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LOCAL K-CONVOLUTED C-SEMIGROUPS AND ABSTRACT CAUCHY PROBLEMS

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Abstract. Let $K : [0, T_0) \to \mathbb{F}$ be a locally integrable function, and $C : X \to X$ a bounded linear operator on a Banach space X over the field $\mathbb{F}$(=$\mathbb{R}$ or $\mathbb{C}$). In this paper, we will deduce some basic properties of a nondegenerate local K-convoluted C-semigroup on X and some generation theorems of local K-convoluted C-semigroups on X with or without the nondegeneracy, which can be applied to obtain some equivalence relations between the generation of a nondegenerate local K-convoluted C-semigroup on X with subgenerator A and the unique existence of solutions of the abstract Cauchy problem:

$$\mathrm{ACP}(A, f, x) \qquad \begin{cases} u'(t) = Au(t) + f(t) & \text{for a.e. } t \in (0, T_0), \\ u(0) = x \end{cases}$$

when K is a kernel on $[0, T_0)$, $C : X \to X$ an injection, and $A : \mathrm{D}(A) \subset X \to X$ a closed linear operator in X such that $CA \subset AC$. Here $0 < T_0 \leq \infty$, $x \in X$, and $f \in \mathrm{L}^1_{loc}([0, T_0), X)$.

1. INTRODUCTION

Let X be a Banach space over the field $\mathbb{F}$(=$\mathbb{R}$ or $\mathbb{C}$) with norm $\|\cdot\|$, and let $\mathrm{L}(X)$ denote the family of all bounded linear operators from X into itself. For each $0 < T_0 \leq \infty$, we consider the following abstract Cauchy problem:

$$\text{(1.1)} \qquad \mathrm{ACP}(A, f, x) \qquad \begin{cases} u'(t) = Au(t) + f(t) & \text{for a.e. } t \in (0, T_0), \\ u(0) = x, \end{cases}$$

where $x \in X$, $A : \mathrm{D}(A) \subset X \to X$ is a closed linear operator, and $f \in \mathrm{L}^1_{loc}([0, T_0), X)$. A function u is called a (strong) solution of $\mathrm{ACP}(A, f, x)$ if $u \in C([0, T_0), X)$ satisfies $\mathrm{ACP}(A, f, x)$ (that is, $u(0) = x$ and for a.e. $t \in (0, T_0)$, $u(t)$ is differentiable

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and $u(t) \in \mathrm{D}(A)$, and $u'(t)=Au(t)+f(t)$ for a.e. $t \in (0,T_0)$). For each $C \in \mathrm{L}(X)$ and $K \in \mathrm{L}^1_{loc}([0,T_0),\mathbb{F})$, a family $S(\cdot)(=\{S(t)\,|\,0 \le t < T_0\})$ in $\mathrm{L}(X)$ is called a local K-convoluted C-semigroup on X if it is strongly continuous, $S(\cdot)C = CS(\cdot)$, and satisfies

$$(1.2) \qquad S(t)S(s)x = (\int_0^{t+s} - \int_0^t - \int_0^s)K(t+s-r)S(r)Cxdr$$

for all $0 \le t, s, t+s < T_0$ and $x \in X$ (see [8]). In particular, $S(\cdot)$ is called a local (0-times integrated) C-semigroup on X if $K = j_{-1}$(the Dirac measure at 0) or equivalently, $S(\cdot)$ is strongly continuous, $S(\cdot)C = CS(\cdot)$, and satisfies

$$(1.3) \qquad S(t)S(s)x = S(t+s)Cx \quad \text{for all } 0 \le t, s, t+s < T_0 \text{ and } x \in X$$

(see [1, 3-4, 26, 28, 30]). Moreover, we say that $S(\cdot)$ is nondegenerate, if $x = 0$ whenever $S(t)x = 0$ for all $0 \le t < T_0$. The nondegeneracy of a local K-convoluted C-semigroup $S(\cdot)$ on X implies that

$$(1.4) \qquad S(0) = C \text{ if } K = j_{-1}, \text{ and } S(0) = 0 \text{ (the zero operator on X) otherwise,}$$

and the (integral) generator $A : \mathrm{D}(A) \subset X \to X$ of $S(\cdot)$ is a closed linear operator in X defined by

$$\begin{aligned} &\mathrm{D}(A) = \{x \in X \,|\, \text{ there exists a } y_x \in X \text{ such that} \\ &S(\cdot)x - K_0(\cdot)Cx = \widetilde{S}(\cdot)y_x \text{ on } [0,T_0)\} \end{aligned}$$

and $Ax = y_x$ for all $x \in \mathrm{D}(A)$. Here $K_\beta(t) = K * j_\beta(t) = \int_0^t K(t-s)j_\beta(s)ds$ for $\beta > -1$ with $j_\beta(t) = \frac{t^\beta}{\Gamma(\beta+1)}$ and the Gamma function $\Gamma(\cdot)$, and $\widetilde{S}(t)z = \int_0^t S(s)zds$. In general, a local K-convoluted C-semigroup on X is called a K-convoluted C-semigroup on X if $T_0 = \infty$ (see [8, 17]); a (local) K-convoluted C-semigroup on X is called a (local) K-convoluted semigroup on X if $C = I$(the identity operator on X) or a (local) α-times integrated C-semigroup on X if $K = j_{\alpha-1}$ for some $\alpha \ge 0$ (see [2, 5, 12-16, 21-25, 29, 31]). Some basic properites of a nondegenerate (local) α-times integrated C-semigroup on X have been established by many authors (see results in [2, 3, 26-28] for the case $\alpha = 0$, in [19] for the case $\alpha \in \mathbb{N}$, in [14] for the case $\alpha > 0$ is arbitrary with $T_0 = \infty$ and in [18] for the general case $0 < T_0 \le \infty$), which can be applied to deduce some equivalence relations between the generation of a nondegenerate (local) α-times integrated C-semigroup on X with subgenerator A (see Definition 2.4 below) and the unique existence of strong or weak solutions of the abstract Cauchy problem $\mathrm{ACP}(A, f, x)$ (see the results in [2-3, 26-27] for the case $\alpha = 0$, in [19] when $\alpha \in \mathbb{N}$ and in [11, 14-15, 18, 29] when $\alpha > 0$ is arbitrary). The purpose of this paper is to investigate the following basic properties of a nondegenerate local K-convoluted C-semigroup $S(\cdot)$ on X just as results in [18] concerning local α-times integrated C-semigroups on X when C is injective and some additional conditions are taken into consideration.

$$C^{-1}AC = A; \tag{1.5}$$

$$\widetilde{S}(t)x \in D(A) \text{ and } A\widetilde{S}(t)x = S(t)x - K_0(t)Cx \quad \text{for all } x \in X \quad \text{and } 0 \leq t < T_0; \tag{1.6}$$

$$S(t)x \in D(A) \text{ and } AS(t)x = S(t)Ax \text{ for all } x \in D(A) \text{ and } 0 \leq t < T_0; \tag{1.7}$$

