

LIMITING BEHAVIOR OF THE KOBAYASHI DISTANCE

Monika Budzyńska, Tadeusz Kuczumow and Simeon Reich

Abstract. Given a sequence $\{D_m\}$ of bounded and convex domains in a complex Banach space, we describe the limiting behavior of the corresponding sequence $\{k_{D_m}\}$ of Kobayashi distances when the given sequence $\{D_m\}$ is either monotonic or convergent in the Hausdorff metric.

1. INTRODUCTION

It is well known that the Kobayashi distance (and more generally, invariant functions and pseudodistances), its properties and especially its limiting behavior play an important role in holomorphic function theory. They are also used in the theory of semigroups of holomorphic mappings, which is closely connected with the study of differential equations (see, for example, [1, 3, 5, 7, 9, 11, 12, 13, 16, 17, 19, 20] and [21]).

The main results of the present paper are concerned with the limiting behavior of the Kobayashi distance on bounded and convex domains in a complex Banach space. We show that under appropriate conditions, $k_D = \lim_m k_{D_m}$ in the compact-open topology. Our results extend those previously established in [2, 10, 13] and [14] (see also [8]).

Our paper is organized as follows. In Section 2 we recall basic properties of the Kobayashi distance k_D on a bounded and convex domain D in a complex Banach space. We also recall connections between the Kobayashi distance and holomorphic mappings. In Section 3 we assume that a sequence of bounded and convex domains converges to a bounded and convex domain in the Hausdorff metric and study the convergence of the corresponding sequence of Kobayashi distances (see Theorem 3.1 below). The next section is devoted to the case of monotonic (either increasing or decreasing) sequences of bounded and convex domains (see Theorems 4.1 and 4.2, respectively). Finally, in Section 5 we state and prove several auxiliary results which are used in the proofs of our main theorems. We also observe that the convexity assumption imposed on the domains in question is crucial in our considerations.

Received April 23, 2014, accepted July 13, 2014.

Communicated by Der-Chen Chang.

2010 *Mathematics Subject Classification*: 32A10, 32F45, 46G20, 46T25.

Key words and phrases: Hausdorff metric, Holomorphic mapping, Kobayashi distance, Monotonic sequences of bounded and convex domains.

2. THE CARATHÉODORY DISTANCE, THE LEMPERS FUNCTION, THE KOBAYASHI DISTANCE AND HOLOMORPHIC MAPPINGS

Unless explicitly stated otherwise, throughout this paper all Banach spaces $(X, \|\cdot\|)$ are complex, all domains $D \subset X$ are bounded and convex, and $H(D_1, D_2)$ denotes the set of all holomorphic mappings from D_1 into D_2 , where D_1 and D_2 are bounded and convex domains in the complex Banach spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, respectively.

Let Δ be the open unit disc in the complex plane \mathbb{C} . Recall that the *Poincaré distance* $k_\Delta = \omega$ on Δ is given by

$$k_\Delta(z, w) = \omega(z, w) := \arg \tanh \left| \frac{z - w}{1 - z\bar{w}} \right| = \arg \tanh (1 - \sigma(z, w))^{\frac{1}{2}},$$

where

$$\sigma(z, w) := \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\bar{w}|^2}, \quad z, w \in \Delta;$$

see, for example, [7] and [17].

Now let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. We first recall the definitions of the Carathéodory distance, the Lempert function and the Kobayashi distance on D .

The function c_D , defined on $D \times D$ by the formula

$$c_D(x, y) := \sup\{k_\Delta(f(x), f(y)) : f \in H(D, \Delta)\},$$

is called the *Carathéodory distance* ([13]).

The Lempert function δ_D is defined as follows:

$$\delta_D(x, y) :=$$

$\inf\{k_\Delta(0, \lambda) : \lambda \in [0, 1) \text{ and there exists } f \in H(\Delta, D) \text{ so that } f(0) = x, f(\lambda) = y\},$

where $x, y \in D$ [18] (see also [4] and [13]).

Finally, the *Kobayashi distance* ([15]; see also [16]) between $x, y \in D$ is defined by

$$k_D(x, y) := \inf\left\{\sum_{j=1}^m \delta_D(x_j, x_{j+1}) : m \in \mathbb{N}, \{x = x_1, \dots, x_{m+1} = y\} \subset D\right\}.$$

We proceed with a few more definitions. Let D_1 and D_2 be two bounded and convex domains in two complex Banach spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, respectively. A mapping $f : D_1 \rightarrow D_2$ is said to be nonexpansive with respect to the Kobayashi distance (the Carathéodory distance or the Lempert function, respectively) if

$$k_{D_2}(f(x), f(y)) \leq k_{D_1}(x, y)$$

$$(c_{D_2}(f(x), f(y)) \leq c_{D_1}(x, y), \quad \delta_{D_2}(f(x), f(y)) \leq \delta_{D_1}(x, y))$$

for all $x, y \in D_1$. If $D_1 = D_2 = D$, then we say that f is k_D -nonexpansive (c_D -nonexpansive or δ_D -nonexpansive, respectively). It is not difficult to observe that every $f \in H(D_1, D_2)$, that is, every holomorphic mapping $f : D_1 \rightarrow D_2$ is simultaneously nonexpansive with respect to the Kobayashi distance, the Carathéodory distance and the Lempert function [7].

Directly from the definitions of the Carathéodory distance, the Kobayashi distance and the Lempert function we get

$$c_D \leq k_D \leq \delta_D,$$

but it turns out that much stronger results are true. The first result of this type is due to L. Lempert [18].

Theorem 2.1. ([18]; see also [13]). *If D is a bounded and convex domain in \mathbb{C}^n , then*

$$c_D = k_D = \delta_D.$$

In [4] S. Dineen, R. M. Timoney and J.-P. Vigué proved the following result, which we present here in a version that is weaker than the original one.

Theorem 2.2. ([4]). *For a bounded and convex domain D in a complex Banach space $(X, \|\cdot\|)$, the following equality is valid:*

$$c_D(x, y) = \inf_{Y \in \mathcal{Y}_{x,y}} c_{D \cap Y}(x, y),$$

where $x, y \in D$ and $\mathcal{Y}_{x,y}$ denotes the family of all linear subspaces $Y \subset X$, which are finite-dimensional and contain both x and y .

Observe that for the Lempert function we have, in general,

$$\delta_D(x, y) \leq \inf_{Y \in \mathcal{Y}_{x,y}} \delta_{D \cap Y}(x, y),$$

where $x, y \in D$ and $\mathcal{Y}_{x,y}$ denotes the family of all linear subspaces $Y \subset X$, which are finite-dimensional and contain both x and y .

Therefore, directly from Theorems 2.1 and 2.2, and the inequalities

$$c_D \leq k_D \leq \delta_D,$$

we get the following generalization of Lempert's theorem.

Theorem 2.3. ([4]). *For a bounded and convex domain D in a complex Banach space $(X, \|\cdot\|)$, we have*

$$c_D = k_D = \delta_D.$$

From the above observations we also get the following corollary.

Corollary 2.4. *For a bounded and convex domain D in a complex Banach space $(X, \|\cdot\|)$, we have*

$$k_D(x, y) = \inf_{Y \in \mathcal{Y}_{x,y}} k_{D \cap Y}(x, y),$$

where $x, y \in D$ and $\mathcal{Y}_{x,y}$ denotes the family of all linear subspaces $Y \subset X$, which are finite-dimensional and contain both x and y .

