

HALPERN TYPE ITERATIONS FOR STRONGLY QUASI-NONEXPANSIVE SEQUENCES AND ITS APPLICATIONS

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Abstract. In this paper, we study the strong convergence of the Halpern type algorithms for a strongly quasi-nonexpansive sequence of operators. These results extend the results of Saejung [11]. Some applications in infinite family of firmly quasi-nonexpansive mappings, multiparameter proximal point algorithm, constraint minimization and subgradient projection are presented.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and C be a nonempty, closed and convex subset of H . We denote weak convergence in H by \rightharpoonup and strong convergence by \rightarrow . Let $T : C \rightarrow C$ be a mapping and $F(T) := \{x \in C : Tx = x\}$. T is said to be nonexpansive (resp. quasi-nonexpansive) iff $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$ (resp. $F(T) \neq \emptyset$ and $\|Tx - q\| \leq \|x - q\|$, $\forall (x, q) \in C \times F(T)$). Also, T is called a firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|x - Tx - (y - Ty)\|^2, \quad \forall x, y \in C,$$

and T is called a firmly quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2, \quad \forall (x, p) \in C \times F(T).$$

A nonlinear (possibly multivalued) operator $A : \mathbb{D}(A) \subset H \rightarrow H$ is called monotone iff

$$\langle u - v, x - y \rangle \geq 0, \quad \forall x, y \in \mathbb{D}(A), \quad \forall u \in A(x), \quad \forall v \in A(y).$$

The monotone operator A is called maximal iff $\mathbb{R}(I + A) = H$, where I is the identity operator on H . $J_\lambda^A := (I + \lambda A)^{-1}$ is called the resolvent of A of order λ , for each

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$\lambda > 0$. It is a well-known result that if A is maximal monotone, then J_λ^A is a single-valued firmly nonexpansive mapping on H . Two of the most important problems in nonlinear analysis are finding the solutions of the following nonlinear stationary equations:

$$(FP) \quad \text{Find } x \in C, \text{ such that } Tx = x,$$

$$(MP) \quad \text{Find } x \in \mathbb{D}(A), \text{ such that } 0 \in A(x),$$

specially when $T : C \rightarrow C$ is a nonexpansive mapping and $A : \mathbb{D}(A) \rightarrow H$ is a maximal monotone operator. A simple example shows that if T is nonexpansive, the picard iteration $(T^n x)$, where $x \in C$, is not weakly convergent in general even if C is convex and compact. Mann [9] proposed the following iterative method

$$(MI) \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, 3, \dots,$$

and proved the weak convergence of the sequence (x_n) generated by (MI) to a fixed point of T . In order to obtain the strong convergence, Halpern [7] suggested the following iteration

$$x_{n+1} = n^{-\theta}u + (1 - n^{-\theta})Tx_n, \quad n = 1, 2, 3, \dots,$$

where $\theta \in (0, 1)$ and $x_1, u \in C$. Wittmann [16] and Xu [17, 18] proposed the following extension of Halpern iteration

$$(HI) \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n = 1, 2, 3, \dots,$$

where $\alpha_n \in (0, 1)$ satisfies the following conditions

- (1) $\lim_n \alpha_n = 0$,
- (2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

It has been shown that the conditions (1) and (2) are necessary for the strong convergence of the Halpern's iteration. The following open question has been arose:

Q) Are conditions (1) and (2) sufficient to prove the strong convergence of the sequence (x_n) in (HI) ?

Suzuki [14] and Chidume and Chidume [5], independently, proved that the conditions (1) and (2) are sufficient for the strong convergence of the sequence (x_n) in (HI) to a fixed point of T if the nonexpansive operator T is a convex combination of the identity operator and another nonexpansive operator. This type of iterations is called the Halpern-Mann type algorithm. Saejung [11] proved sufficiency of the conditions (1) and (2) for the strong convergence of the sequence (x_n) in (HI) when T is a strongly nonexpansive mapping.