and

$$S(t)S(s) = S(s)S(t) \quad \text{for all } 0 \leq t, s, t+s < T_0 \tag{1.8}$$

(see Theorems 2.7 and 2.11, and Corollary 2.12 below). We then deduce some equivalence relations between the generation of a nondegenerate local K-convoluted C-semigroup on X with subgenerator A and the unique existence of strong solutions of $\mathrm{ACP}(A, f, x)$ in section 3 just as some results in [14, 15] concerning some equivalence relations between the generation of a nondegenerate local α-times C-semigroup on X with subgenerator A and the unique existence of strong solutions of $\mathrm{ACP}(A, f, x)$. To do these, we will prove an important lemma which shows that a strongly continuous family $S(\cdot)$ in $\mathrm{L}(X)$ is a local K-convoluted C-semigroup on X is equivalent to say that $\widetilde{S}(\cdot)$ is a local K_0-convoluted C-semigroup on X (see Lemma 2.1 below), and then show that a strongly continuous family $S(\cdot)$ in $\mathrm{L}(X)$ which commutes with C on X is a local K-convoluted C-semigroup on X is equivalent to say that $\widetilde{S}(t)[S(s) - K_0(s)C]$=$[S(t) - K_0(t)C]\widetilde{S}(s)$ for all $0 \leq t, s, t+s < T_0$ (see Theorem 2.2 below). In order, we show that $a * S(\cdot)$ is a local $a * K$-convoluted C-semigroup on X if $S(\cdot)$ is a local K-convoluted C-semigroup on X and $a \in \mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$. In particular, $j_\beta * S(\cdot)$ is a local K_β-convoluted C-semigroup on X if $S(\cdot)$ is a local K-convoluted C-semigroup on X and $\beta > -1$ (see Proposition 2.3 below). Here $f * S(t)x = \int_0^t f(t-s)S(s)xds$ for all $x \in X$ and $f \in \mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$. We also show that a strongly continuous family in $\mathrm{L}(X)$ which commutes with C on X is a local K-convoluted C-semigroup on X when $S(\cdot)$ has a subgenerator (see Theorem 2.5 below), which had been proven in [8] by another method similar to that already employed in [14] in the case that $S(\cdot)$ has a closed subgenerator and C is injective; and the generator of a nondegenerate local K-convoluted C-semigroup $S(\cdot)$ on X is the unique subgenerator of $S(\cdot)$ which contains all subgenerators of $S(\cdot)$ and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$ when $S(\cdot)$ has a subgenerator (see Theorems 2.7 and 2.11, and Corollary 2.12 below). This can be applied to show that $CA \subset AC$ and $S(\cdot)$ is a nondegenerate local K-convoluted C-semigroup on X with generator $C^{-1}AC$ when C is injective, K_0 a kernel on $[0, T_0)$ (that is, $f = 0$ on $[0, T_0)$ whenever $f \in \mathrm{C}([0, T_0), \mathbb{F})$ with $\int_0^t K_0(t-s)f(s)ds = 0$ for all $0 \leq t < T_0$) and $S(\cdot)$ a strongly continuous family in $\mathrm{L}(X)$ with closed subgenerator A. In this case, $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator A_0 of $S(\cdot)$ (see Theorem 2.13 below). Some illustrative examples concerning these theorems are also presented in the final part of paper.

2. Basic Properties of Local K-convoluted C-semigroups

We will deduce an important lemma which can be applied to obtain an equivalence relation between the generation of a local K-convoluted C-semigroup $S(\cdot)$ on X and the equation

$$\widetilde{S}(t)[S(s) - K_0(s)C] = [S(t) - K_0(t)C]\widetilde{S}(s) \text{ for all } 0 \leq t, s, t+s < T_0 \tag{2.1}$$

(see a result in [18] for the case of local α-times integrated C-semigroup and a corresponding statement in [9] for the case of (a, k)-regularized (C_1, C_2)-existence and uniqueness family).

Lemma 2.1. *Let $S(\cdot)$ be a strongly continuous family in $\mathrm{L}(X)$. Then $S(\cdot)$ is a local K-convoluted C-semigroup on X if and only if $\widetilde{S}(\cdot)$ is a local K_0-convoluted C-semigroup on X.*

Proof. We will show that

$$\begin{aligned} &\frac{d}{dt}[(\int_0^{t+s} - \int_0^t - \int_0^s)K_0(t+s-r)\widetilde{S}(r)Cxdr] + K_0(s)\widetilde{S}(t)Cx \\ &= (\int_0^{t+s} - \int_0^t - \int_0^s)K(t+s-r)\widetilde{S}(r)Cxdr \end{aligned} \tag{2.2}$$

for all $x \in \mathrm{X}$ and $0 \leq t, s, t+s < T_0$. Indeed, for $0 \leq t, s, t+s < T_0$, we have

$$\begin{aligned} &\frac{d}{dt}[(\int_0^{t+s} - \int_0^t - \int_0^s)K_0(t+s-r)\widetilde{S}(r)Cxdr \\ &= [(\int_0^{t+s} - \int_0^t - \int_0^s)K(t+s-r)\widetilde{S}(r)Cxdr - K_0(s)\tilde{S}(t)Cx]. \end{aligned}$$

That is, (2.2) holds for all $0 \leq t, s, t+s < T_0$. Clearly, the right-hand side of (2.2) is symmetric in t, s with $0 \leq t, s, t+s < T_0$. It follows that

$$\begin{aligned} &\frac{d}{ds}[(\int_0^{t+s} - \int_0^t - \int_0^s)K_0(t+s-r)\widetilde{S}(r)Cxdr] + K_0(t)\widetilde{S}(s)Cx \\ &= (\int_0^{t+s} - \int_0^t - \int_0^s)K(t+s-r)\widetilde{S}(r)Cxdr \end{aligned} \tag{2.3}$$

for all $x \in$ X and $0 \leq t, s, t+s < T_0$. Using integration by parts, we obtain

$$
\begin{aligned}
&(\int_0^{t+s} - \int_0^t - \int_0^s) K(t+s-r)\widetilde{S}(r)Cxdr \\
(2.4) \qquad &= (\int_0^{t+s} - \int_0^t - \int_0^s) K_0(t+s-r)S(r)Cxdr \\
&+K_0(s)\widetilde{S}(t)Cx + K_0(t)\widetilde{S}(s)Cx
\end{aligned}
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Suppose that $\widetilde{S}(\cdot)$ is a local K_0-convoluted C-semigroup on X. Then we have by $(2.3) - (2.4)$ that

$$
\begin{aligned}
&\widetilde{S}(t)S(s)x = \frac{d}{ds}\widetilde{S}(t)\widetilde{S}(s)x \\
&= (\int_0^{t+s} - \int_0^t - \int_0^s) K_0(t+s-r)S(r)Cxdr + K_0(s)\widetilde{S}(t)Cx \\
&\quad + K_0(t)\widetilde{S}(s)Cx - K_0(t)\widetilde{S}(s)Cx \\
&= (\int_0^{t+s} - \int_0^t - \int_0^s) K_0(t+s-r)S(r)Cxdr + K_0(s)\widetilde{S}(t)Cx
\end{aligned}
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$, so that

$$
(2.5) \qquad S(t)S(s)x = \frac{d}{dt}\widetilde{S}(t)S(s)x = (\int_0^{t+s} - \int_0^t - \int_0^s) K(t+s-r)S(r)Cxdr
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Hence, $S(\cdot)$ is a local K-convoluted C-semigroup on X. Conversely, suppose that $S(\cdot)$ is a local K-convoluted C-semigroup on X. We will apply Fubini's theorem for double integrals to obtain

$$
(2.6) \qquad S(t)\widetilde{S}(s)x = (\int_0^{t+s} - \int_0^t - \int_0^s) K_0(t+s-r)S(r)Cxdr + K_0(t)\widetilde{S}(s)Cx
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Let $x \in X$ be given, then for $0 \leq t, \tau, t+\tau < T_0$, we have

$$
\begin{aligned}
&\int_0^{\tau} \int_t^{t+\lambda} K(t+\lambda-r)S(r)Cxdrd\lambda \\
(2.7) \qquad &= \int_t^{t+\tau} \int_{r-t}^{\tau} K(t+\lambda-r)S(r)Cxd\lambda dr \\
&= \int_t^{t+\tau} K_0(t+\tau-r)S(r)Cxdr,
\end{aligned}
$$

and

$$\begin{aligned}&\int_0^\tau\int_0^\lambda K(t-\lambda+r)S(r)Cxdrd\lambda\\ &=\int_0^\tau\int_r^\tau K(t-\lambda+r)S(r)Cxd\lambda dr\\ &=\int_0^\tau K_0(t-\tau+r)S(r)Cxdr-K_0(t)\widetilde{S}(\tau)Cx.\end{aligned}\tag{2.8}$$

Combining (1.2) with (2.7) and (2.8), we get

$$S(t)\widetilde{S}(\tau)x=(\int_0^{t+\tau}-\int_0^t-\int_0^\tau)K_0(t+\tau-r)S(r)Cxdr+K_0(t)\widetilde{S}(\tau)Cx.$$

That is, (2.6) holds for all $x\in X$ and $0\leq t,s,t+s<T_0$. Combining (2.2) with (2.4) and (2.6), we have

$$\begin{aligned}&S(t)\widetilde{S}(s)x\\ &=(\int_0^{t+s}-\int_0^t-\int_0^s)K(t+s-r)\widetilde{S}(r)Cxdr-K_0(s)\widetilde{S}(t)Cx\\ &=\frac{d}{dt}[(\int_0^{t+s}-\int_0^t-\int_0^s)K_0(t+s-r)\widetilde{S}(r)Cxdr]\end{aligned}$$

for all $x\in X$ and $0\leq t,s,t+s<T_0$. Combining this and (2.2) with $t=0$, we conclude that $\widetilde{S}(\cdot)$ is a local K_0-convoluted C-semigroup on X.

