}^{k}(x+y)^{2\varepsilon-\sum_{r=1,r\neq l}^{j}e_{i_r}-2e_{i_l}}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\beta^k_{d,\alpha+\varepsilon}(\xi)\xi^{-\frac{k-j}{2}}\sqrt{\exp(q\pm)}d\xi.$$

Thus it suffices to bound suitably the last term of the sum, witch we denote by $J^5_{y_{i_l}}$, making use of (6) and then applying item (5) of Lemma (1) gives

$$J^5_{y_{i_l}}\lesssim\sum_{j=l}^{k}(x+y)^{2(\varepsilon-\sum_{r=1,r\neq l}^{j}\frac{e_{i_r}}{2}-e_{i_l})}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)$$
$$\int_0^1\beta^k_{d,\alpha+\varepsilon-\sum_{r=1,r\neq l}^{j}\frac{e_{i_r}}{2}-e_{i_l}}(\xi)\xi^{-\frac{k+1}{2}}\sqrt{\exp(q\pm)}d\xi$$
$$\lesssim\sum_{j=l}^{k}(x+y)^{2(\varepsilon-\sum_{r=1,r\neq l}^{j}\frac{e_{i_r}}{2}-e_{i_l})}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q+)^{-d-|\alpha|-|\varepsilon-\sum_{r=1,r\neq l}^{j}\frac{e_{i_r}}{2}-e_{i_l}|-\frac{1}{2}}.$$

Now Lemma $(2)-(b)$ employed with $\gamma = \varepsilon - \sum_{r=1, r\neq l}^{j} \frac{e_{i_r}}{2} - e_{i_l}$ and $\kappa = \sum_{r=1, r\neq l}^{j} \frac{e_{i_r}}{2} + e_{i_l}$ implies the desired bound of $J^5_{y_{i_l}}$.
For $k+1 \leq l \leq d$, passing to $J_{x_{i_l}}$, items $(2)-(3)-(4)$ of Lemma (1) reveal that

$$\begin{aligned} J_{x_{i_l}} &\lesssim (x+y)^{2\varepsilon - e_{i_l}} \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \beta^k_{d,\alpha+\varepsilon}(\xi) \xi^{-\frac{k}{2}} \sqrt{\exp(q\pm)} d\xi \\ &+ \sum_{j=1}^{k} (x+y)^{2\varepsilon - \sum_{r=1}^{j} e_{i_r} - e_{i_l}} \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \beta^k_{d,\alpha+\varepsilon}(\xi) \xi^{-\frac{k-j}{2}} \sqrt{\exp(q\pm)} d\xi \\ &+ \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \beta^k_{d,\alpha+\varepsilon}(\xi) \xi^{-\frac{1}{2}} |\Phi^{\alpha,\varepsilon}_{\xi,\tau_k}| \frac{1}{\sqrt{\exp(q\pm)}} d\xi \\ &\equiv J^1_{x_{i_l}} + \sum_{j=1}^{k} J^{2,j}_{x_{i_l}} + J^3_{x_{i_l}}. \end{aligned}$$

Treatment of $J^1_{x_{i_l}}$ and $J^3_{x_{i_l}}$ is the same as in case of $1 \leq l \leq k$. Thus it remains to deal with $J^{2,j}_{x_{i_l}}$ for $k+1 \leq l \leq d$.

Let $\gamma = \varepsilon - \sum_{r=1}^{j} \frac{e_{i_r}}{2} - \frac{e_{i_l}}{2}$, in view of (6) and Lemma (1), inequality (5), we have

$$\begin{aligned} J^{2,j}_{x_{i_l}} &\lesssim (x+y)^{2\gamma} \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \beta^k_{d,\alpha+\gamma}(\xi) \xi^{-\frac{k+1}{2}} \sqrt{\exp(q\pm)} d\xi \\ &\lesssim (x+y)^{2\gamma} \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) (q+)^{-d-|\alpha|-|\gamma|-\frac{1}{2}}. \end{aligned}$$

This, together with Lemma $(2)-(b)$ taken with $\kappa = \sum_{r=1}^{j} \frac{e_{i_r}}{2} + \frac{e_{i_l}}{2}$ provides the relevant estimates of $J_{x_{i_l}}$ for $k+1 \leq l \leq d$.