We recall that $T : C \rightarrow C$ is strongly nonexpansive (resp. strongly quasi-nonexpansive) iff T is nonexpansive and $x_n - y_n - (Tx_n - Ty_n) \rightarrow 0$, whenever (x_n) and (y_n) are sequences in C such that $(x_n - y_n)$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \rightarrow 0$ (resp. T is quasi-nonexpansive and $x_n - Tx_n \rightarrow 0$, whenever (x_n) is a bounded sequence in C such that $\|x_n - q\| - \|Tx_n - q\| \rightarrow 0$, for some $q \in F(T)$). We also recall strongly nonexpansive and strongly quasi-nonexpansive sequences from [4] that play an essential role in this paper. The sequence (T_n) of nonexpansive mappings is said to be strongly nonexpansive sequence iff $x_n - y_n - (T_n x_n - T_n y_n) \rightarrow 0$, whenever (x_n) and (y_n) are sequences in C such that $(x_n - y_n)$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \rightarrow 0$. The sequence (T_n) of quasi-nonexpansive mappings is said to be strongly quasi-nonexpansive sequence iff $\bigcap_n F(T_n) \neq \emptyset$ and $x_n - T_n x_n \rightarrow 0$, whenever (x_n) is a bounded sequence in C such that $\|x_n - q\| - \|T_n x_n - q\| \rightarrow 0$ for some $q \in \bigcap_n F(T_n)$. It is clear that a strongly nonexpansive sequence (T_n) with $\bigcap_n F(T_n) \neq \emptyset$ is a strongly quasi-nonexpansive sequence.

In this paper, we obtain the strong convergence of the Halpern type algorithm

$$(1.1) \quad x_{n+1} = \alpha_n u + (1 - \alpha_n) T_n x_n, \quad n = 1, 2, 3, \dots,$$

where $u, x_1 \in C$, (T_n) is a strongly quasi-nonexpansive sequence, and (α_n) is a sequence in $(0, 1)$ such that satisfy $\lim_n \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$. This result extends the results that were presented in [11] in the case of Hilbert spaces.

This paper is organized as follows. In Section 2, we give some lemmas that we need in the sequel. Section 3 is devoted to the main result of the paper. In this section, we prove the strong convergence of (1.1) to an element of $\bigcap_n F(T_n)$ when (T_n) is a strongly quasi-nonexpansive sequence. In Section 4, we apply the results of Section 3 to obtain the strong convergence for infinite family of firmly nonexpansive mappings. In Section 5, we apply our main theorem to study the strong convergence of multiparameter proximal point algorithm. Finally, in Section 6 of the paper, some other applications in minimization and subgradient projection are presented.

2. SOME LEMMAS

Lemma 2.1. [1] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$ with $\sum_{n=1}^\infty \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^\infty u_n < \infty$, and $\{t_n\}$ a sequence of real numbers with $\limsup_n t_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n t_n + u_n,$$

for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.2. [13] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{t_n\}$ be a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n \text{ for all } n \geq 1.$$

If $\limsup_{k \rightarrow \infty} t_{m_k} \leq 0$ for every subsequence $\{s_{m_k}\}$ of $\{s_n\}$ satisfying $\liminf_k (s_{m_k+1} - s_{m_k}) \geq 0$, then $\lim_{n \rightarrow \infty} s_n = 0$.

The following lemma is elementary, then we omit its proof.

Lemma 2.3. Suppose $a_n > 0$, $\forall n \in \mathbb{N}$, $s_n = \sum_{k=1}^n a_k$ and $\sum_{n=1}^{\infty} a_n < \infty$. Then $\sum_{n=1}^{\infty} \frac{a_n}{s_n} < \infty$.

The following lemma is a well-known elementary lemma in Hilbert spaces.

Lemma 2.4. Let $x, y \in H$. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle x + y, y \rangle.$$

3. CONVERGENCE RESULTS FOR A STRONGLY QUASI-NONEXPANSIVE SEQUENCE

In this section, we prove the strong convergence of the Halpern type iteration for a strongly quasi-nonexpansive sequence $(T_n) : C \rightarrow C$ that satisfies the following condition:

$$(3.1) \quad \begin{cases} \text{if } (x_{n_j}) \subset C \text{ and } (T_{n_j}) \subset (T_n) \text{ such that} \\ x_{n_j} \rightharpoonup x \in C \text{ and } x_{n_j} - T_{n_j}x_{n_j} \rightarrow 0, \text{ then } x \in \bigcap_{n=1}^{\infty} F(T_n). \end{cases}$$

Note that the condition (3.1) can be regarded as a kind of demi-closedness property for the sequence (T_n) , which reduces to the classical demi-closedness property when $T_n \equiv T$ (see [6]), in the other word

$$(3.2) \quad \begin{cases} \text{for any sequence } (z_k) \subset H \text{ and } z \in H, \\ z_k \rightharpoonup z, (I - T)z_k \rightarrow 0 \Rightarrow z \in F(T). \end{cases}$$

The following theorem is a generalization of Corollary 8 in [11].