Here and subsequently we consider bounded and convex domains and use the Kobayashi distance.

It is known that the Kobayashi distance k_D is locally equivalent to the norm $\|\cdot\|$ in X [9]. Indeed, if for $\emptyset \neq A \subset D$ we denote by

$$\text{dist}(x, A) := \inf\{\|x - y\| : y \in A\}$$

the distance in $(X, \|\cdot\|)$ between a point x and the set A , and if

$$\text{diam}D := \sup\{\|x - y\| : x, y \in D\}$$

denotes the diameter of D in $(X, \|\cdot\|)$, then the following theorem holds.

Theorem 2.5. ([9]). *If D is a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$, then*

$$\arg \tanh \left(\frac{\|x - y\|}{\text{diam}D} \right) \leq k_D(x, y)$$

for all $x, y \in D$ and

$$k_D(x, y) \leq \arg \tanh \left(\frac{\|x - y\|}{\text{dist}(x, \partial D)} \right)$$

whenever $\|x - y\| < \text{dist}(x, \partial D)$.

We also use the following notation. If C_1 and C_2 are nonempty and bounded subsets of a Banach space $(X, \|\cdot\|)$, then $\text{dist}(C_1, C_2) := \inf\{\|x - y\| : x \in C_1 \text{ and } y \in C_2\}$. The open ball of center x and radius r is denoted by $B(x, r)$.

Finally, we recall the definition of the Hausdorff metric.

Definition 2.1. ([6]). Let $(X, \|\cdot\|)$ be a Banach space and let \mathcal{M} denote the family of all nonempty, bounded and convex domains in X . For $D_1, D_2 \in \mathcal{M}$, set

$$d(D_1, D_2) := \sup\{\text{dist}(y, D_1); y \in D_2\},$$

$$d(D_2, D_1) := \sup\{\text{dist}(x, D_2); x \in D_1\}.$$

With these notations, the function $d_H : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$, defined by

$$d_H(D_1, D_2) := \max\{d(D_1, D_2), d(D_2, D_1)\}$$

for $D_1, D_2 \in \mathcal{M}$, is called the *Hausdorff metric*.

3. CONVERGENCE OF THE KOBAYASHI DISTANCES IN THE CASE WHERE A SEQUENCE OF BOUNDED AND CONVEX DOMAINS IS CONVERGENT IN THE HAUSDORFF METRIC

This section is devoted to the behavior of the corresponding sequence of Kobayashi distances when a given sequence $\{D_m\}$ of bounded and convex domains tends to a bounded and convex domain D in the Hausdorff metric. We first state and prove the main result of this section and then derive two corollaries.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a complex Banach space. Let D be a bounded and convex domain in X , and let $\{D_m\}$ be a sequence of bounded and convex domains in X . If $\lim_{n \rightarrow \infty} d_H(D, D_m) = 0$, then for each $x, y \in D$, there exists $\tilde{m} \in \mathbb{N}$ such that for all $\tilde{m} \leq m \in \mathbb{N}$, we have $x, y \in D_m$ and $k_D(x, y) = \lim_{\tilde{m} \leq m \rightarrow \infty} k_{D_m}(x, y)$, uniformly on compact sets.*

Proof. Without any loss of generality we may assume that $0 \in D$. Take two points $x, y \in D$. By Lemma 5.7, there exist $r > 0$ and $\tilde{m} \in \mathbb{N}$ such that $B(0, r) \subset D \cap D_m$, $B(x, r) \subset D \cap D_m$ and $B(y, r) \subset D \cap D_m$ for each $m \geq \tilde{m}$. Hence by Lemma 5.8, there exist numerical sequences $\{s_m\}_{m \geq \tilde{m}}$ and $\{t_m\}_{m \geq \tilde{m}}$ such that $\lim_{\tilde{m} \leq m \rightarrow \infty} s_m = 1$, $\lim_{\tilde{m} \leq m \rightarrow \infty} t_m = 1$, $0 \leq s_m \leq 1$, $0 \leq t_m \leq 1$, $s_m D \subset D_m$ and $t_m D_m \subset D$ for $m \geq \tilde{m}$. Therefore, applying Theorem 2.5, we obtain

$$\lim_{\tilde{m} \leq m \rightarrow \infty} |k_{D_m}(s_m x, s_m y) - k_{D_m}(x, y)| = 0$$

and

$$\lim_{\tilde{m} \leq m \rightarrow \infty} k_D(t_m x, t_m y) = k_D(x, y).$$

Finally, we also have

$$k_D(x, y) = k_{s_m D}(s_m x, s_m y) \geq k_{D_m}(s_m x, s_m y)$$

for each $m \geq \tilde{m}$, which implies that

$$k_D(x, y) \geq \limsup_{\tilde{m} \leq m \rightarrow \infty} k_{D_m}(x, y).$$

Similarly, the inequalities

$$k_{D_m}(x, y) = k_{t_m D_m}(t_m x, t_m y) \geq k_D(t_m x, t_m y),$$

which are valid for each $m \geq \tilde{m}$, yield

$$\liminf_{\tilde{m} \leq m \rightarrow \infty} k_{D_m}(x, y) \geq k_D(x, y).$$

Thus

$$\lim_{\tilde{m} \leq m \rightarrow \infty} k_{D_m}(x, y) = k_D(x, y),$$

as asserted.

Now let $C \subset D$ be a compact set. By Lemma 5.5, for each $x \in D$, there exist $r_x > 0$ and $M_x \in \mathbb{N}$ such that $B(x, r_x) \subset D$ and $B(x, r_x) \subset D_m$ for each $m \geq M_x$. Since C is compact, the open cover $\{B(x, \frac{1}{2}r_x)\}_{x \in C}$ of C contains a finite subcover $\{B(x_j, \frac{1}{2}r_{x_j})\}_{j=1}^N$ of C . Let $\tilde{r} := \min\{r_{x_1}, \dots, r_{x_N}\}$ and $M := \max\{M_{x_1}, \dots, M_{x_N}\}$. It is clear that for each $x \in C$, we have $\text{dist}(x, \partial D) > \frac{\tilde{r}}{2}$ and $\text{dist}(x, \partial D_m) > \frac{\tilde{r}}{2}$ for all $m \geq M$. Next, take two arbitrary sequences $\{x'_m\}$ and $\{x''_m\}$ in C which converge in $(X, \|\cdot\|)$ to x' and x'' , respectively. It follows from Theorem 2.5 that

$$\begin{aligned} & |k_{D_m}(x'_m, x''_m) - k_D(x', x'')| \\ & \leq |k_{D_m}(x'_m, x')| + |k_{D_m}(x'', x''_m)| + |k_{D_m}(x', x'') - k_D(x', x'')| \\ & \leq \arg \tanh \left(\frac{2\|x'_m - x'\|}{r} \right) + \arg \tanh \left(\frac{2\|x'' - x''_m\|}{r} \right) \\ & \quad + |k_{D_m}(x', x'') - k_D(x', x'')| \rightarrow_m 0 \end{aligned}$$

and this completes the proof. \blacksquare

Combining Theorem 3.1 and Corollary 5.4, we get the following results regarding the behavior of the sequences of Kobayashi distances corresponding to given monotonic sequences of bounded and convex domains in \mathbb{C}^n .