Eventually, the analysis related to $J_{y_{i_l}}$ for $k+1 \leq l \leq d$ is analogous to that for $J_{x_{i_l}}$ for $k+1 \leq l \leq d$.

We now come back to explaining the possibility of exchanging $\partial_{x_{i_l}}$ with the integral over $[-1,1]^d \times (0,1)$. To apply Fubini's theorem and then use the argument invoked earlier, it is sufficient to show that

$$\int_\mu^\nu \Big(\int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 |\partial_{x_{i_l}} (\delta^{\alpha,\tau_k}_{\pm,x} \psi^\varepsilon_\xi(x,y,s))| \beta^k_{d,\alpha+\varepsilon}(\xi) d\xi \Big) dx_{i_l} < \infty$$

for any $0 < \mu < \nu < \infty$ such that y_{i_l} is not in $[\mu,\nu]$ when $x_{i_j} = y_{i_j}$, $1 \leq j \leq d$, $j \neq l$. This estimate, however, may be easily obtained with the aid of the bound for $J_{x_{i_l}}$ established above.

This completes the proof of Theorem (1). ∎

3. Higher Order Riesz Transforms for the Dunkl Harmonic Oscillator

In order to study the Dunkl-Riesz transform $\mathcal{R}_{\pm}^{\alpha,\tau_k}$ of order k, $1 \le k \le d$, we begin by observing that :

$$\delta_{\pm}^{\alpha,\tau_k} h_n^\alpha = m^{\pm}(n,\alpha,\tau_k) h^\alpha_{n \mp \sum_{j=1}^k e_{i_j}},$$

where

$$m^+(n,\alpha,\tau_k) = \prod_{j=1}^k m(n_{i_j},\alpha_{i_j}), \quad m^-(n,\alpha,\tau_k) = \prod_{j=1}^k m(n_{i_j}+1,\alpha_{i_j}).$$

It is readily seen that there exists a constant $C_{k,\alpha}$ such that

$$m^{\pm}(n,\alpha,\tau_k) \le C_{k,\alpha}(2|n|+2|\alpha|+2d)^{\frac{k}{2}}, \tag{7}$$

the higher Dunkl-Riesz transform of h_ν^α, is defined by

$$\mathcal{R}_{\pm}^{\alpha,\tau_k} h_\nu^\alpha = \frac{m^{\pm}(\nu,\alpha,\tau_k)}{(2|\nu|+2|\alpha|+2d)^{\frac{k}{2}}} h^\alpha_{\nu \mp \sum_{j=1}^k e_{i_j}}, \tag{8}$$

therefore the higher Dunkl-Riesz transform of $f = \sum_{\nu \in \mathbb{N}^d} \langle f, h_\nu^\alpha \rangle h_\nu^\alpha$, on $L^2(\mathbb{R}^d, \omega_\alpha)$, is given by

$$\mathcal{R}_{\pm}^{\alpha,\tau_k} f = \sum_{\nu \in \mathbb{N}^d} \frac{m^{\pm}(\nu,\alpha,\tau_k)}{(2|\nu|+2|\alpha|+2d)^{\frac{k}{2}}} \langle f, h_\nu^\alpha \rangle_\alpha h^\alpha_{\nu \mp \sum_{j=1}^k e_{i_j}}. \tag{9}$$