Theorem 3.1. Let $(T_n) : C \rightarrow C$ be a strongly quasi-nonexpansive sequence such that (3.1) is satisfied and (α_n) be a sequence in $(0, 1)$ that satisfies the following conditions:

$$(3.3) \quad \begin{cases} \text{(i) } \lim_n \alpha_n = 0, \\ \text{(ii) } \sum_{n=1}^{\infty} \alpha_n = \infty. \end{cases}$$

If $(u, x_1) \in C^2$ and (x_n) is generated by:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T_n x_n,$$

then (x_n) converges strongly to $z = P_S u \in S$, where $S = \bigcap_{n=1}^{\infty} F(T_n)$ and P is the nearest point projection of C on S .

Proof. Suppose $z = P_S u$, then

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(u - z) + (1 - \alpha_n)(T_n x_n - z)\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|T_n x_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|x_n - z\| \leq \max\{\|u - z\|, \|x_n - z\|\} \\ &\leq \dots \leq \max\{\|u - z\|, \|x_1 - z\|\}. \end{aligned}$$

Thus (x_n) is bounded. By Lemma 2.4, for all $n \in \mathbb{N}$, we have:

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + (1 - \alpha_n)(T_n x_n - z)\|^2 \\ &\leq \|(1 - \alpha_n)^2 (T_n x_n - z)\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z \rangle. \end{aligned}$$

So

$$(3.4) \quad \|x_{n+1} - z\|^2 \leq (1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z \rangle.$$

By Lemma 2.2, it suffices to show that $\limsup_k 2\langle u - z, x_{m_k+1} - z \rangle \leq 0$ for every subsequence $(\|x_{m_k} - z\|)$ of $(\|x_n - z\|)$ satisfying $\liminf_k (\|x_{m_k+1} - z\| - \|x_{m_k} - z\|) \geq 0$.

For this, suppose that $(\|x_{m_k} - z\|)$ is a subsequence of $(\|x_n - z\|)$ such that $\liminf_k (\|x_{m_k+1} - z\| - \|x_{m_k} - z\|) \geq 0$. Then

$$\begin{aligned} 0 &\leq \liminf_k (\|x_{m_k+1} - z\| - \|x_{m_k} - z\|) \\ &\leq \liminf_k (\alpha_{m_k} \|u - z\| + (1 - \alpha_{m_k}) \|T_{m_k} x_{m_k} - z\| - \|x_{m_k} - z\|) \\ &\leq \liminf_k (\|T_{m_k} x_{m_k} - z\| - \|x_{m_k} - z\|) + \limsup_k (\alpha_{m_k} (\|u - z\| - \|T_{m_k} x_{m_k} - z\|)) \\ &= \liminf_k (\|T_{m_k} x_{m_k} - z\| - \|x_{m_k} - z\|) \leq \limsup_k (\|T_{m_k} x_{m_k} - z\| - \|x_{m_k} - z\|) \\ &\leq \limsup_k (\|x_{m_k} - z\| - \|x_{m_k} - z\|) = 0. \end{aligned}$$

Hence $\lim_k (\|T_{m_k} x_{m_k} - z\| - \|x_{m_k} - z\|) = 0$, which by the assumption on the sequence (T_n) , we get: $\lim_k \|T_{m_k} x_{m_k} - x_{m_k}\| = 0$.

Now, we show $\limsup_k \langle u - z, x_{m_k} - z \rangle \leq 0$. There exists subsequence $(x_{m_{k_t}})$ of (x_{m_k}) such that $x_{m_{k_t}} \rightarrow x \in C$ and $\limsup_k \langle u - z, x_{m_k} - z \rangle = \lim_t \langle u - z, x_{m_{k_t}} - z \rangle = \langle u - z, x - z \rangle$.

Since $x_{m_{k_t}} \rightarrow x$ and $T_{m_{k_t}} x_{m_{k_t}} - x_{m_{k_t}} \rightarrow 0$, thus, the condition (3.1) implies $x \in \bigcap_{n=1}^\infty F(T_n) = S$. Therefore by $z = P_S u$, we obtain $\limsup_k \langle u - z, x_{m_k} - z \rangle \leq 0$.

Moreover, we have:

$$\|x_{m_k+1} - x_{m_k}\| \leq \alpha_{m_k} \|u - x_{m_k}\| + (1 - \alpha_{m_k}) \|T_{m_k} x_{m_k} - x_{m_k}\|,$$

which implies $\|x_{m_k+1} - x_{m_k}\| \rightarrow 0$. Hence

$$\begin{aligned} \limsup_k \langle u - z, x_{m_k+1} - z \rangle &= \limsup_k (\langle u - z, x_{m_k+1} - x_{m_k} \rangle + \langle u - z, x_{m_k} - z \rangle) \\ &\leq \limsup_k (\|u - z\| \|x_{m_k+1} - x_{m_k}\|) + \limsup_k \langle u - z, x_{m_k} - z \rangle \leq 0. \end{aligned}$$

Thus $\limsup_k 2\langle u - z, x_{m_k+1} - z \rangle \leq 0$. Hence, by Lemma 2.2, we have $\|x_n - z\| \rightarrow 0$. That is the desired result. ■

Remark 3.1. Theorem 3.1 is satisfied for a strongly nonexpansive sequence (T_n) with a common fixed point.