Theorem 3.2. ([13, 16]). *Let D be a bounded and convex domain in \mathbb{C}^n , and let $D = \bigcup_{m=1}^{\infty} D_m$, where $\{D_m\}_{m=1}^{\infty}$ is an increasing sequence of bounded and convex domains. Then the corresponding sequence $\{k_{D_m}\}_{m=1}^{\infty}$ of Kobayashi distances converges, as m tends to infinity, to k_D , uniformly on compact sets.*

Theorem 3.3. ([10, 13, 16, 2]). *Let D be a bounded and convex domain in \mathbb{C}^n , and let $\{D_m\}_{m=1}^{\infty}$ be a sequence of bounded and convex domains in \mathbb{C}^n such that $D_{m+1} \subset D_m$ for all $m \in \mathbb{N}$ and $\bigcap_{m=1}^{\infty} D_m = \overline{D}$. Then the corresponding sequence $\{k_{D_m}\}_{m=1}^{\infty}$ of Kobayashi distances converges, as m tends to infinity, to k_D , uniformly on compact sets.*

4. THE CASE OF MONOTONIC SEQUENCES OF BOUNDED AND CONVEX DOMAINS IN INFINITE DIMENSIONAL BANACH SPACES

We begin this section with two examples which show that in *infinite dimensional* complex Banach spaces we are not able to apply Theorem 3.1 in order to prove results which are analogous to those established in Theorems 3.2 and 3.3.

Example 4.1. Consider the complex Banach space ℓ^2 . Let $D = B(0, 1)$, $m \in \mathbb{N}$ and

$$D_m := \left\{ x = \{x^j\}_j \in \ell^2 : \|x\| < \frac{3}{2}, \sum_{j=1}^m |x^j|^2 < 1 \text{ and } |x^j| < \frac{3}{2} \text{ for } j \geq m+1 \right\}.$$

Then we have $D \subset D_{m+1} \subset D_m$ for each $m \in \mathbb{N}$, $\overline{D} = \overline{\bigcap_{m=1}^{\infty} D_m}$ and $d_H(D, D_m) = \frac{1}{2}$ for each $m \in \mathbb{N}$.

Example 4.2. Take the complex Banach space ℓ^2 . Let $D = B(0, 1)$, $m \in \mathbb{N}$ and

$$D_m := \left\{ x = \{x^j\}_j \in \ell^2 : \|x\| < 1 \text{ and } |x^j| < \frac{1}{2} \text{ for } j \geq m+1 \right\}.$$

Then we have $D_m \subset D_{m+1} \subset D$ for each $m \in \mathbb{N}$, $D = \bigcup_{m=1}^{\infty} D_m$ and $d_H(D, D_m) = \frac{1}{2}$ for each $m \in \mathbb{N}$.

Thus we see that we need a new approach in order to establish theorems analogous to Theorems 3.2 and 3.3.

Theorem 4.1. *Let D be a bounded and convex domain in an infinite dimensional complex Banach space $(X, \|\cdot\|)$ and let $D = \bigcup_{m=1}^{\infty} D_m$, where $\{D_m\}_{m=1}^{\infty}$ is an increasing sequence of bounded and convex domains in X . Then the corresponding sequence $\{k_{D_m}\}_{m=1}^{\infty}$ of Kobayashi distances converges, as m tends to infinity, to k_D , uniformly on compact sets.*

Proof. Fix $x, y \in D$. Since $D_m \subset D_{m+1} \subset D$ for all $m \in \mathbb{N}$, there exists $\tilde{m} \in \mathbb{N}$ such that $x, y \in D_m$ for each $m \geq \tilde{m}$. Therefore we have

$$k_D(x, y) \leq k_{D_{m+1}}(x, y) \leq k_{D_m}(x, y)$$

for all $m \geq \tilde{m}$. It follows that the sequence $\{k_{D_m}(x, y)\}_{m \geq \tilde{m}}$ converges.

Next, by Corollary 2.4 we have

$$k_D(x, y) = \inf_{Y \in \mathcal{Y}_{x,y}} k_{D \cap Y}(x, y),$$

where $\mathcal{Y}_{x,y}$ denotes the family of all linear subspaces $Y \subset X$, which are finite-dimensional and contain both x and y . Hence there exists an increasing sequence $\{Y_j\}$ in $\mathcal{Y}_{x,y}$ such that

$$k_D(x, y) \leq k_{D \cap Y_j}(x, y) < k_D(x, y) + \frac{1}{j}$$

for $j = 1, 2, \dots$. Since

$$D_m \cap Y_j \subset D \cap Y_j \subset D,$$

we have

$$k_D(x, y) \leq k_{D \cap Y_j}(x, y) \leq k_{D_m \cap Y_j}(x, y)$$

for each $j \in \mathbb{N}$ and each $m \geq \tilde{m}$. Next, by Theorem 3.2,

$$\lim_{\tilde{m} \leq m \rightarrow \infty} k_{D_m \cap Y_j}(x, y) = k_{D \cap Y_j}(x, y)$$

for each $j \in \mathbb{N}$. Therefore there exists a strictly increasing sequence $\{m_j\}$ such that

$$|k_{D_{m_j} \cap Y_j}(x, y) - k_{D \cap Y_j}(x, y)| < \frac{1}{j}$$

for $j = 1, 2, \dots$. This implies that

$$k_D(x, y) \leq k_{D_{m_j}}(x, y) \leq k_{D_{m_j} \cap Y_j}(x, y) < k_{D \cap Y_j}(x, y) + \frac{1}{j} < k_D(x, y) + \frac{2}{j}$$

and finally we obtain

$$\lim_{\tilde{m} \leq m \rightarrow \infty} k_{D_m}(x, y) = \lim_{j \rightarrow \infty} k_{D_{m_j}}(x, y) = k_D(x, y),$$

as asserted.

Now let $C \subset D$ be a compact set. Since $D = \bigcup_{m=1}^{\infty} D_m$ and the sequence $\{D_m\}$ is increasing, there exists $m' \in \mathbb{N}$ such that C is a subset of D_m for each $m \geq m'$. Let $\tilde{r} = \text{dist}(C, \partial D_{m'})$. Then for each $x \in C$, we have $\text{dist}(x, \partial D) \geq \tilde{r}$ and $\text{dist}(x, \partial D_m) \geq \tilde{r}$ for all $m \geq m'$. Now take two arbitrary sequences $\{x'_m\}$ and $\{x''_m\}$ in C which converge to x' and x'' in $(X, \|\cdot\|)$, respectively. We have $x', x'' \in C \subset D_m$ for each $m \geq m'$. Next, there exists $r > 0$ such that $B(x', r) \subset D_m$ and $B(x'', r) \subset D_m$ for each $m \geq m'$. Therefore there exists $m'' > m'$ such that $x'_m \in B(x', \frac{r}{2}) \subset D_m$ and $x''_m \in B(x'', \frac{r}{2}) \subset D_m$ for each $m \geq m''$. From Theorem 2.5, we now get