Theorem 2.2. *Let $S(\cdot)$ be a strongly continuous family in $L(X)$ which commutes with C on X. Then $S(\cdot)$ is a local K-convoluted C-semigroup on X if and only if (2.1) holds for all $0\leq t,s,t+s<T_0$.*

Proof. Suppose that $S(\cdot)$ is a local K-convoluted C-semigroup on X. By Lemma 2.1, (2.2) and (2.3), we have $S(t)\widetilde{S}(s)x+K_0(s)\widetilde{S}(t)Cx=\widetilde{S}(t)S(s)x+K_0(t)\widetilde{S}(s)Cx$ for all $x\in X$ and $0\leq t,s,t+s<T_0$ or equivalently, $\widetilde{S}(t)[S(s)-K_0(s)C]$=$[S(t)-K_0(t)C]\widetilde{S}(s)$ for all $0\leq t,s,t+s<T_0$. Conversely, suppose that (2.1) holds for all $0\leq t,s,t+s<T_0$. Then $\widetilde{S}(t)S(s)x-S(t)\widetilde{S}(s)x=K_0(s)\widetilde{S}(t)Cx-K_0(t)\widetilde{S}(s)Cx$ for all $x\in X$ and $0\leq t,s,t+s<T_0$. Fix $x\in X$ and $0\leq t,s,t+s<T_0$, we have

$$\begin{aligned}&\widetilde{S}(t+s-r)S(r)x-S(t+s-r)\widetilde{S}(r)x\\ &=K_0(r)\widetilde{S}(t+s-r)Cx-K_0(t+s-r)\widetilde{S}(r)Cx\end{aligned}\tag{2.9}$$

for all $0\leq r\leq t$. Using integration by parts to the left-hand side of the integration of (2.9) and change of variables to the right-hand side of the integration of (2.9), we obtain

$$\widetilde{S}(t)\widetilde{S}(s)x = \int_0^t [\widetilde{S}(t+s-r)S(r)x - S(t+s-r)\widetilde{S}(r)x]dr$$

$$= \int_0^t [K_0(r)\widetilde{S}(t+s-r)Cx - K_0(t+s-r)\widetilde{S}(r)Cx]dr$$

$$= (\int_0^{t+s} - \int_0^t - \int_0^s) K_0(t+s-r)\widetilde{S}(r)Cxdr$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Consequently, $\widetilde{S}(\cdot)$ is a local K_0-convoluted C-semigroup on X. Combining this with Lemma 2.1, we get that $S(\cdot)$ is a local K-convoluted C-semigroup on X.

By slightly modifying the proof of [18, Corollary 2.4], the next result concerning local K-convoluted C-semigroups on X is also attained.

Proposition 2.3. *Let $S(\cdot)$ be a local K-convoluted C-semigroup on X and $a \in L^1_{loc}([0,T_0),\mathbb{F})$. Then $a * S(\cdot)$ is a local $a * K$-convoluted C-semigroup on X. In particular, for each $\beta > -1$ $j_\beta * S(\cdot)$ is a local K_β-convoluted C-semigroup on X.*

Definition 2.4. Let $S(\cdot)$ be a strongly continuous family in L(X). A linear operator A in X is called a subgenerator of $S(\cdot)$ if

$$S(t)x - K_0(t)Cx = \int_0^t S(r)Axdr \tag{2.10}$$

for all $x \in D(A)$ and $0 \leq t < T_0$, and

$$\int_0^t S(r)xdr \in D(A) \quad \text{and } A\int_0^t S(r)xdr = S(t)x - K_0(t)Cx \tag{2.11}$$

for all $x \in X$ and $0 \leq t < T_0$. A subgenerator A of $S(\cdot)$ is called the maximal subgenerator of $S(\cdot)$ if it is an extension of each subgenerator of $S(\cdot)$ to $D(A)$.

Applying Theorem 2.2, we can obtain the next result concerning the generation of a local K-convoluted C-semigroup $S(\cdot)$ on X, which had been proven in [8] by another method similar to that already employed in [14] in the case that $S(\cdot)$ has a closed subgenerator and C is injective.

Theorem 2.5. *Let $S(\cdot)$ be a strongly continuous family in $L(X)$ which commutes with C on X. Assume that $S(\cdot)$ has a subgenerator. Then $S(\cdot)$ is a local K-convoluted C-semigroup on X. Moreover, $S(\cdot)$ is nondegenerate if the injectivity of C is added and K_0 is a non-zero function on $[0,T_0)$.*

Proof. Let A be a subgenerator of $S(\cdot)$. By (2.11), we have

$$[S(t) - K_0(t)C]\widetilde{S}(\cdot)x = \widetilde{S}(t)A\widetilde{S}(\cdot)x = \widetilde{S}(t)[S(\cdot) - K_0(\cdot)C]x$$

on $[0, T_0 - t)$ for all $x \in X$ and $0 \le t < T_0$. Applying Theorem 2.2, we get that $S(\cdot)$ is a local K-convoluted C-semigroup on X. Suppose that C is injective, K_0 is a nonzero function, $x \in X$ and $S(t)x = 0$, $t \in [0, T_0)$. By (2.11), we have $K_0(\cdot)Cx = 0$ on $[0, T_0)$, and so $Cx = 0$. Hence, $x = 0$, which implies that $S(\cdot)$ is nondegenerate.

Lemma 2.6. *Let A be a closed subgenerator of a strongly continuous family $S(\cdot)$ in $L(X)$, and K_0 a kernel on $[0, t_0)$ (or equivalently, K is a kernel on $[0, t_0)$). Assume that C is injective and $u \in C([0, t_0), X)$ satisfies $u(\cdot) = Aj_0 * u(\cdot)$ on $[0, t_0)$ for some $0 < t_0 < T_0$. Then $u = 0$ on $[0, t_0)$.*

Proof. We know from (2.10)-(2.11) that $A\int_0^t S(r)xdr = \int_0^t S(r)Axdr$ for all $x \in D(A)$ and $0 \le t < T_0$. Combining this with the closedness of A, we have $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $0 \le t < T_0$, and so $\int_0^t S(t-s)u(s)ds = \int_0^t S(t-s)Aj_0 * u(s)ds = \int_0^t AS(t-s)j_0 * u(s)ds = A\int_0^t S(t-s)j_0 * u(s)ds = A\widetilde{S} * u(t) = \int_0^t S(t-s)u(s)ds - C\int_0^t K_0(t-s)u(s)ds$ for all $0 \le t < t_0$. Hence, $\int_0^t K_0(t-s)u(s)ds = 0$ for all $0 \le t < t_0$, which implies that $u(t) = 0$ for all $0 \le t < t_0$.

Theorem 2.7. Let $S(\cdot)$ be a nondegenerate local K-convoluted C-semigroup on X with generator A. Assume that $S(\cdot)$ has a subgenerator. Then A is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover, if C is injective. Then (1.5)-(1.7) hold, and (1.8) also holds when K_0 is a kernel on $[0, T_0)$ or $T_0 = \infty$.