The following condition on the sequence (T_n) was introduced by Aoyama et al. [1] in a Banach space. Let $(T_n)_{n=1}^\infty : C \rightarrow C$ be a countable family of mappings. Family (T_n) satisfies AKTT-condition if

$$(3.5) \quad \sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty,$$

for each bounded subset B of C .

If (T_n) satisfies AKTT-condition, then we can define a nonexpansive mapping $T : C \rightarrow C$ such that

$$(3.6) \quad Tx = \lim_{n \rightarrow \infty} T_n x, \quad (x \in C).$$

In this case, we also say (T_n, T) satisfies AKTT-condition. The following condition is used in some literature such as [1, 11] for a sequence of nonexpansive mappings (T_n) .

$$(3.7) \quad \begin{cases} (A) (T_n, T) \text{ satisfies AKTT-condition,} \\ (B) F(T) = \bigcap_{n=1}^{\infty} F(T_n). \end{cases}$$

In the following, it is shown that the condition (3.1) is strictly weaker than (3.7) for a sequence of nonexpansive mappings.

Proposition 3.2. *If the condition (3.7) is satisfied for sequence of nonexpansive mappings (T_n) , then (T_n) satisfy the condition (3.1).*

Proof. Suppose that the condition (3.7) is satisfied and $(T_{n_j}) \subset (T_n)$, $(x_{n_j}) \subset C$ are subsequences such that $x_{n_j} \rightarrow x$ and $x_{n_j} - T_{n_j}x_{n_j} \rightarrow 0$. Then

$$\begin{aligned} 0 &\leq \|x_{n_j} - Tx_{n_j}\| \leq \|x_{n_j} - T_{n_j}x_{n_j}\| + \|T_{n_j}x_{n_j} - Tx_{n_j}\| \\ &\leq \|x_{n_j} - T_{n_j}x_{n_j}\| + \sup\{\|T_{n_j}y - Ty\| : y \in (x_{n_j})\} \\ &\leq \|x_{n_j} - T_{n_j}x_{n_j}\| + \sup\left\{\sum_{i=n_j}^{\infty} \|T_i y - T_{i+1}y\| : y \in (x_{n_j})\right\} \\ &\leq \|x_{n_j} - T_{n_j}x_{n_j}\| + \sum_{i=n_j}^{\infty} \sup\{\|T_i y - T_{i+1}y\| : y \in (x_{n_j})\}. \end{aligned}$$

Thus by the assumptions, we get $\|x_{n_j} - Tx_{n_j}\| \rightarrow 0$. Therefore, demiclosedness of T implies that $x \in F(T)$. Hence, by part (B) of (3.7), we get $x \in \bigcap_{n=1}^{\infty} F(T_n)$. Consequently, the condition (3.1) is satisfied for (T_n) . ■

Example 3.1. Set $T_1x = x$ and $T_nx = \begin{cases} \frac{x}{n}, & \text{if } n \text{ is even,} \\ \frac{2x}{n}, & \text{if } n \text{ is odd,} \end{cases}$ for $n \geq 2$.

Then, clearly, the condition (3.1) holds for the sequence (T_n) but the sequence (T_n) does not satisfy the condition (3.7).

Remark 3.2. If T is a strongly nonexpansive selfmapping on C , then the condition (3.1) is satisfied when $T_n \equiv T$. Thus, by Remark 3.1 and Proposition 3.2, Theorem 3.1 extends and improves Theorem 4 and Theorem 10 of [11] in Hilbert spaces setting.

Theorem 3.3. Let $(T_n) : H \rightarrow H$ be a strongly nonexpansive sequence such that (3.1) is satisfied and $S = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose (α_n) is a sequence in $(0, 1)$ which satisfies the condition (3.3) and (e_n) is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\lim_n \frac{\|e_n\|}{\alpha_n} = 0$. If $(u, x_1) \in H^2$ and (x_n) is generated by:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T_n x_n + e_n,$$

then (x_n) converges strongly to $z = P_S u \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Suppose that sequence (y_n) is generated by the exact algorithm:

$$y_{n+1} = \alpha_n u + (1 - \alpha_n)T_n y_n \text{ and } y_1 = x_1,$$

then by Remark 3.1, (y_n) converges strongly to $z = P_S u \in \bigcap_{n=1}^{\infty} F(T_n)$. Since for all $n \in \mathbb{N}$, T_n is nonexpansive, we have