$$\begin{aligned} & |k_{D_m}(x'_m, x''_m) - k_D(x', x'')| \\ & \leq |k_{D_m}(x'_m, x')| + |k_{D_m}(x'', x''_m)| + |k_{D_m}(x', x'') - k_D(x', x'')| \\ & \leq \arg \tanh \left(\frac{2 \|x'_m - x'\|}{r} \right) + \arg \tanh \left(\frac{2 \|x'' - x''_m\|}{r} \right) \\ & \quad + |k_{D_m}(x', x'') - k_D(x', x'')| \rightarrow_m 0 \end{aligned}$$

and this completes the proof. ■

Theorem 4.2. *Let $(X, \|\cdot\|)$ be an infinite dimensional complex Banach space, D be a bounded and convex domain in X , and let $\{D_m\}_{m=1}^{\infty}$ be a sequence of bounded and convex domains in X such that $D_{m+1} \subset D_m$ for all $m \in \mathbb{N}$ and $\overline{\bigcap_{m=1}^{\infty} D_m} = \overline{D}$. Then the corresponding sequence $\{k_{D_m}\}_{m=1}^{\infty}$ of Kobayashi distances converges, as m tends to infinity, to k_D , uniformly on compact sets.*

Proof. Fix $x, y \in D$. We know that $D \subset D_{m+1} \subset D_m$ (see Lemma 5.1 below) and

$$k_{D_m}(x, y) \leq k_{D_{m+1}}(x, y) \leq k_D(x, y)$$

for each m . It follows that the sequence $\{k_{D_m}(x, y)\}$ is convergent. Next, again by Lemma 5.1, we have $\overline{D \cap Y} = \overline{\bigcap_{m=1}^\infty D_m \cap Y} = \bigcap_{m=1}^\infty \overline{D_m \cap Y}$ for each $Y \in \mathcal{Y}_{x,y}$, where $\mathcal{Y}_{x,y}$ denotes the family of all linear subspaces $Y \subset X$, which are finite-dimensional and contain both x and y . Therefore for an arbitrary $Y \in \mathcal{Y}_{x,y}$, we can apply Theorem 3.3 to the sequence $\{D_m \cap Y\}$. Next, by Corollary 2.4 we have

$$k_D(x, y) = \inf_{Y \in \mathcal{Y}_{x,y}} k_{D \cap Y}(x, y)$$

for each $Y \in \mathcal{Y}_{x,y}$. Hence there exists an increasing sequence $\{Y_j\}$ in $\mathcal{Y}_{x,y}$ such that

$$k_D(x, y) \leq k_{D \cap Y_j}(x, y) < k_D(x, y) + \frac{1}{j}$$

for $j = 1, 2, \dots$. Since $D \cap Y_j \subset D_m \cap Y_j \subset D_m$, we also have

$$k_{D_m}(x, y) \leq k_{D_m \cap Y_j} \leq k_{D \cap Y_j}$$

for each j and each m . As we have mentioned earlier, for each $j = 1, 2, \dots$, we can apply Theorem 3.3 to the sequence $\{D_m \cap Y_j\}_m$. Hence we get

$$\lim_{m \rightarrow \infty} k_{D_m \cap Y_j}(x, y) = k_{D \cap Y_j}(x, y)$$

for each $j \in \mathbb{N}$. This implies that there exists a strictly increasing sequence $\{m_j\}$ such that

$$|k_{D_{m_j} \cap Y_j}(x, y) - k_{D \cap Y_j}(x, y)| < \frac{1}{j}$$

for $j = 1, 2, \dots$. It follows that

$$k_D(x, y) + \frac{2}{j} > k_{D \cap Y_j}(x, y) + \frac{1}{j} \geq k_{D_{m_j} \cap Y_j}(x, y) + \frac{1}{j} > k_{D \cap Y_j}(x, y) \geq k_D(x, y)$$

and finally we obtain

$$\lim_{m \rightarrow \infty} k_{D_m}(x, y) = \lim_{j \rightarrow \infty} k_{D_{m_j}}(x, y) = k_D(x, y),$$

as asserted.

Now, let $C \subset D$ be a compact set. Then we have $\text{dist}(C, \partial D) \geq \tilde{r} > 0$. Since $D \subset D_m$ for each m , we get $\text{dist}(C, \partial D_m) \geq \tilde{r} > 0$ for each m . Now take two arbitrary sequences $\{x'_m\}$ and $\{x''_m\}$ in C which converge to x' and x'' in $(X, \|\cdot\|)$, respectively. It follows from Theorem 2.5 that

$$\begin{aligned} & |k_{D_m}(x'_m, x''_m) - k_D(x', x'')| \\ & \leq |k_{D_m}(x'_m, x')| + |k_{D_m}(x'', x''_m)| + |k_{D_m}(x', x'') - k_D(x', x'')| \\ & \leq \arg \tanh \left(\frac{\|x'_m - x'\|}{r} \right) + \arg \tanh \left(\frac{\|x'' - x''_m\|}{r} \right) \\ & \quad + |k_{D_m}(x', x'') - k_D(x', x'')| \longrightarrow_m 0 \end{aligned}$$

and this completes the proof. \blacksquare

5. AUXILIARY RESULTS

In this section we prove a few auxiliary lemmata, which were used in the proofs of the theorems we established in Sections 3 and 4. In these lemmata it does not matter if the Banach spaces are real or complex.

We begin with the following observations. It is generally known that if D is a bounded and convex domain in a Banach space $(X, \|\cdot\|)$, $0 \in D$ and $x \in \overline{D}$, then $tx \in D$ for each $0 < t < 1$. Hence we get

$$D = \bigcup_{0 < t < 1} tD = \bigcup_{0 < t < 1} t\overline{D}.$$

Also, if $C = \overline{C}$ is a bounded, closed and convex set in a Banach space $(X, \|\cdot\|)$, and $\text{int } C \neq \emptyset$, then $C = \overline{\text{int } C}$. Note that these observations no longer hold in the case of star-shaped sets (see also Remark 5.1 below).

Now we are ready to state and prove the first result of this section.

Lemma 5.1. *Let $(X, \|\cdot\|)$ be a Banach space, D be a bounded and convex domain in X , $\tilde{x} \in D$, and let $\{D_m\}_{m=1}^{\infty}$ be a sequence of bounded and convex domains in X such that $D_{m+1} \subset D_m$ for all $m \in \mathbb{N}$ and $\overline{D} = \overline{\bigcap_{m=1}^{\infty} D_m}$. Then $D \subset D_m$ for each m , $\overline{D} = \overline{\bigcap_{m=1}^{\infty} D_m} = \bigcap_{m=1}^{\infty} \overline{D_m}$ and $\overline{D \cap Y} = \overline{\bigcap_{m=1}^{\infty} (D_m \cap Y)} = \bigcap_{m=1}^{\infty} \overline{D_m \cap Y}$ for each finite-dimensional linear subspace $Y \subset X$ with $D \cap Y \neq \emptyset$.*