Proof. Let B be a subgenerator of $S(\cdot)$. Clearly, $B \subset A$. It follows that $C(t)z - K_0(t)Cz = B\int_0^t\int_0^s C(r)zdrds = A\int_0^t\int_0^s C(r)zdrds$ for all $z \in X$ and $0 \le t < T_0$, which together with the definition of A implies that A is also a subgenerator of $S(\cdot)$. To show that each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. We will show that B is closable. Let $x_k \in D(B)$, $x_k \to 0$, and $Bx_k \to y$ in X. Then $x_k \in D(A)$ and $Ax_k = Bx_k \to y$. By the closedness of A, we have $y = 0$. In order to show that $\overline{B}$ is a subgenerator of $S(\cdot)$. Let $x \in D(\overline{B})$ be given, then $x_k \to x$ and $Bx_k \to \overline{B}x$ in X for sequence $\{x_k\}_{k=1}^\infty$ in $D(B)$. By (2.10), we have $S(t)x_k - K_0(t)Cx_k = \int_0^t S(r)Bx_kdr$ for all $k \in \mathbb{N}$ and $0 \le t < T_0$. Letting $k \to \infty$, we get that $S(t)x - K_0(t)Cx = \int_0^t S(r)\overline{B}xdr$ for all $0 \le t < T_0$.

Since $B \subset \overline{B}$, we also have $S(t)z - K_0(t)Cz = B\int_0^t S(r)zdr = \overline{B}\int_0^t S(r)zdr$ for all $z \in X$ and $0 \le t < T_0$. Consequently, the closure of B is a subgenerator of $S(\cdot)$. To show that A is the maximal subgenerator of $S(\cdot)$. Let $\mathcal{F}$ be the family of all subgenerators of $S(\cdot)$. We define a partial order "$\subset$" on $\mathcal{F}$ by $f \subset g$ if g is an extension of f to $D(g)$. By Zorn's lemma, ($\mathcal{F}$,$\subset$) has a maximal element B which is a subgenerator of $S(\cdot)$, and does not have a proper extension that is still a subgenerator of $S(\cdot)$. In particular, $B \subset A$. Similarly, we can show that B is the maximal subgenerator of $S(\cdot)$, which implies that $A \subset B$. Clearly,(1.6) and (1.7) both hold because A is the maximal subgenerator of $S(\cdot)$. To show that (1.5) holds when C is injective. We will show that $A \subset C^{-1}AC$ or equivalently, $CA \subset AC$. Let $x \in D(A)$ be given, then $K_1(t)Cx = \widetilde{S}(t)x - j_0 * \widetilde{S}(t)Ax \in D(A)$ and

$$\begin{aligned} AK_1(t)Cx =& A\widetilde{S}(t)x - Aj_0 * \widetilde{S}(t)Ax \\ =& A\widetilde{S}(t)x - [\widetilde{S}(t)Ax - K_1(t)CAx] \\ =& K_1(t)CAx \end{aligned}$$

for all $0 \le t < T_0$, so that $CAx = ACx$. Hence, $CA \subset AC$. In order to show that $C^{-1}AC \subset A$. Let $x \in D(C^{-1}AC)$ be given, then $Cx \in D(A)$ and $ACx \in R(C)$. By the definition of generator and the commutativity of C with $S(\cdot)$, we have $C[S(t)x - K_0(t)Cx] = S(t)Cx - K_0(t)C^2x = \int_0^t S(r)ACxdr = \int_0^t S(r)CC^{-1}ACxdr = C\int_0^t S(r)C^{-1}ACxdr$. Since C is injective, we have $x \in D(A)$ and $Ax = C^{-1}ACx$. Consequently, $A \subset C^{-1}AC$. Finally, we will show that (1.8) holds when K_0 is a kernel on $[0, T_0)$. Clearly, it suffices to show that $\widetilde{S}(t)\widetilde{S}(s)x$=$\widetilde{S}(s)\widetilde{S}(t)x$ for all $x \in X$ and $0 \le t, s < T_0$. Let $x \in X$ and $0 \le s < T_0$ be given. By (1.7) and the closedness of A, we have

$$\begin{aligned} & \widetilde{S}(\cdot)\widetilde{S}(s)x - Aj_0 * \widetilde{S}(\cdot)\widetilde{S}(s)x \\ =& K_1(\cdot)C\widetilde{S}(s)x \\ =& \widetilde{S}(s)K_1(\cdot)Cx \\ =& \widetilde{S}(s)[\widetilde{S}(\cdot)x - Aj_0 * \widetilde{S}(\cdot)x] \\ =& \widetilde{S}(s)\widetilde{S}(\cdot)x - \widetilde{S}(s)Aj_0 * \widetilde{S}(\cdot)x \\ =& \widetilde{S}(s)\widetilde{S}(\cdot)x - Aj_0 * \widetilde{S}(s)\widetilde{S}(\cdot)x \end{aligned}$$

on $[0, T_0)$, and so $[\widetilde{S}(\cdot)\widetilde{S}(s)x - \widetilde{S}(s)\widetilde{S}(\cdot)x]$ =$Aj_0 * [\widetilde{S}(\cdot)\widetilde{S}(s)x - \widetilde{S}(s)\widetilde{S}(\cdot)x]$ on $[0, T_0)$. Hence, $\widetilde{S}(\cdot)\widetilde{S}(s)x$=$\widetilde{S}(s)\widetilde{S}(\cdot)x$ on $[0, T_0)$, which implies that $\widetilde{S}(t)\widetilde{S}(s)x$=$\widetilde{S}(s)\widetilde{S}(t)x$ for all $0 \le t, s < T_0$.

Lemma 2.8. *Let $S(\cdot)$ be a local K-convoluted C-semigroup on X, and $0 \in$ suppK_0 (the support of K_0). Assume that $S(\cdot)x = 0$ on $[0, t_0)$ for some $x \in X$ and*

$0 < t_0 < T_0$. *Then* $CS(\cdot)x = 0$ *on* $[0, T_0)$. *In particular,* $S(t)x = 0$ *for all* $0 \le t < T_0$ *if the injectivity of* C *is added.*

Proof. Let $0 \le t < T_0$ be given, then $t + s < T_0$ and $K_0(s)$ is nonzero for some $0 < s < t_0$, so that $\widetilde{S}(s)S(t)x = S(t)\widetilde{S}(s)x = 0$, $S(s)\widetilde{S}(t)x = \widetilde{S}(t)S(s)x = 0$ and $\widetilde{S}(s)K_0(t)Cx = K_0(t)C\widetilde{S}(s)x = 0$. By Theorem 2.2, we have $K_0(s)\widetilde{S}(t)Cx = K_0(s)C\widetilde{S}(t)x = 0$. Hence, $\widetilde{S}(t)Cx = 0$. Since $0 \le t < T_0$ is arbitrary, we have $CS(t)x = S(t)Cx = 0$ for all $0 \le t < T_0$. In particular, $S(t)x = 0$ for all $0 \le t < T_0$ if the injectivity of C is added.

Theorem 2.9. *Let* $S(\cdot)$ *be a local* K*-convoluted* C*-semigroup on* X*, and* $0 \in$ *supp*K_0*. Assume that* C *is injective. Then* $S(\cdot)$ *is nondegenerate if and only if it has a subgenerator.*

Proof. By Theorem 2.5, we need only to show that A is a subgenerator of $S(\cdot)$ when $S(\cdot)$ is a nondegenerate local K-convoluted C-semigroup on X with generator A and $0 \in \text{supp}K_0$. Observe (2.10)-(2.11) and the definition of A, we need only to show that (2.10) holds. Let $0 \le t_0 < T_0$ be fixed. Then for each $x \in X$ and $0 \le s < T_0$, we set $y = \widetilde{S}(t_0)x$. By Theorem 2.2, we have

$$\begin{aligned}
&\widetilde{S}(r)[S(s) - K_0(s)C]y \\
=&[S(r) - K_0(r)C]\widetilde{S}(s)y \\
=&\widetilde{S}(s)[S(r) - K_0(r)C]y \\
=&\widetilde{S}(s)([S(r) - K_0(r)C]\widetilde{S}(t_0)x) \\
=&\widetilde{S}(s)(\widetilde{S}(r)[S(t_0) - K_0(t_0)C]x) \\
=&[\widetilde{S}(s)\widetilde{S}(r)][S(t_0) - K_0(t_0)C]x \\
=&\widetilde{S}(r)\widetilde{S}(s)[S(t_0) - K_0(t_0)C]x
\end{aligned}$$

for all $0 \le r < T_0$ with $r + s, r + t_0 < T_0$ or equivalently, $S(r)[S(s) - K_0(s)C]y = \widetilde{S}(r)\widetilde{S}(s)[S(t_0) - K_0(t_0)C]x$ for all $0 \le r < T_0$ with $r + s, r + t_0 < T_0$. It follows from Lemma 2.8 and the nondegeneracy of $S(\cdot)$ that we have $[S(s) - K_0(s)C]y = \widetilde{S}(s)[S(t_0) - K_0(t_0)C]x$. Since $0 \le s < T_0$ is arbitrary, we have $y \in D(A)$ and $Ay = [S(t_0) - K_0(t_0)C]x$. Since $0 \le t_0 < T_0$ is arbitrary, we conclude that (2.10) holds.