$$\|x_{n+1} - y_{n+1}\| \leq (1 - \alpha_n)\|T_n x_n - T_n y_n\| + \|e_n\| \leq (1 - \alpha_n)\|x_n - y_n\| + \|e_n\|,$$

which by condition (ii) of (3.3) and Lemma 2.1, implies $\|x_n - y_n\| \rightarrow 0$. Thus, the inequality $\|x_n - z\| \leq \|x_n - y_n\| + \|y_n - z\|$ implies that the sequence (x_n) converges strongly to $z = P_S u \in \bigcap_{n=1}^{\infty} F(T_n)$. ■

4. INFINITE FAMILY OF FIRMLY QUASI-NONEXPANSIVE MAPPINGS

In this section, we apply the results of Theorems 3.1 and 3.3 to prove the strong convergence of iterations of Halpern type for a infinite family of firmly quasi-nonexpansive mappings.

Proposition 4.1. Let R_1, R_2, R_3, \dots be firmly quasi-nonexpansive selfmappings on C such that $\bigcap_{k=1}^{\infty} F(R_k) \neq \emptyset$ and (θ_n^k) be a family of nonnegative real numbers with indices $k, n \in \mathbb{N}$ such that $k \leq n$. Consider the following conditions:

$$(4.1) \quad \begin{cases} \text{(i)} & \sum_{k=1}^n \theta_n^k = 1, \quad \forall n \in \mathbb{N}, \\ \text{(ii)} & \lim_{n \rightarrow \infty} \theta_n^k > 0, \quad \forall k \in \mathbb{N}, \\ \text{(iii)} & \sum_{n=1}^{\infty} \sum_{k=1}^n |\theta_{n+1}^k - \theta_n^k| < \infty. \end{cases}$$

Set $T_n = \sum_{k=1}^n \theta_n^k R_k$, $\forall n \in \mathbb{N}$.

- (1) If (θ_n^k) satisfies (i), then for all $n \in \mathbb{N}$, T_n is a well-defined selfmapping on C .
- (2) If (θ_n^k) satisfies (i), (ii), then for all $n \in \mathbb{N}$, $F(T_n) = \bigcap_{k \in J} F(R_k)$, where $J = \{1 \leq k \leq n : \theta_n^k \neq 0\}$. Hence $\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(R_k)$.
- (3) If (θ_n^k) satisfies (i), (ii), then the sequence (T_n) is a strongly quasi-nonexpansive sequence.
- (4) If (θ_n^k) satisfies (i), (ii) and (iii), then (T_n) satisfies (3.7). Consequently, by Proposition 3.2, (T_n) satisfies (3.1).
- (5) If for all $k \in \mathbb{N}$, R_k satisfies (3.2) and (θ_n^k) satisfies (i), (ii), then (T_n) satisfies (3.1).

Proof. For (1), (2) and (4), we refer the reader to [1, 8]. Notice that the proofs of (2) and (4) that were presented in [1] for nonexpansive mappings remain true for quasi-nonexpansive mappings.

In order to prove (3), suppose $p \in \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and (x_n) is a bounded sequence in C such that $\|x_n - p\| - \|T_n x_n - p\| \rightarrow 0$. By the condition (i) and part (2), we get

$$\begin{aligned} \|x_n - p\| - \|T_n x_n - p\| &= \|x_n - p\| - \left\| \sum_{k=1}^n \theta_n^k R_k x_n - \sum_{k=1}^n \theta_n^k p \right\| \\ &\geq \|x_n - p\| - \sum_{k=1}^n \theta_n^k \|R_k x_n - p\| = \sum_{k=1}^n \theta_n^k (\|x_n - p\| - \|R_k x_n - p\|). \end{aligned}$$

Thus, $\lim_n \sum_{k=1}^n \theta_n^k (\|x_n - p\| - \|R_k x_n - p\|) = 0$. Moreover, there exists $L > 0$ such that $\|x_n - p\| \leq L$. Thus, by firmly quasi-nonexpansivity of (R_k) , we have

$$\begin{aligned} \|x_n - R_k x_n\|^2 &\leq \|x_n - p\|^2 - \|R_k x_n - p\|^2 \\ &= (\|x_n - p\| - \|R_k x_n - p\|)(\|x_n - p\| + \|R_k x_n - p\|) \\ &\leq 2L(\|x_n - p\| - \|R_k x_n - p\|) \end{aligned}$$