Proof. Without loss of generality we may assume that $0 \in D$. Since $\overline{D} = \overline{\bigcap_{m=1}^{\infty} D_m} \subset \overline{D_m}$ for each m , both D and D_m are bounded and convex domains, and since for each $x \in D$, there is $r > 0$ such that $B(x, r) \subset D \subset \overline{D} \subset \overline{D_m}$, it is obvious that for each m , we have $B(x, r) \subset D_m$ and this implies that $D \subset D_m$ for each m . Next, we have $D \subset \text{int } \overline{\bigcap_{m=1}^{\infty} D_m} \subset \text{int } \bigcap_{m=1}^{\infty} \overline{D_m}$. Moreover, it follows from the observations made just before Lemma 5.1, applied to the bounded, closed and convex set $\bigcap_{m=1}^{\infty} \overline{D_m}$, that $\text{int } \overline{\bigcap_{m=1}^{\infty} D_m} = \bigcap_{m=1}^{\infty} \overline{D_m}$. On the other hand, for each $0 \leq t < 1$ and each m , we have $tD_m \subset D_m$, which implies

$$\begin{aligned} \overline{D} &= \overline{\bigcap_{m=1}^{\infty} D_m} \subset \bigcap_{m=1}^{\infty} \overline{D_m} = \overline{\text{int } \bigcap_{m=1}^{\infty} \overline{D_m}} \\ &= \overline{\bigcup_{0 \leq t < 1} t \bigcap_{m=1}^{\infty} \overline{D_m}} = \overline{\bigcup_{0 \leq t < 1} \bigcap_{m=1}^{\infty} t\overline{D_m}} \subset \overline{\bigcap_{m=1}^{\infty} D_m} = \overline{D}. \end{aligned}$$

Hence we obtain $\overline{D} = \overline{\bigcap_{m=1}^{\infty} D_m} = \bigcap_{m=1}^{\infty} \overline{D_m}$.

To prove the second part of our lemma, it is sufficient to observe that we can apply the equalities we have just proved to Y , $D \cap Y$ and to the sequence $\{D_m \cap Y\}$, which replace X , D and the sequence $\{D_m\}$, respectively. \blacksquare

We now recall (a slightly modified version of) [2, Lemma 4.3].

Lemma 5.2. ([2]). *Let D be a bounded and convex domain in a finite-dimensional Banach space $(X, \|\cdot\|)$, $\tilde{x} \in D$, and let $\{D_m\}_{m=1}^\infty$ be a sequence of bounded and convex domains in X such that $D_{m+1} \subset D_m$ for all $m \in \mathbb{N}$ and $\overline{\bigcap_{m=1}^\infty D_m} = \overline{D}$. Then*

$$\lim_{m \rightarrow \infty} \inf_{x \in D_m} \sup\{0 \leq t \leq 1 : \tilde{x} + t(x - \tilde{x}) \in D\} = 1.$$

An analogous result is true for an increasing sequence of bounded and convex domains. For the convenience of the reader we now state and prove it.

Lemma 5.3. *Let D be a bounded and convex domain in a finite-dimensional Banach space $(X, \|\cdot\|)$, and let $\{D_m\}_{m=1}^\infty$ be a sequence of bounded and convex domains in X such that $D_m \subset D_{m+1}$ for all $m \in \mathbb{N}$ and $\bigcup_{m=1}^\infty D_m = D$. If $\tilde{x} \in D_1$, then*

$$\lim_{m \rightarrow \infty} \inf_{x \in D} \sup\{0 \leq s \leq 1 : \tilde{x} + s(x - \tilde{x}) \in D_m\} = 1.$$

Proof. Without any loss of generality we may assume that $\tilde{x} = 0 \in D_1$ and that $\overline{B_{\|\cdot\|}(0, r)} \subset D \subset D_1 \subset \overline{B_{\|\cdot\|}(0, R)}$ for some $0 < r < R$. Assume that

$$\lim_{m \rightarrow \infty} \inf_{x \in D} \sup\{0 \leq s \leq 1 : sx \in D_m\} = \tilde{s} < 1.$$

We obviously have $\tilde{s} \geq \frac{r}{R} > 0$. Also, there exist sequences $\{s_m\}_{m=1}^\infty$ and $\{x_m\}_{m=1}^\infty$ such that $x_m \in D$, $s_m x_m \in D_m \setminus \overline{B_{\|\cdot\|}(0, r)}$ and

$$\begin{aligned} & \inf_{x \in D} \sup\{0 \leq s \leq 1 : sx \in D_m\} - \frac{1}{m} \leq s_m \\ & \leq \sup\{0 \leq s \leq 1 : sx_m \in D_m\} \leq \inf_{x \in D} \sup\{0 \leq s \leq 1 : sx \in D_m\} + \frac{1}{m} \end{aligned}$$

for each $m = 1, 2, \dots$. Hence $\lim_m s_m = \tilde{s}$. Using the compactness of \overline{D} and passing, if need be, to a subsequence, we may assume that there exists a point $x \in \overline{D}$ such that $\lim_m \|x_m - x\| = 0$. Next, since $\|x\| \geq r > 0$ and D is a convex domain containing the closed ball $\overline{B_{\|\cdot\|}(0, r)}$, we have $B_{\|\cdot\|}(\tilde{s}x, (1 - \tilde{s})r) \subset D$. Therefore $\tilde{s}x + (1 - \tilde{s})\frac{r}{2\|x\|}x$ is an element of D and hence $\tilde{s}x + (1 - \tilde{s})\frac{r}{2\|x\|}x \in D_{\tilde{m}}$ for some $\tilde{m} \in \mathbb{N}$. This implies that

$$s_m x_m + (1 - \tilde{s})\frac{r}{2\|x\|}x_m = [s_m + (1 - \tilde{s})\frac{r}{2\|x\|}]x_m \in D_m$$

and

$$\begin{aligned} & \inf_{x \in D} \sup\{0 \leq s \leq 1 : sx \in D_m\} - \frac{1}{m} \leq s_m < s_m + (1 - \tilde{s})\frac{r}{2\|x\|} \\ & \leq \sup\{0 \leq s \leq 1 : sx_m \in D_m\} \leq \inf_{x \in D} \sup\{0 \leq s \leq 1 : sx \in D_m\} + \frac{1}{m} \end{aligned}$$

for all sufficiently large m . Taking m to infinity, we arrive at the following contradiction:

$$\begin{aligned} \tilde{s} &= \lim_{m \rightarrow \infty} s_m < \tilde{s} + (1 - \tilde{s}) \frac{r}{2\|x\|} = \lim_{m \rightarrow \infty} \left[s_m + (1 - \tilde{s}) \frac{r}{2\|x\|} \right] \\ &\leq \lim_{m \rightarrow \infty} \left[\inf_{x \in D} \sup \{ 0 \leq s \leq 1 : sx \in D_m \} + \frac{1}{m} \right] = \tilde{s}. \end{aligned}$$

This completes our proof. ■

The following corollary is a direct consequence of the last two lemmata.

Corollary 5.4. *Let D be a bounded and convex domain in a finite-dimensional Banach space $(X, \|\cdot\|)$, and let $\{D_m\}_{m=1}^\infty$ be a sequence of bounded and convex domains in X . If either $\{D_m\}_{m=1}^\infty$ is an increasing sequence of domains and $\bigcup_{m=1}^\infty D_m = D$, or $\{D_m\}_{m=1}^\infty$ is a decreasing sequence of domains and $\overline{\bigcap_{m=1}^\infty D_m} = \overline{D}$, then $\lim_{m \rightarrow \infty} d_H(D, D_m) = 0$.*

We now consider the connections between the Hausdorff metric and bounded and convex domains in Banach spaces.