By slightly modifying the proof of Theorem 2.9, we can apply (1.2) to obtain the next result concerning nondegenerate K-convoluted C-semigroups.

Theorem 2.10. *Let* $S(\cdot)$ *be a nondegenerate* K*-convoluted* C*-semigroup on* X*. Then* C *is injective, and* $S(\cdot)$ *has a subgenerator.*

Combining Theorem 2.10 with Theorem 2.7, the next result concerning nondegenerate K-convoluted C-semigroups is also obtained.

Theorem 2.11. *Let $S(\cdot)$ be a nondegenerate K-convoluted C-semigroup on X with generator A. Then A is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover, (1.5)-(1.8) hold.*

Since $0 \in \text{supp}K_0$ implies that K_0 is a kernel on $[0, T_0)$, we can apply Theorems 2.7 and 2.9 to obtain the next corollary.

Corollary 2.12. *Let $S(\cdot)$ be a nondegenerate local K-convoluted C-semigroup on X with generator A, and $0 \in \text{supp}K_0$. Assume that C is injective. Then A is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover, (1.5)-(1.8) hold.*

Theorem 2.13. *Let A be a closed subgenerator of a strongly continuous family $S(\cdot)$ in $L(X)$, and K_0 a kernel on $[0, T_0)$. Assume that C is injective. Then $CA \subset AC$, and $S(\cdot)$ is a nondegenerate local K-convoluted C-semigroup on X with generator $C^{-1}AC$. In particular, $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator A_0 of $S(\cdot)$.*

Proof. To show that $S(\cdot)$ is a nondegenerate local K-convoluted C-semigroup on X. By Theorem 2.5, we need only to show that $CS(\cdot) = S(\cdot)C$ or equivalently, $C\widetilde{S}(\cdot) = \widetilde{S}(\cdot)C$. Just as in the proof of Theorem 2.7, we have $CA \subset AC$ and $[\widetilde{S}(\cdot)Cx - C\widetilde{S}(\cdot)x] = Aj_0 * [\widetilde{S}(\cdot)Cx - C\widetilde{S}(\cdot)x]$ on $[0, T_0)$. By Lemma 2.6, we also have $\widetilde{S}(\cdot)Cx = C\widetilde{S}(\cdot)x$ on $[0, T_0)$. We will prove that $C^{-1}AC$ is the generator of $S(\cdot)$. Let B denote the generator of $S(\cdot)$. By Theorem 2.7, we have $A \subset B$. By (1.5), we also have $C^{-1}AC \subset C^{-1}BC = B$. Conversely, let $x \in D(B)$ be given, then $K_1(t)Cx = \widetilde{S}(t)x - j_0 * \widetilde{S}(t)Bx \in D(A)$ for all $0 \le t < T_0$, so that $Cx \in D(A)$ and

$$
\begin{aligned}
AK_1(\cdot)Cx &= A\widetilde{S}(\cdot)x - Aj_0 * \widetilde{S}(\cdot)Bx \\
&= A\widetilde{S}(\cdot)x - [\widetilde{S}(\cdot)Bx - K_1(\cdot)CBx] \\
&= A\widetilde{S}(\cdot)x - [B\widetilde{S}(\cdot)x - K_1(\cdot)CBx] \\
&= K_1(\cdot)CBx
\end{aligned}
$$

on $[0, T_0)$. Hence, $ACx = CBx \in R(C)$, which implies that $x \in D(C^{-1}AC)$ and $C^{-1}ACx = Bx$. Consequently, $B \subset C^{-1}AC$.

Corollary 2.14. *Let $S(\cdot)$ be a nondegenerate local K-convoluted C-semigroup on X, and $0 \in \text{supp}K_0$. Assume that C is injective. Then $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator A_0 of $S(\cdot)$.*

Remark 2.15. Let $S(\cdot)$ be a local K-convoluted C-semigroup on X. Then

(i) $S(\cdot)$ is nondegenerate if and only if $\widetilde{S}(\cdot)$ is;

(ii) A is the generator of $S(\cdot)$ if and only if it is the generator of $\widetilde{S}(\cdot)$;

(iii) A is a closed subgenerator of $S(\cdot)$ if and only if it is a closed subgenerator of $\widetilde{S}(\cdot)$.

Remark 2.16. A strongly continuous family in $\mathrm{L}(X)$ may not have a subgenerator; a local K-convoluted C-semigroup on X is degenerate when it has a subgenerator but does not have a maximal subgenerator; and a closed linear operator in X generates at most one nondegenerate local K-convoluted C-semigroup on X when C is injective and K_0 a kernel on $[0, T_0)$.

3. Abstract Cauchy Problems

In the following, we always assume that $C \in \mathrm{L}(X)$ is injective, K_0 a kernel on $[0, T_0)$, and A a closed linear operator in X such that $CA \subset AC$. We also note some basic properties concerning the strong solutions of $\mathrm{ACP}(A, f, x)$ just as results in [14] when A is the generator of a nondegenerate (local) α-times integrated C-semigroup on X.

Proposition 3.1. *Let A be a subgenerator of a nondegenerate local K_0-convoluted C-semigroup $S(\cdot)$ on X. Then for each $x \in D(A)$ $S(\cdot)x$ is the unique solution of* $\mathrm{ACP}(A, K_0(\cdot)Cx, 0)$ *in $C([0, T_0), [D(A)])$. Here $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $|x|_A = \|x\| + \|Ax\|$ for $x \in D(A)$.*

Proposition 3.2. *Let A be a subgenerator of a nondegenerate local K-convoluted C-semigroup $S(\cdot)$ on X and $C^1 = \{x \in X \,|\, S(\cdot)x$ is continuously differentiable on $(0, T_0)\}$. Then*

(i) *for each $x \in C^1$ $S(t)x \in D(A)$ for a.e. $t \in (0, T_0)$;*

(ii) *for each $x \in C^1$ $S(\cdot)x$ is the unique solution of* $\mathrm{ACP}(A, K(\cdot)Cx, 0)$*;*

(iii) *for each $x \in D(A)$ $S(\cdot)x$ is the unique solution of* $\mathrm{ACP}(A, K(\cdot)Cx, 0)$ *in $C([0, T_0), [D(A)])$.*

Proposition 3.3. *Let A be the generator of a nondegenerate local K-convoluted C-semigroup $S(\cdot)$ on X and $x \in X$. Assume that $S(t)x \in R(C)$ for all $0 \leq t < T_0$, and $C^{-1}S(\cdot)x \in C([0, T_0), X)$ is differentiable a.e. on $(0, T_0)$. Then $C^{-1}S(t)x \in D(A)$ for a.e. $t \in (0, T_0)$, and $C^{-1}S(\cdot)x$ is the unique solution of* $\mathrm{ACP}(A, K(\cdot)x, 0)$.