We know that the negative power of $\mathcal{L}_\alpha$ is defined by

$$\mathcal{L}_\alpha^{-\frac{k}{2}} f(x) = \frac{1}{\Gamma(\frac{k}{2})} \int_0^{+\infty} e^{-t\mathcal{L}_\alpha} f(x) t^{\frac{k}{2}-1} dt,$$

so

$$\mathcal{L}_\alpha^{-\frac{k}{2}} f(x) = \int_{\mathbb{R}^d} \Big(\frac{1}{\Gamma(\frac{k}{2})} \int_0^{+\infty} G_t^\alpha(x,y) t^{\frac{k}{2}-1} dt \Big) f(y) d\omega_\alpha(y),$$

and

$$\mathcal{R}_{\pm}^{\alpha,\tau_k} f(x) = \delta_{\pm}^{\alpha,\tau_k} \mathcal{L}_\alpha^{-\frac{k}{2}} f(x).$$

Note that the proof of Theorem (1) above contains the proof of

$$\int_0^{+\infty} |\delta_{\pm,x}^{\alpha,\tau_k} G_t^\alpha(x,y)| t^{\frac{k}{2}-1} dt \lesssim \frac{1}{\omega_\alpha(B(x,\|y-x\|))}, \quad x \ne y \tag{10}$$

because

$$G_t^\alpha(x,y) = \sum_{\varepsilon\in\{0,1\}^d} G_t^{\alpha,\varepsilon}(x,y),$$

and for each $\varepsilon \in \{0,1\}^d$, $G_t^{\alpha,\varepsilon}(x,y)$ verify the last inequality.

To proceed to a deeper analysis of these definitions, in particular to consider $\mathcal{R}_{\pm}^{\alpha,\tau_k}$ on a wider class of functions, we define the kernels $\mathcal{R}_{\pm}^{\alpha,\tau_k}(x,y)$ by

$$\mathcal{R}_{\pm}^{\alpha,\tau_k}(x,y) = \frac{1}{\Gamma(\frac{k}{2})}\int_0^{+\infty} \delta_{\pm,x}^{\alpha,\tau_k} G_t^\alpha(x,y) t^{\frac{k}{2}-1} dt. \tag{11}$$

The following result shows that the kernel $\mathcal{R}_{\pm}^{\alpha,\tau_k}(x,y)$ is associated, in the Calderón-Zygmund theory sense, with the operator $\mathcal{R}_{\pm}^{\alpha,\tau_k}$. Its proof is an immediate modification of the corresponding Proposition 4.1 in [6].

Proposition 1. *Let $\alpha \in [-\frac{1}{2},\infty)^d$, $1 \le k \le d$ and $\tau_k = (i_1,...,i_k) \in \{1,...,d\}^k$ such that $i_j \neq i_m$ if $j \neq m$. Then for $f,\ g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with disjoint supports.*

$$\langle \mathcal{R}_{\pm}^{\alpha,\tau_k} f, g\rangle_\alpha = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{R}_{\pm}^{\alpha,\tau_k}(x,y) f(y)\overline{g(x)} d\omega_\alpha(y) d\omega_\alpha(x). \tag{12}$$

The next theorem says that the kernel of the higher Dunkl-Riesz tranform satisfy standard estimates in the sense of the homogeneous space $(\mathbb{R}^d, \omega_\alpha, \|.\|)$. It corresponds to Theorem 4.2 in [6].

Theorem 2. *Let $\alpha \in [-\frac{1}{2},\infty)^d$, $1 \le k \le d$ and $\tau_k = (i_1,...,i_k) \in \{1,...,d\}^k$ such that $i_j \neq i_m$ if $j \neq m$. Then the kernel $\mathcal{R}_{\pm}^{\alpha,\tau_k}(x,y)$, satisfy*

$i)$

$$|\mathcal{R}_{\pm}^{\alpha,\tau_k}(x,y)| \lesssim \frac{1}{\omega_\alpha(B(x,\|y-x\|))}, \quad x \neq y$$

$ii)$

$$\|\nabla_{x,y}\mathcal{R}_{\pm}^{\alpha,\tau_k}(x,y)\| \lesssim \frac{1}{\|x-y\|}\frac{1}{\omega_\alpha(B(x,\|y-x\|))}, \quad x \neq y$$