Lemma 5.5. *Let $(X, \|\cdot\|)$ be a Banach space and let $r > 0$. Assume that D is a bounded and convex domain in X , $\tilde{x} \in D$ and $\overline{B(\tilde{x}, r)} \subset D$. Let $D_0 \subset X$ be a bounded and convex domain satisfying $d_H(D, D_0) < \frac{r}{4}$ and let $\tilde{x} \in D_0$. Then $r_0 := \sup\{r_1 > 0 : \overline{B(\tilde{x}, r_1)} \subset D_0\} > \frac{r}{4}$. Hence D_0 contains the closed ball $\overline{B(\tilde{x}, \frac{r}{4})}$.*

Proof. Without loss of generality we may assume that $\tilde{x} = 0$. To obtain a contradiction, suppose that there exists a bounded and convex domain $D_0 \subset X$ with $d_H(D, D_0) < \frac{r}{4}$, $0 \in D_0$, and $r_0 = \sup\{r_1 > 0 : \overline{B(0, r_1)} \subset D_0\} \leq \frac{r}{4}$. Then there exists $x \in D$ such that $\|x\| = r$ and $\frac{5r_0}{4r}x \notin D_0$. From our assumption that $d_H(D, D_0) < \frac{r}{4}$ it follows that there is a point $x_0 \in D_0$ with $\|x - x_0\| < r/4$. Observe now that the point $\frac{4r_0}{r}(x - x_0)$ lies in $B(0, r_0) \subset D_0$ and therefore

$$y = \alpha \frac{4r_0}{r}(x - x_0) + (1 - \alpha)x_0 = \frac{4r_0}{4r_0 + r}x \in D_0,$$

where $0 < \alpha = \frac{r}{4r_0 + r} < 1$. This, however, contradicts the assumption that $\frac{5r_0}{4r}x \notin D_0$ because

$$\frac{4r_0}{4r_0 + r} \geq \frac{4r_0}{4\frac{r}{4} + r} = 2\frac{r_0}{r} > \frac{5r_0}{4r}.$$

The contradiction we have reached finishes the proof. ■

Lemma 5.6. *Let $(X, \|\cdot\|)$ be a Banach space and let $r > 0$. If D is a bounded and convex domain in X , $\tilde{x} \in D$ and $\overline{B(\tilde{x}, r)} \subset D$, then for each $0 < \varepsilon < r$, there exists $\tilde{\eta} > 0$ such that each bounded and convex domain $D_0 \subset X$ with $d_H(D, D_0) < \tilde{\eta}$ and $\tilde{x} \in D_0$ contains the closed ball $\overline{B(\tilde{x}, r - \varepsilon)}$.*

Proof. Let $\tilde{x} = 0$ for simplicity. Assume, without loss of generality, that $0 < \varepsilon < \frac{r}{4}$. Take $0 < \tilde{\eta} = \frac{\varepsilon r}{4(r-\varepsilon)} < \varepsilon$ and let $D_0 \subset X$ be a bounded and convex domain with $d_H(D, D_0) < \tilde{\eta}$ and $0 \in D_0$. Observe that $0 < \tilde{\eta} = \frac{\varepsilon r}{4(r-\varepsilon)} < \frac{1}{4}r$. Therefore Lemma 5.5 implies that the domain D_0 contains the closed ball $\overline{B(0, \frac{r}{4})}$. Choose an arbitrary $x \in X$ with $\|x\| = r$. Then x is an element of D and since $d_H(D, D_0) < \tilde{\eta}$, there exists a point $x_0 \in D_0$ such that $\|x - x_0\| < \tilde{\eta}$. In addition, the point $\frac{r-\varepsilon}{\varepsilon}(x - x_0)$ lies in $B(0, \frac{r}{4}) \subset D_0$ and therefore

$$y = \alpha \frac{r-\varepsilon}{\varepsilon}(x - x_0) + (1 - \alpha)x_0 = \frac{r-\varepsilon}{r}x \in D_0,$$

where $0 < \alpha = \frac{\varepsilon}{r} < 1$. Since $\|x\| = r$ was chosen in an arbitrary way, this means that $\overline{B(0, r - \varepsilon)} \subset D_0$. The proof is complete. ■

Lemma 5.7. *Let $(X, \|\cdot\|)$ be a Banach space and let $r > 0$. If D is a bounded and convex domain in X , $\tilde{x} \in D$ and $\overline{B(\tilde{x}, r)} \subset D$, then for each $0 < \varepsilon < r$, there exists a number $\tilde{\eta}$ such that for each bounded and convex domain $D_0 \subset X$ with $d_H(D, D_0) < \tilde{\eta}$, we have $\overline{B(\tilde{x}, r - \varepsilon)} \subset D_0$.*

Proof. Without any loss of generality, we may assume that $\tilde{x} = 0$. Let $0 < \varepsilon_1 < \varepsilon < r$ and $\varepsilon_2 = \varepsilon - \varepsilon_1$. By the previous lemma, for $0 < \varepsilon_2 < r$, there exists a number $0 < \tilde{\eta} < \varepsilon_1$ such that for each bounded and convex domain $D_1 \subset X$ with $d_H(D, D_1) < 2\tilde{\eta}$ and $0 \in D_1$, we have $\overline{B(0, r - \varepsilon_2)} \subset D_1$. Now let $D_0 \subset X$ be a bounded and convex domain with $d_H(D, D_0) < \tilde{\eta}$. Since $0 \in D$, there exists $x_0 \in D_0$ such that $\|x_0\| < \tilde{\eta} < \varepsilon_1$. Hence for the bounded and convex domain $\tilde{D}_1 = -x_0 + D_0$, we have $d_H(D, \tilde{D}_1) < 2\tilde{\eta}$ and $0 \in \tilde{D}_1$. Hence $\overline{B(0, r - \varepsilon_2)} \subset \tilde{D}_1$. This implies that $\overline{B(x_0, r - \varepsilon_2)} \subset D_0$ and finally, we obtain $\overline{B(0, r - \varepsilon)} \subset D_0$. ■

Lemma 5.8. *Let $(X, \|\cdot\|)$ be a Banach space. Let D be a bounded and convex domain in X , and let $\{D_m\}$ be a sequence of bounded and convex domains in X . If $\lim_{m \rightarrow \infty} d_H(D, D_m) = 0$, then for each $\tilde{x} \in D$, there exist $\tilde{m} \in \mathbb{N}$, and sequences $\{s_m\}_{m \geq \tilde{m}}$ and $\{t_m\}_{m \geq \tilde{m}}$ such that $\lim_{\tilde{m} \leq m \rightarrow \infty} s_m = 1$, $\lim_{\tilde{m} \leq m \rightarrow \infty} t_m = 1$, $0 < s_m < 1$, $0 < t_m < 1$, $(1 - s_m)\tilde{x} + s_m D \subset D_m$ and $(1 - t_m)\tilde{x} + t_m D_m \subset D$ for $m \geq \tilde{m}$.*

Proof. Without any loss of generality, we may assume that $\tilde{m} = 1$, $\tilde{x} = 0$ and that there exists a number $r > 0$ such that $\overline{B(0, r)} \subset D$ and $\overline{B(0, r)} \subset D_m$ for $m = 1, 2, \dots$ (see Lemma 5.7). By our assumption, the domain D and all the domains D_m are uniformly bounded in X , that is, for some $R > r > 0$, we have $D \subset B(0, R)$ and $D_m \subset B(0, R)$ for $m = 1, 2, \dots$