which implies $\|x_n - R_k x_n\|^2 \leq 2L(\|x_n - p\| - \|R_k x_n - p\|)$. Multiplying this inequality by θ_n^k and then summing up from $k = 1$ to n , we obtain

$$\sum_{k=1}^n \theta_n^k \|x_n - R_k x_n\|^2 \leq 2L \sum_{k=1}^n \theta_n^k (\|x_n - p\| - \|R_k x_n - p\|)$$

which implies $\lim_n \sum_{k=1}^n \theta_n^k \|x_n - R_k x_n\|^2 = 0$. Now, by Cauchy-Schwarz inequality and (i), we have

$$\begin{aligned} 0 &\leq \|x_n - T_n x_n\| = \left\| \sum_{k=1}^n \theta_n^k (x_n - R_k x_n) \right\| \\ &\leq \sum_{k=1}^n \theta_n^k \|x_n - R_k x_n\| \leq \left(\sum_{k=1}^n \theta_n^k \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \theta_n^k \|x_n - R_k x_n\|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^n \theta_n^k \|x_n - R_k x_n\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

that follows $\|x_n - T_n x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

On the other hand, by (i), for all $x \in C$, $p \in \bigcap_{n=1}^\infty F(T_n)$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} \|T_n x - p\| &= \left\| \sum_{k=1}^n \theta_n^k R_k x - p \right\| \leq \sum_{k=1}^n \theta_n^k \|R_k x - p\| \\ &\leq \sum_{k=1}^n \theta_n^k \|x - p\| = \|x - p\|. \end{aligned}$$

Hence for all $n \in \mathbb{N}$, T_n is quasi-nonexpansive, as desired.

In order to prove (5), suppose that there exist $(T_{n_j}) \subset (T_n)$ and $(x_{n_j}) \subset C$ such that $x_{n_j} \rightarrow x \in C$ and $x_{n_j} - T_{n_j} x_{n_j} \rightarrow 0$ and $q \in \bigcap_{k=1}^\infty F(R_k)$. Since each R_k is quasi-nonexpansive, by (i), we have

$$\begin{aligned} &\frac{1}{2} \sum_{k=1}^{n_j} \theta_{n_j}^k \|x_{n_j} - R_k x_{n_j}\|^2 \\ &\leq \sum_{k=1}^{n_j} \theta_{n_j}^k \langle x_{n_j} - R_k x_{n_j}, x_{n_j} - q \rangle \\ &= \left\langle x_{n_j} - \sum_{k=1}^{n_j} \theta_{n_j}^k R_k x_{n_j}, x_{n_j} - q \right\rangle = \langle x_{n_j} - T_{n_j} x_{n_j}, x_{n_j} - q \rangle. \end{aligned}$$

Consequently, by using the boundedness of (x_{n_j}) , we easily deduced that $\lim_j \sum_{k=1}^{n_j} \theta_{n_j}^k \|x_{n_j} - R_k x_{n_j}\|^2 = 0$. Thus for each $k \in \mathbb{N}$, $\lim_j \theta_{n_j}^k \|x_{n_j} - R_k x_{n_j}\|^2 = 0$. Hence by (ii), for each $k \in \mathbb{N}$, $\lim_j \|x_{n_j} - R_k x_{n_j}\| = 0$. Since for each $k \in \mathbb{N}$, R_k satisfies (3.2), then $x \in \bigcap_{k=1}^\infty F(R_k)$. Therefore by (2), $x \in \bigcap_{n=1}^\infty F(T_n)$. Hence (T_n) satisfies (3.1). ■

Theorem 4.2. Let R_1, R_2, R_3, \dots be firmly quasi-nonexpansive mappings of C into itself such that $S = \bigcap_{k=1}^{\infty} F(R_k) \neq \emptyset$ and (θ_n^k) be a family of nonnegative real numbers with indices $k, n \in \mathbb{N}$ with $k \leq n$ such that satisfies the conditions (i) and (ii) of (4.1). Suppose one of the following conditions holds

$$\begin{aligned} \text{(iii)} \quad & \sum_{n=1}^{\infty} \sum_{k=1}^n |\theta_{n+1}^k - \theta_n^k| < \infty, \\ \text{(iv)} \quad & \forall k \in \mathbb{N}, R_k \text{ satisfies (3.2)}. \end{aligned}$$

Also, let (α_n) be a sequence in $(0, 1)$ such that satisfies (3.3). If $(u, x_1) \in C^2$ and (x_n) is generated by:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \sum_{k=1}^n \theta_n^k R_k x_n,$$

then (x_n) converges strongly to $z = P_S u \in \bigcap_{k=1}^{\infty} F(R_k)$.