Proof. We have

$$\mathcal{R}_{\pm}^{\alpha,\tau_k}(x,y) = \sum_{\varepsilon\in\{0,1\}^d} \mathcal{R}_{\pm,\varepsilon}^{\alpha,\tau_k}(x,y).$$

We can see that

$$|\mathcal{R}_{\pm,\varepsilon}^{\alpha,\tau_k}(x,y)| = |\mathcal{R}_{\pm,\varepsilon}^{\alpha,\tau_k}(\eta x,\xi y)|, \quad \eta,\xi \in \{-1,1\}^d.$$

Similarly, we have

$$\|\nabla_{x,y}\mathcal{R}_{\pm,\varepsilon}^{\alpha,\tau_k}(x,y)\| = \|\nabla_{x,y}\mathcal{R}_{\pm,\varepsilon}^{\alpha,\tau_k}(\eta x,\xi y)\|, \quad \eta,\xi \in \{-1,1\}^d.$$

This, together with the symmetry of ω_α,

$$\omega_\alpha(x) = \omega_\alpha(\xi x), \ \xi \in \{-1,1\}^d,$$

proves that it is enough to show the relevant bounds only for $x, \ y \in \mathbb{R}^d_+$.

The proof of the first estimate in Theorem (1) justify the application of Fubini's theorem that was nacessary to get

$$\mathcal{R}^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y) = R^{\alpha,\tau_k}_{\pm,\varepsilon}(x,y), \ \ foreach\, \varepsilon \in \{0,1\}^d.$$

So we deduce the result from Theorem (1).

We denote by $A^\alpha_p = A^\alpha_p(\mathbb{R}^d, \omega_\alpha)$ the Muckenhoupt class of A_p weights related to the space $(\mathbb{R}^d, \omega_\alpha, \|.\|)$. As a consequence of the Theorem 2, Proposition 1 and the general theory (see [3]) we can deduce the main result which corresponds to Theorem 4.3 in [6].

Theorem 3. *Assume that* $\alpha \in [-\frac{1}{2}, \infty)^d, \ 1 \leq k \leq d$ *and* $\tau_k = (i_1, ..., i_k) \in \{1, ..., d\}^k$ *such that* $i_j \neq i_m$ *if* $j \neq m$.

Then the higher Dunkl-Riesz transform $\mathcal{R}^{\alpha,\tau_k}_{\pm}$ *defined on* $L^2(\mathbb{R}^d, \omega_\alpha)$ *by* (9), *are Calderón-Zygmund operators associated with the higher kernels defined by* (11). *In consequence,* $\mathcal{R}^{\alpha,\tau_k}_{\pm}$ *extend uniquely to bounded linear operators on* $L^p(\mathbb{R}^d, \ Wd\omega_\alpha), \ 1 < p < \infty, \ W \in A^\alpha_p$, *and to bounded linear operators from* $L^1(\mathbb{R}^d, \ Wd\omega_\alpha)$ *to* $L^{1,\infty}(\mathbb{R}^d, \ Wd\omega_\alpha), \ W \in A^\alpha_1$.

The analogue of Proposition 4.4 in [6] is given in the following proposition which we omit its proof.

Proposition 2. *Let* $\alpha \in [-\frac{1}{2}, \infty)^d, \ 1 \leq k \leq d, \ 1 < p < \infty$, *and* $i_1, ..., i_k, j_1, ..., j_k \in \{1, ..., d\}$, *with* $i_n \neq i_m$ *and* $j_n \neq j_m$ *if* $n \neq m$.
Then

$$\|(\delta^\alpha_{\pm i_1}.....\delta^\alpha_{\pm i_k})^*(\delta^\alpha_{\pm j_1}.....\delta^\alpha_{\pm j_k})f\|_{L^p(\mathbb{R}^d,\omega_\alpha)} \lesssim \|\mathcal{L}^k_\alpha f\|_{L^p(\mathbb{R}^d,\omega_\alpha)}, \quad f \in \mathcal{C}^\infty_c(\mathbb{R}^d).$$

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