Assume that

$$\liminf_{m \rightarrow \infty} \inf_{x \in D_m} \sup\{0 \leq t \leq 1 : tx \in D\} = \tilde{t} < 1.$$

We obviously have $\tilde{t} \geq \frac{r}{R} > 0$. Also, there exist sequences $\{t'_{m_j}\}_{j=1}^\infty$, $\{x_{m_j}\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ such that $x_{m_j} \in D_{m_j}$, $t'_{m_j}x_{m_j} \in D \setminus \overline{B(0, r)}$, $0 < t'_{m_j} < 1$, $\lim_{j \rightarrow \infty} t'_{m_j} = \tilde{t}$, $y_j \in D$, $\|y_j - x_{m_j}\| < \frac{r}{j}$, and

$$\begin{aligned} \tilde{t} - \frac{2}{j} &< \inf_{x \in D_{m_j}} \sup\{0 \leq t \leq 1 : tx \in D\} - \frac{1}{j} \leq t'_{m_j} \\ &\leq \sup\{0 \leq t \leq 1 : tx_{m_j} \in D\} \leq \inf_{x \in D_{m_j}} \sup\{0 \leq t \leq 1 : tx \in D\} + \frac{1}{j} < \tilde{t} + \frac{2}{j} \end{aligned}$$

for each $j = 1, 2, \dots$. Next, we have $\|x_{m_j}\| \leq R$ and therefore

$$t'_{m_j}x_{m_j} + (1 - t'_{m_j})\frac{r}{2R}x_{m_j} = [t'_{m_j} + (1 - t'_{m_j})\frac{r}{2R}]x_{m_j} \in D_{m_j}.$$

Since $y_j \in D$ and D is a convex domain containing the closed ball $\overline{B(0, r)}$, we have $B(t'_{m_j}y_{m_j}, (1 - t'_{m_j})r) \subset D$. Now observe that there exists a natural number \tilde{j} such that

$$\|t'_{m_j}y_j - [t'_{m_j}x_{m_j} + (1 - t'_{m_j})\frac{r}{2R}x_{m_j}]\| \leq t'_{m_j}\frac{r}{j} + (1 - t'_{m_j})\frac{r}{2} < (1 - t'_{m_j})r$$

for each $j \geq \tilde{j}$ and this means that

$$t'_{m_j}x_{m_j} + (1 - t'_{m_j})\frac{r}{2R}x_{m_j} \in D.$$

Hence we get

$$\begin{aligned} \tilde{t} - \frac{2}{j} &< \inf_{x \in D_{m_j}} \sup\{0 \leq t \leq 1 : tx \in D\} - \frac{1}{j} \leq t'_{m_j} \leq t'_{m_j} + (1 - t'_{m_j})\frac{r}{2R} \\ &\leq \sup\{0 \leq t \leq 1 : tx_{m_j} \in D\} \leq \inf_{x \in D_{m_j}} \sup\{0 \leq t \leq 1 : tx \in D\} + \frac{1}{j} < \tilde{t} + \frac{2}{j}. \end{aligned}$$

Taking j to infinity, we get the following contradiction:

$$\tilde{t} \leq \tilde{t} + (1 - \tilde{t})\frac{r}{2R} \leq \tilde{t}.$$

Thus we have

$$\lim_{m \rightarrow \infty} \inf_{x \in D_m} \sup\{0 \leq t \leq 1 : tx \in D\} = 1$$

and therefore there exists a sequence $\{t_m\}$ such that $\lim_{m \rightarrow \infty} t_m = 1$, $0 < t_m < 1$ and $t_mD_m \subset D$ for $m = 1, 2, \dots$

The proof of the second part of our lemma is analogous. Indeed, assume that

$$\liminf_{m \rightarrow \infty} \inf_{y \in D} \sup\{0 \leq s \leq 1 : sy \in D_m\} = \tilde{s} < 1.$$

We have $\tilde{s} \geq \frac{r}{R} > 0$ and there exist sequences $\{s'_{m_j}\}_{j=1}^\infty$, $\{y_{m_j}\}_{j=1}^\infty$ and $\{x_j\}_{j=1}^\infty$ such that $y_{m_j} \in D$, $s'_{m_j}y_{m_j} \in D_{m_j} \setminus \overline{B(0, r)}$, $0 < s'_{m_j} < 1$, $\lim_{j \rightarrow \infty} s'_{m_j} = \tilde{s}$, $x_j \in D_{m_j}$, $\|x_j - y_{m_j}\| < \frac{r}{j}$, and

$$\tilde{s} - \frac{2}{j} < \inf_{y \in D} \sup\{0 \leq s \leq 1 : sy \in D_{m_j}\} - \frac{1}{j} \leq s'_{m_j}$$

$$\leq \sup\{0 \leq s \leq 1 : sy_{m_j} \in D_{m_j}\} \leq \inf_{y \in D} \sup\{0 \leq s \leq 1 : sy \in D_{m_j}\} + \frac{1}{j} < \tilde{s} + \frac{2}{j}$$

for each $j = 1, 2, \dots$. Next, $\|y_{m_j}\| \leq R$ and therefore

$$s'_{m_j}y_{m_j} + (1 - s'_{m_j})\frac{r}{2R}y_{m_j} = [s'_{m_j} + (1 - s'_{m_j})\frac{r}{2R}]y_{m_j} \in D.$$

Since $x_j \in D_{m_j}$ and D_{m_j} is a convex domain containing the closed ball $\overline{B(0, r)}$, we have $B(s'_{m_j}x_{m_j}, (1 - s'_{m_j})r) \subset D_{m_j}$. Now observe that there exists a natural number \tilde{j} such that

$$\|s'_{m_j}x_j - [s'_{m_j}y_{m_j} + (1 - s'_{m_j})\frac{r}{2R}y_{m_j}]\| \leq s'_{m_j}\frac{r}{j} + (1 - s'_{m_j})\frac{r}{2} < (1 - s'_{m_j})r$$

for each $j \geq \tilde{j}$. This means that

$$s'_{m_j}y_{m_j} + (1 - s'_{m_j})\frac{r}{2R}y_{m_j} \in D_{m_j}.$$

Hence we get

$$\tilde{s} - \frac{2}{j} < \inf_{y \in D} \sup\{0 \leq s \leq 1 : sy \in D_{m_j}\} - \frac{1}{j} \leq s'_{m_j} \leq s'_{m_j} + (1 - s'_{m_j})\frac{r}{2R}$$

$$\leq \sup\{0 \leq s \leq 1 : sy_{m_j} \in D\} \leq \inf_{y \in D} \sup\{0 \leq s \leq 1 : sy \in D_{m_j}\} + \frac{1}{j} < \tilde{s} + \frac{2}{j}.$$

Taking j to infinity, we get the following contradiction:

$$\tilde{s} \leq \tilde{s} + (1 - \tilde{s})\frac{r}{2R} \leq \tilde{s}.$$

So, we have

$$\lim_{m \rightarrow \infty} \inf_{y \in D} \sup\{0 \leq s \leq 1 : sy \in D_{m_j}\} = 1$$

and therefore there exists a sequence $\{s_m\}$ such that $\lim_{m \rightarrow \infty} s_m = 1$, $0 < s_m < 1$ and $s_mD \subset D_m$ for $m = 1, 2, \dots$. This completes the proof. ■

Next, we note the following consequence of Lemma 5.8.