Proof. By setting $T_n = \sum_{k=1}^n \theta_n^k R_k, \forall n \in \mathbb{N}$, the proof is an immediate consequence of Proposition 4.1 and Theorem 3.1. \blacksquare

Theorem 4.3. Let R_1, R_2, R_3, \dots be firmly quasi-nonexpansive mappings of C into itself such that $S = \bigcap_{k=1}^{\infty} F(R_k) \neq \emptyset$ and $(\lambda_n) \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \lambda_n < \infty$. Suppose (α_n) is a sequence in $(0, 1)$ such that satisfies (3.3). If $(u, x_1) \in C^2$ and (x_n) is generated by:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \left(\left(\sum_{i=1}^n \lambda_i \right)^{-1} \sum_{k=1}^n \lambda_k R_k \right) x_n,$$

then (x_n) converges strongly to $z = P_S u \in \bigcap_{k=1}^{\infty} F(R_k)$.

$$\text{Proof. Set } \theta_n^k = \begin{cases} \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}, & k \leq n, \\ 0, & k > n, \end{cases}$$

then for each $n \in \mathbb{N}$, $\sum_{k=1}^n \theta_n^k = \sum_{k=1}^n \frac{\lambda_k}{\sum_{i=1}^n \lambda_i} = 1$ and since $(\lambda_n) \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \lambda_n < \infty$, we get

$$\lim_{n \rightarrow \infty} \theta_n^k = \lim_{n \rightarrow \infty} \frac{\lambda_k}{\sum_{i=1}^n \lambda_i} > 0, \quad \forall k \in \mathbb{N}.$$

Also, by Lemma 2.3

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{k=1}^n |\theta_{n+1}^k - \theta_n^k| &= \sum_{n=1}^{\infty} \sum_{k=1}^n \left| \frac{\lambda_k}{\sum_{i=1}^{n+1} \lambda_i} - \frac{\lambda_k}{\sum_{i=1}^n \lambda_i} \right| \\ &= \sum_{n=1}^{\infty} \left(1 - \frac{\sum_{k=1}^n \lambda_k}{\sum_{i=1}^{n+1} \lambda_i} \right) = \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\sum_{i=1}^n \lambda_i} < \infty. \end{aligned}$$

Hence, Theorem 4.2 implies the interested result. ■

Remark 4.1. By using Theorem 3.3 and Proposition 4.1, the inexact versions of Theorems 4.2 and 4.3 are true when R_1, R_2, R_3, \dots are firmly nonexpansive selfmappings on H and the error sequence $(e_n) \subset H$ satisfies the condition $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\lim_n \frac{\|e_n\|}{\alpha_n} = 0$.

5. PROXIMAL POINT METHODS

By the main result of the paper (Theorem 3.1), we study the Halpern-Mann type of the proximal point algorithm in the following.

Lemma 5.1. *Let $A \subset H \times H$ be a maximal monotone operator such that $A^{-1}(0) \neq \emptyset$.*

- (i1) *If (λ_n) is a sequence in $(0, \infty)$, then (J_{λ_n}) is a strongly nonexpansive sequence.*
- (i2) *If (λ_n) is a sequence in $(0, \infty)$ such that $\liminf_n \lambda_n > 0$, then the sequence (J_{λ_n}) satisfies (3.1).*

Proof. We can find the proof of part (i1) in [11]. For the proof of (i2), suppose $(x_{n_k}) \subset H$ such that $x_{n_k} \rightharpoonup x$ and $x_{n_k} - J_{\lambda_{n_k}} x_{n_k} \rightarrow 0$, then for all $y \in H$, we have $\langle J_{\lambda_{n_k}} x_{n_k} - x, y \rangle = \langle J_{\lambda_{n_k}} x_{n_k} - x_{n_k}, y \rangle + \langle x_{n_k} - x, y \rangle$. Hence $J_{\lambda_{n_k}} x_{n_k} \rightharpoonup x$. Since $\liminf_n \lambda_n > 0$, we get $\lambda_{n_k}^{-1}(x_{n_k} - J_{\lambda_{n_k}} x_{n_k}) \rightarrow 0$. On the other hand, $\lambda_{n_k}^{-1}(x_{n_k} - J_{\lambda_{n_k}} x_{n_k}) \in A(J_{\lambda_{n_k}} x_{n_k})$ and A is demiclosed (see [10]), thus $0 \in A(x)$. In other words $x \in A^{-1}(0) = F(J_\lambda)$, $\forall \lambda > 0$. ■

Lemma 5.2. *Let (T_n) be a sequence of selfmappings on C and (δ_n) is a sequence in $[0, 1]$ such that $\limsup_n \delta_n < 1$. For each $n \in \mathbb{N}$, set*

$$S_n x = \delta_n x + (1 - \delta_n) T_n x,$$

then

- (1) If (T_n) is strongly quasi-nonexpansive sequence (strongly nonexpansive sequence), then (S_n) is also strongly quasi-nonexpansive sequence (resp. strongly nonexpansive sequence).
- (2) If (T_n) satisfies (3.1), then (S_n) satisfies (3.1).