Proof. Clearly, $S(\cdot)x = CC^{-1}S(\cdot)x$ is differentiable a.e. on $(0, T_0)$. By Theorem 2.11, we have $C\frac{d}{dt}C^{-1}S(t)x = \frac{d}{dt}S(t)x = AS(t)x + K(t)Cx = ACC^{-1}S(t)x + K(t)Cx$ for a.e. $t \in (0, T_0)$. Hence, for a.e. $t \in (0, T_0)$, $C^{-1}S(t)x \in \mathrm{D}(C^{-1}AC) = \mathrm{D}(A)$ and $\frac{d}{dt}C^{-1}S(t)x = (C^{-1}AC)C^{-1}S(t)x + K(t)x = AC^{-1}S(t)x + K(t)x$, which implies that $C^{-1}S(\cdot)x$ is a solution of $\mathrm{ACP}(A, K(\cdot)x, 0)$.

Applying Theorem 2.13, we can prove an important result concerning the relation between the generation of a nondegenerate local K-convoluted C-semigroup on X with subgenerator A and the unique existence of strong solutions of $\mathrm{ACP}(A, f, x)$, which has been established in [18] when $K = j_{\alpha-1}$, in [15] when $K = j_{\alpha-1}$ with $T_0 = \infty$, and in [26] when $K = j_{-1}$ with $T_0 = \infty$.

Theorem 3.4. *The following statements are equivalent:*

(i) *A is a subgenerator of a nondegenerate local K-convoluted C-semigroup $S(\cdot)$ on X;*

(ii) *for each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$ the problem $\mathrm{ACP}(A, K_0(\cdot)Cx + K_0 * Cg(\cdot), 0)$ has a unique solution in $C^1([0, T_0), X) \cap C([0, T_0), [D(A)])$;*

(iii) *for each $x \in X$ the problem $\mathrm{ACP}(A, K_0(\cdot)Cx, 0)$ has a unique solution in $C^1([0, T_0), X) \cap C([0, T_0), [D(A)])$;*

(iv) for each $x \in X$ the integral equation $v(\cdot) = Aj_0 * v(\cdot) + K_0(\cdot)Cx$ has a unique solution $v(\cdot; x)$ in $\mathrm{C}([0, T_0), X)$.

In this case, $\widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$ is the unique solution of $\mathrm{ACP}(A, K_0(\cdot)Cx + K_0 * Cg(\cdot), 0)$ and $v(\cdot; x) = S(\cdot)x$.

Proof. We will prove that (i) implies (ii). Let $x \in X$ and $g \in \mathrm{L}^1_{loc}([0, T_0), X)$ be given. We set $u(\cdot) = \widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$, then $u \in \mathrm{C}^1([0, T_0), X) \cap \mathrm{C}([0, T_0), [\mathrm{D}(A)])$, $u(0) = 0$, and

$$\begin{aligned} Au(t) &= A\widetilde{S}(t)x + A\int_0^t \widetilde{S}(t-s)g(s)ds \\ &= S(t)x - K_0(t)Cx + \int_0^t [S(t-s) - K_0(t-s)C]g(s)ds \\ &= S(t)x + \int_0^t S(t-s)g(s)ds - [K_0(t)Cx + K_0 * Cg(t)] \\ &= u'(t) - [K_0(t)Cx + K_0 * Cg(t)] \end{aligned}$$

for all $0 \leq t < T_0$. Hence, u is a solution of $\mathrm{ACP}(A, K_0(\cdot)Cx + K_0 * Cg(\cdot), 0)$ in $\mathrm{C}^1([0, T_0), X) \cap \mathrm{C}([0, T_0), [\mathrm{D}(A)])$. The uniqueness of solutions for $\mathrm{ACP}(A, K_0(\cdot)Cx + K_0 * Cg(\cdot), 0)$ follows directly from the uniqueness of solutions for $\mathrm{ACP}(A, 0, 0)$. Clearly, "$(ii) \Rightarrow (iii)$" holds, and (iii) and (iv) both are equivalent. We remain only to show that "$(iv) \Rightarrow (i)$" holds. Let $S(t) : X \to X$ be defined by $S(t)x = v(t; x)$ for all $x \in X$ and $0 \leq t < T_0$. Clearly, $S(\cdot)$ is strongly continuous, and satisfies (2.11). Combining the uniqueness of solutions for the integral equation $v(\cdot) = Aj_0 * v(\cdot) + K_0(\cdot)Cx$ with the assumption $CA \subset AC$, we have $v(\cdot; Cx) = Cv(\cdot; x)$ for each $x \in X$, which implies that $S(t)$ for $0 \leq t < T_0$ are linear, and commute with C. Let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence in $(0, T_0)$ such that $t_k \to T_0$, and $\mathrm{C}([0, T_0), X)$

a Frechet space with the quasi-norm $|\cdot|$ defined by $|v| = \sum_{k=1}^{\infty} \frac{\|v\|_k}{2^k(1+\|v\|_k)}$ for $v \in$ C$([0,T_0),X)$. Here $\|v\|_k = \max_{t\in[0,t_k]} \|v(t)\|$ for all $k \in \mathbb{N}$. To show that $S(\cdot)$ is a family in L(X), we need only to show that the linear map $\eta : X \to$ C$([0,T_0),X)$ defined by $\eta(x) = v(\cdot;x)$ for $x \in X$, is continuous or equivalently, $\eta : X \to$ C$([0,T_0),X)$ is a closed linear operator. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in X such that $x_k \to x$ in X and $\eta(x_k) \to v$ in C$([0,T_0),X)$, then $v(\cdot;x_k) = Aj_0 * v(\cdot;x_k) + K_0(\cdot)Cx_k$ on $[0,T_0)$. Combining the closedness of A with the uniform convergence of $\{\eta(x_k)\}_{k=1}^{\infty}$ on $[0,t_k]$, we have $v(\cdot)$=$Aj_0 * v(\cdot) + K_0(\cdot)Cx$ on $[0,T_0)$. By the uniqueness of solutions for integral equations, we have $v(\cdot)$=$v(\cdot;x)$=$\eta(x)$. Consequently, $\eta : X \to$ C$([0,T_0),X)$ is a closed linear operator. To show that A is a subgenerator of $S(\cdot)$, we remain only to show that $\widetilde{S}(t)A \subset A\widetilde{S}(t)$ for all $0 \le t < T_0$. Let $x \in D(A)$ be given, then $\widetilde{S}(t)x - K_1(t)Cx$=$Aj_0 * \widetilde{S}(t)x$=$j_0 * A\widetilde{S}(t)x$ for all $0 \le t < T_0$, and so $\widetilde{S}(t)Ax - Aj_0 * \widetilde{S}(t)Ax = K_1(t)CAx = AK_1(t)Cx = A\widetilde{S}(t)x - Aj_0 * \widetilde{S}(t)Ax$ for all $0 \le t < T_0$. Hence, $Aj_0 * [\widetilde{S}(\cdot)Ax - A\widetilde{S}(\cdot)x]$=$\widetilde{S}(\cdot)Ax - A\widetilde{S}(\cdot)x$ on $[0,T_0)$. By the uniqueness of solutions for ACP$(A,0,0)$, we have $\widetilde{S}(\cdot)Ax = A\widetilde{S}(\cdot)x$ on $[0,T_0)$. Applying Theorem 2.5, we get that $S(\cdot)$ is a nondegenerate local K-convoluted C-semigroup on X with subgenerator A.

By slightly modifying the proof of [15, Corollary 2.5], we can apply Theorem 3.4 to obtain the next result.