Lemma 5.9. *Let $(X, \|\cdot\|)$ be a Banach space. Let D be a bounded and convex domain in X and let $\{D_m\}$ be a sequence of bounded and convex domains in X . The following statements are equivalent:*

- (1) $\lim_{m \rightarrow \infty} d_H(D, D_m) = 0$;
- (2) *there exist a point $\tilde{x} \in X$, $\tilde{m} \in \mathbb{N}$, and numerical sequences $\{s_m\}_{m \geq \tilde{m}}$ and $\{t_m\}_{m \geq \tilde{m}}$ such that $\lim_{\tilde{m} \leq m \rightarrow \infty} s_m = 1$, $\lim_{\tilde{m} \leq m \rightarrow \infty} t_m = 1$, $0 < s_m < 1$, $0 < t_m < 1$, $(1 - s_m)\tilde{x} + s_m D \subset D_m$, and $(1 - t_m)\tilde{x} + t_m D_m \subset D$ for $m \geq \tilde{m}$.*

Proof. It is obvious that 2) \Rightarrow 1). On the other hand, the previous lemma shows that 1) \Rightarrow 2). ■

In conclusion, we make the following remark.

Remark 5.1. If the domains in question are not convex, then the results in this section no longer hold (even if the domains are star-shaped with a common center) whenever X is a real Banach space with $\dim X \geq 2$. Indeed, let X be, for example, the plane \mathbb{R}^2 with the standard ℓ^2 -norm and let

$$\Omega_k = \text{int conv}\left\{\left(\cos \frac{1}{4k-1}, \sin \frac{1}{4k-1}\right), \left(2 \cos \frac{1}{4k}, 2 \sin \frac{1}{4k}\right), \left(\cos \frac{1}{4k+1}, \sin \frac{1}{4k+1}\right)\right\},$$

where $k = 1, 2, \dots$. Taking, respectively, either $D_n := B(0, 1) \cup \bigcup_{k=n}^{\infty} \Omega_k$, where $n = 1, 2, \dots$, and $D := B(0, 1)$, or $D_n := B(0, 1) \cup \bigcup_{k=1}^n \Omega_k$, where again $n = 1, 2, \dots$, and $D := B(0, 1) \cup \bigcup_{k=1}^{\infty} \Omega_k$, we see that Lemmata 5.1–5.3 and Corollary 5.4 are no longer true (note that the domains D_n are star-shaped, but not convex). Similarly, without the assumption that the domain D is convex, Lemmata 5.5–5.9 are no longer valid. Indeed, it is not difficult to construct counterexamples using the following sets in the real plane \mathbb{R}^2 : $B(0, 1)$ and $(B(0, 1) \setminus A) \cup B(0, \tilde{r})$, where $A := \{x = (r \cos \alpha, r \sin \alpha) \in \mathbb{R}^2 : 0 < r < 1, \Theta_1 \leq \alpha \leq \Theta_2\}$ with $0 \leq \Theta_1 < \Theta_2 < 2\pi$, and $0 < \tilde{r} < 1$. Finally, we observe that if $\dim X = 1$, then all the results in the present section are valid for arbitrary bounded domains in X because in this case all bounded domains are simply bounded and open segments.

ACKNOWLEDGMENTS

These results have been partially achieved within the framework of the STREVCOMS Project No. 612669 with funding from the IRSES Scheme of the FP7 Programme of the European Union.

The third author was partially supported by the Israel Science Foundation (Grant 389/12), the Fund for the Promotion of Research at the Technion, and by the Technion General Research Fund.

Part of this research was carried out when the first two authors were visiting the Technion. They are grateful to their colleagues for their kind hospitality.

REFERENCES

1. M. Abate, *Iteration Theory of Holomorphic Maps on Taut Manifolds*, Mediterranean Press, Rende, 1989.
2. M. Budzyńska, T. Kuczumow and S. Reich, *The Common Fixed Point Set of Commuting Holomorphic Mappings in Cartesian Products of Banach Spaces*, Fixed Point Theory, accepted for publication.
3. S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.
4. S. Dineen, R. M. Timoney and J.-P. Vigué, Pseudodistances invariantes sur les domaines d'un espace localement convexe, *Ann. Scuola Norm. Sup. Pisa*, **12** (1985), 515-529.
5. T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam, 1980.
6. K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.
7. K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.
8. R. E. Greene and S. G. Krantz, *Stability of the Carathéodory and Kobayashi Metrics and Applications to Biholomorphic Mappings*, Complex Analysis of Several Variables (Madison, WI, 1982), Proc. Sympos. Pure Math. **41**, Amer. Math. Soc., Providence, RI, 1984, pp. 77-93.
9. L. A. Harris, *Schwarz-Pick Systems of Pseudometrics for Domains in Normed Linear Spaces*, Advances in Holomorphy, North Holland, Amsterdam, 1979, pp. 345-406.
10. H. V. Hristov, Limits of Carathéodory and Kobayashi pseudometrics, *C. R. Acad. Bulgare Sci.*, **29** (1976), 951-954.
11. M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, Berlin, 1993.
12. M. Jarnicki and P. Pflug, Invariant distances and metrics in complex analysis - revisited, *Dissertationes Math.*, **430** (2005), 1-192.
13. M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Second extended edition, Walter de Gruyter, Berlin, 2013.
14. M. Kobayashi, Convergence of invariant distances on decreasing domains, *Complex Variables*, **47** (2002), 155-165.
15. S. Kobayashi, Invariant distances on complex manifolds and holomorphic mappings, *J. Math. Soc. Japan*, **19** (1967), 460-480.
16. S. Kobayashi, *Hyperbolic Complex Spaces*, Springer, Berlin, 1998.
17. T. Kuczumow, S. Reich and D. Shoikhet, *Fixed points of holomorphic mappings: a metric approach*, Handbook of Metric Fixed Point Theory, (W. A. Kirk and B. Sims eds.), Kluwer Academic Publishers, Dordrecht, 2001, pp. 437-515.

18. L. Lempert, Holomorphic retracts and intrinsic metrics in convex domains, *Anal. Math.*, **8** (1982), 257-261.
19. N. Nikolov, Invariant functions and metrics in complex analysis, *Dissertationes Math.*, **486** (2012), 1-100.
20. S. Reich and D. Shoikhet, *Nonlinear Semigroups, Fixed Points and Geometry of Domains in Banach Spaces*, Imperial College Press, London, 2005.
21. H. Royden, *Remarks on the Kobayashi metric*, Lecture Notes in Math. **185**, Springer, Berlin, 1971, pp. 125-137.

Monika Budzyńska and Tadeusz Kuczumow
Instytut Matematyki UMCS
20-031 Lublin
Poland
E-mail: monikab1@hektor.umcs.lublin.pl
tadek@hektor.umcs.lublin.pl

Simeon Reich
Department of Mathematics
The Technion - Israel Institute of Technology
32000 Haifa
Israel
E-mail: sreich@tx.technion.ac.il