Theorem 3.5. *Assume that R$(C) \subset$ R$(\lambda - A)$ for some $\lambda \in \mathbb{F}$, and* ACP$(A, K(\cdot)x, 0)$ *has a unique solution in C$([0,T_0),[D(A)])$ for each $x \in D(A)$ with $(\lambda - A)x \in$ R(C). Then A is a subgenerator of a nondegenerate local K-convoluted C-semigroup on X.*

Proof. Clearly, it suffices to show that for each $x \in X$ the integral equation

$$v(\cdot) = A\int_0^{\cdot} v(r)dr + K_0(\cdot)Cx \tag{3.1}$$

has a (unique) solution $v(\cdot;x)$ in C$([0,T_0),X)$ for each $x \in X$. Indeed, if $x \in X$ is given, then there exists a $y_x \in$ D(A) such that $(\lambda - A)y_x = Cx$. By hypothesis, ACP$(A, K(\cdot)y_x, 0)$ has a unique solution $u(\cdot;y_x)$ in C$([0,T_0),[$D$(A)])$. In particular, $u'(\cdot;y_x) = Au(\cdot;y_x) + K(\cdot)y_x \in$ L$^1_{loc}([0,T_0),X)$. By the closedness of A and the continuity of $Au(\cdot;y_x)$, we have $\int_0^t u(r;y_x)dr \in$ D(A) and $A\int_0^t u(r;y_x)dr = \int_0^t Au(r;y_x)dr = u(t;y_x) - K_0(t)y_x \in$ D(A) for all $0 \le t < T_0$, so that

$$(3.2)\qquad \begin{aligned}(\lambda - A)u(t;y_x) &= (\lambda - A)[A\int_0^t u(r;y_x)dr + K_0(t)y_x] \\ &= A\int_0^t (\lambda - A)u(r;y_x)dr + K_0(t)Cx\end{aligned}$$

for all $0 \le t < T_0$. Hence, $v(\cdot;x) = (\lambda - A)u(\cdot;y_x)$ is a solution of (3.1) in $\mathrm{C}([0,T_0),X)$.

Since $C^{-1}AC = A$ and $R((\lambda - A)^{-1}C) = C(\mathrm{D}(A))$ if $\rho(A) \neq \emptyset$, we can apply Proposition 3.1 and Theorem 3.5 to obtain the next corollary.

Corollary 3.6. *Assume that the resolvent set of A is nonempty. Then A is the generator of a nondegenerate local K-convoluted C-semigroup on X if and only if for each $x \in D(A)$ $\mathrm{ACP}(A, K(\cdot)Cx, 0)$ has a unique solution in $C([0,T_0),[D(A)])$.*

Just as results in [15] for the case of α-times integrated C-semigroup, we can apply Theorem 3.4 to obtain the next theorem. The wellposedness of abstract fractional Cauchy problems and abstract Cauchy problems associated with various classes of Volterra integro-differential equations in locally convex spaces have been recently considered in [10].

Theorem 3.7. *Assume that A is densely defined. Then the following are equivalent:*

(i) *A is a subgenerator of a nondegenerate local K-convoluted C-semigroup $S(\cdot)$ on X;*

(ii) *for each $x \in D(A)$ $\mathrm{ACP}(A, K(\cdot)Cx, 0)$ has a unique solution $u(\cdot;Cx)$ in $C([0,T_0),[D(A)])$ which depends continuously on x. That is, if $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $(D(A), \|\cdot\|)$, then $\{u(\cdot;Cx_n)\}_{n=1}^\infty$ converges uniformly on compact subsets of $[0,T_0)$.*

Proof. $(i)\Rightarrow(ii)$. It is easy to see from the definition of a subgenerator of $S(\cdot)$ that $S(\cdot)x$ is the unique solution of $\mathrm{ACP}(A, K(\cdot)Cx, 0)$ in $\mathrm{C}([0,T_0),[\mathrm{D}(A)])$ which depends continuously on $x \in \mathrm{D}(A)$. $(ii)\Rightarrow(i)$. In view of Theorem 3.4, we need only to show that for each $x \in X$ (3.1) has a unique solution $v(\cdot;x)$ in $\mathrm{C}([0,T_0),X)$. Let $x \in X$ be given. By the denseness of $\mathrm{D}(A)$, we have $x_m \to x$ in X for some sequence $\{x_m\}_{m=1}^\infty$ in $D(A)$. We set $u(\cdot;Cx_m)$ to denote the unique solution of $\mathrm{ACP}(A, K(\cdot)Cx_m, 0)$ in $\mathrm{C}([0,T_0),[\mathrm{D}(A)])$. Then $u(\cdot;Cx_m) \to u(\cdot)$ uniformly on compact subsets of $[0,T_0)$ for some $u \in \mathrm{C}([0,T_0),X)$, and so $\int_0^\cdot u(r;Cx_m)dr \to \int_0^\cdot u(r)dr$ uniformly on compact subsets of $[0,T_0)$. Since $u'(\cdot;Cx_m) = Au(\cdot;Cx_m) + K(\cdot)Cx_m$ a.e. on $(0,T_0)$, we have

$$(3.3)\qquad A\int_0^\cdot u(r;Cx_m)dr = \int_0^\cdot Au(r;Cx_m)dr = u(\cdot;Cx_m) - K_0(\cdot)Cx_m$$

on $[0, T_0)$ for all $m \in \mathbb{N}$. Clearly, the right-hand side of the last equality of (3.3) converges uniformly to $u(\cdot) - K_0(\cdot)Cx$ on compact subsets of $[0, T_0)$. It follows from the closedness of A that $\int_0^t u(r)dr \in \mathrm{D}(A)$ for all $0 \le t < T_0$ and $A\int_0^{\cdot} u(r)dr = u(\cdot) - K_0(\cdot)Cx$ on $[0, T_0)$, which implies that $u(\cdot)$ is a (unique) solution of (3.1) in $\mathrm{C}([0, T_0), X)$.

We end this paper with several illustrative examples.

Example 1. Let $X = C_b(\mathbb{R})$, and $S(t)$ for $t \ge 0$ be bounded linear operators on X defined by $S(t)f(x) = f(x+t)$ for all $x \in \mathbb{R}$. Then for each $K \in \mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$ and $\beta > -1$, $K_\beta * S(\cdot) = \{K_\beta * S(t) | 0 \le t < T_0\}$ is local a K_β-convoluted semigroup on X which is also nondegenerate with a closed subgenerator $\frac{d}{dx}$ acting with its maximal distributional domain when K_0 is not the zero function on $[0, T_0)$ (or equivalently, K is not the zero in $\mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$), but $K * S(\cdot)$ may not be a local K-convoluted semigroup on X except for $K \in \mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$ so that $K * S(\cdot)$ is a strongly continuous family in $\mathrm{L}(X)$ for which $\frac{d}{dx}$ is a closed subgenerator of $K * S(\cdot)$ when K_0 is not the zero function on $[0, T_0)$. Moreover, (1.5)-(1.8) hold and $\frac{d}{dx}$ is its generator and maximal subgenerator when K_0 is a kernel on $[0, T_0)$. In this case, $\frac{d}{dx} = \overline{A_0}$ for each subgenerator A_0 of $S(\cdot)$.

Example 2. Let k be a fixed nonnegative integer and K_0 a kernel on $[0, \infty)$, and let $S(t)$ for $t \ge 0$ and C be bounded linear operators on c_0 (the family of all convergent sequences in $\mathbb{F}$ with limit 0) defined by $S(t)x = \{x_n(n-k)e^{-n}\int_0^t K(t-s)e^{ns}ds\}_{n=1}^\infty$ and $Cx = \{x_n(n-k)e^{-n}\}_{n=1}^\infty$ for all $x = \{x_n\}_{n=1}^\infty \in c_0$, then $\{S(t) | 0 \le t < 1\}$ is a local K-convoluted C-semigroup on c_0 which is degenerate except for $k = 0$ and generator A defined by $Ax = \{nx_n\}_{n=1}^\infty$ for all $x = \{x_n\}_{n=1}^\infty \in c_0$ with $\{nx_n\}_{n=1}^\infty \in c_0$, and for each $r > 1$ $\{S(t) | 0 \le t < r\}$ is not a local K-convoluted C-semigroup on c_0. Suppose that $k \in \mathbb{N}$. Then $A_a : c_0 \to c_0$ for $a \in \mathbb{F}$ defined by $A_a x = \{ny_n\}_{n=1}^\infty$ for all $x = \{x_n\}_{n=1}^\infty \in c_0$ with $\{nx_n\}_{n=1}^\infty \in c_0$, are subgenerators of $\{S(t) | 0 \le t < 1\}$ which do not have proper extensions that are still subgenerators of $\{S(t) | 0 \le t < 1\}$. Here $y_n = akx_k$ if $n = k$, and $y_n = nx_n$ otherwise. Consequently, $\{S(t) | 0 \le t < 1\}$ does not have a maximal subgenerator when $k \in \mathbb{N}$.

Example 3. Let $X = \mathrm{C}_b(\mathbb{R})$(or $\mathrm{L}^\infty(\mathbb{R})$), and A be the maximal differential operator in X defined by $Au = \sum_{j=0}^k a_j D^j u$ on $\mathbb{R}$ for all $u \in D(A)$, then $\mathrm{UC}_b(\mathbb{R})$ (or $\mathrm{C}_0(\mathbb{R})$) $= \overline{D(A)}$. Here $a_0, a_1, \cdots, a_k \in \mathbb{C}$ and $D^j u(x) = u^{(j)}(x)$ for all $x \in \mathbb{R}$. It is shown in [2,19] that $\{S(t)|0 \le t < T_0\}$ defined by $(S(t)f)(x) = \frac{1}{\sqrt{2\pi}}\int_0^t\int_{-\infty}^{\infty} K(t-s)\widetilde{\phi_s}(x-y)f(y)dyds$ for all $f \in X$ and $0 \le t < T_0$, is a norm continuous local K_0-convoluted semigroup on X with closed subgenerator A if the real-valued polynomial $p(x) = \sum_{j=0}^{k} a_j(ix)^j$ satisfies $\sup_{x\in\mathbb{R}} p(x) < \infty$, and $K \in \mathrm{L}^1_{loc}([0,T_0),\mathbb{F})$ is not the zero function on $[0,T_0)$. Here $\widetilde{\phi_t}$ denotes the inverse Fourier transform of ϕ_t with $\phi_t(x) = \int_0^t e^{p(x)s}ds$ for all $t \ge 0$. Suppose that K_0 is a kernel on $[0,T_0)$. Then A is its generator and maximal subgenerator. Applying Theorem 3.4, we get that for each $f \in X$ and continuous function g on $[0,T_0)\times\mathbb{R}$ with $\int_0^t \sup_{x\in\mathbb{R}} |g(s,x)|ds < \infty$ for all $0 \le t < T_0$, the function u on $[0,T_0)\times\mathbb{R}$ defined by $u(t,x) = \frac{1}{\sqrt{2\pi}}\int_0^t\int_{-\infty}^{\infty} K_0(t-s)\widetilde{\phi_s}(x-y)f(y)dyds + \frac{1}{\sqrt{2\pi}}\int_0^t\int_0^{t-r}\int_{-\infty}^{\infty} K_0(t-r-s)\widetilde{\phi_s}(x-y)g(r,y)dydsdr$ for all $0 \le t < T_0$ and $x \in \mathbb{R}$, is the unique solution of

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \displaystyle\sum_{j=0}^{k} a_j(\frac{\partial}{\partial x})^j u(t,x) + K_1(t)f(x) \\ \qquad\qquad + \displaystyle\int_0^t K_1(t-s)g(s,x)ds \text{ for } t \in (0,T_0) \text{ and a.e. } x \in \mathbb{R}, \\ u(0,x) = 0 \quad \text{for a.e. } x \in \mathbb{R} \end{cases}$$

in $\mathrm{C}^1([0,T_0),\mathrm{X}) \cap \mathrm{C}([0,T_0),[\mathrm{D(A)}])$.

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REFERENCES

1. W. Arendt, C. J. K. Batty, H. Hieber and F. Neubrander, *Vector-valued Laplace Transformes and Cauchy Problem*, Birkhauser Verlag Basel-Boston-Berlin, 2001.
2. R. DeLaubenfuls, Integrated semigroups, C-semigroups and the abstract Cauchy problem, *Semigroup Forum*, **41** (1990), 83-95.
3. M. Gao, Local C-semigroups and local C-cosine functions, *Acta Math. Sci.*, **19** (1999), 201-213.
4. J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford, 1985.

5. M. Hieber, Integrated semigroups and differential operators on L^p spaces, *Math. Ann.*, **291** (1991), 1-16.

6. M. Hieber, Laplace transforms and α-times integrated semigroups, *Forum Math.*, **3** (1991), 595-612.

7. H. Kellerman and M. Hieber, Integrated semigroups, *J. Funct. Anal.*, **84** (1989), 160-180.

8. M. Kostic, *Generalized Semigroups and Cosine Functions*, Mathematical Institute Belgrade, 2011.

9. M. Kostic, (a,k)-regularized (C_1, C_2)-existence and uniqueness families, *Bull. Cl. Sci. Math.*, **38** (2013), 9-26.

10. M. Kostic, *Abstract Volterra Integro-differential Equations*, Taylor and Francis Group, 2014.

11. C.-C. Kuo, On existence and approximation of solutions of abstract Cauchy problem, *Taiwanese J. Math.*, **13** (2009), 137-155.

12. C.-C. Kuo, On perturbation of α-times integrated C-semigroups, *Taiwanese J. Math.*, **14** (2010), 1979-1992.

13. C.-C. Kuo, Perturbation theorems for local integrated semigroups, *Studia Math.*, **197** (2010), 13-26.

14. C.-C. Kuo and S.-Y. Shaw, On α-times integrated C-semigroups and the abstract Cauchy problem, *Studia Math.*, **142** (2000), 201-217.

15. C.-C. Kuo and S.-Y. Shaw, On strong and weak solutions of abstract Cauchy problem, *J. Concrete and Applicable Math.*, **2** (2004), 191-212.

16. M. Li and Q. Zheng, α-times integrated semigroups: local and global, *Studia Math.*, **154** (2003), 243-252.

17. F. Li, T.-J. Xiao, J. Liang and J. Zhang, On perturbation of convoluted C-regularized operator families, *J. Funct. Spaces Appl.*, **2013** (2013), Article ID 579326, 8 pages.

18. Y.-C. Li and S.-Y. Shaw, On local α-times integrated C-semigroups, *Abstract and Applied Anal.*, **2007** (2007), Article ID 34890, 18 pages.

19. Y.-C. Li and S.-Y. Shaw, N-times integrated C-semigroups and the abstract Cauchy problem, *Taiwanese J. Math.*, **1** (1997), 75-102.

20. Y.-C. Li and S.-Y. Shaw, Perturbation of nonexponentially-bounded α-times integrated C-semigroups, *J. Math. Soc. Japan*, **55** (2003), 1115-1136.

21. M. Mijatović and S, Pilipović, α-times integrated semigroups($\alpha \in \mathbb{R}^+$), *J. Math. Anal. Appl.*, **210** (1997), 790-803.

22. I. Miyadera, M. Okubo and N. Tanaka, On integrated semigroups where are not exponentially bounded, *Proc. Japan Acad.*, **69** (1993), 199-204.

23. F. Neubrander, Integrated semigroups and their applications to the abstract Cauchy problem, *Pacific J. Math.*, **135** (1988), 111-155.

24. F. Neubrander, Integrated semigroups and their applications to complete second order Cauchy problems, *Semigroup Forum*, **38** (1989), 233-251.

25. S. Nicasie, The Hille-Yosida and Trotter-Kato theorems for integrated semigroups, *J. Math. Anal. Appl.*, **180** (1993), 303-316.

26. S.-Y. Shaw and C.-C. Kuo, Generation of local C-semigroups and solvability of the abstract Cauchy problems, *Taiwanese J. Math.*, **9** (2005), 291-311.

27. N. Tanaka and I. Miyadera, C-semigroups and the abstract Cauchy problem, *J. Math. Anal. Appl.*, **170** (1992), 196-206.

28. N. Tanaka and N. Okazawa, Local C-semigroups and local integrated semigroups, *Proc. London Math. Soc.*, **61(3)** (1990), 63-90.

29. T.-J. Xiao and J. Liang, *The Cauchy Problem for Higher-order Abstract Differential Equations*, Lectures Notes in Math., 1701, Springer, 1998.

30. T.-J. Xiao and J. Liang, Differential operators and C-wellposedness of complete second order abstract Cauchy problems, *Pacific J. Math.*, **186** (1998), 167-200.

31. T.-J. Xiao and J. Liang, Approximations of Laplace transforms and integrated semigroups, *J. Funct. Anal.*, **172** (2000), 202-220.

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