Proof. It is clear that $F(T_n) = F(S_n), \forall n \in \mathbb{N}$. If (T_n) is strongly quasi-nonexpansive sequence, then $\bigcap_{n=1}^{\infty} F(S_n) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $q \in F(S_n)$, then since $F(T_n) = F(S_n)$ and T_n is quasi-nonexpansive, for each $x \in C$

$$\|S_n x - q\| \leq \delta_n \|x - q\| + (1 - \delta_n) \|T_n x - q\| \leq \|x - q\|.$$

Thus, S_n is quasi-nonexpansive. Suppose $q \in \bigcap_{n=1}^{\infty} F(S_n)$ and (x_n) is a bounded sequence in C such that $\|S_n x_n - q\| - \|x_n - q\| \rightarrow 0$, then we get

$$\|S_n x_n - q\| - \|x_n - q\| \leq (1 - \delta_n)(\|T_n x_n - q\| - \|x_n - q\|) \leq 0,$$

which implies $(1 - \delta_n)(\|T_n x_n - q\| - \|x_n - q\|) \rightarrow 0$. Since $\limsup_n \delta_n < 1$, therefore $(\|T_n x_n - q\| - \|x_n - q\|) \rightarrow 0$. Since (T_n) is strongly quasi-nonexpansive sequence, we get $\|T_n x_n - x_n\| \rightarrow 0$. Moreover, $\|S_n x_n - x_n\| = (1 - \delta_n) \|T_n x_n - x_n\|$ implies $\|S_n x_n - x_n\| \rightarrow 0$. Hence, (S_n) is strongly quasi-nonexpansive sequence.

With a similar process, we deduce that if (T_n) is strongly nonexpansive sequence, then (S_n) is also strongly nonexpansive sequence.

In order to prove (2), suppose $(S_{n_j}) \subset (S_n), (x_{n_j}) \subset C$ are subsequences such that $x_{n_j} \rightarrow x$ and $x_{n_j} - S_{n_j} x_{n_j} \rightarrow 0$. Since $x_{n_j} - S_{n_j} x_{n_j} = (1 - \delta_{n_j})(x_{n_j} - T_{n_j} x_{n_j})$ and $\limsup_n \delta_n < 1$, we have $x_{n_j} - T_{n_j} x_{n_j} \rightarrow 0$. Since (T_n) satisfies (3.1), $x \in \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(S_n)$. Therefore (S_n) satisfies (3.1). ■

The following theorem has essentially proved by Wang and Cui [15], which we recall it as an application of Theorem 3.1.

Theorem 5.1. Let (λ_n) be a sequence in $(0, \infty)$ such that $\liminf_n \lambda_n > 0$ and $A \subset H \times H$ be a maximal monotone operator such that $A^{-1}(0) \neq \emptyset$. Suppose $(\alpha_n), (\beta_n)$ and (γ_n) are sequences in $(0, 1)$ that satisfy the following conditions:

$$(5.1) \quad \begin{cases} \text{(i)} \alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \in \mathbb{N}; \\ \text{(ii)} \lim_n \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \\ \text{(iii)} \liminf_n \gamma_n > 0. \end{cases}$$

and (e_n) is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\lim_n \frac{\|e_n\|}{\alpha_n} = 0$. If $(u, x_1) \in H^2$ and (x_n) is generated by:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{\lambda_n} x_n + e_n,$$

then (x_n) converges strongly to $z = P_{A^{-1}(0)} u \in A^{-1}(0)$.

Proof. By Lemma 5.1, (J_{λ_n}) is a strongly nonexpansive sequence. Set $S_n x = \frac{\beta_n}{1-\alpha_n}x + \frac{\gamma_n}{1-\alpha_n}J_{\lambda_n}x, \forall x \in H$. By the assumptions, Lemmas 5.1 and 5.2, (S_n) is a strongly nonexpansive sequence that satisfies (3.1). Moreover, we have

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)S_n x_n + e_n.$$

Consequently, by Theorem 3.3, (x_n) converges strongly to $z = P_{A^{-1}(0)}u \in A^{-1}(0)$. ■

The following corollary improves the main result of Yao and Shahzad [20], that gave an answer to an open question by Boikanyo and Morosanu [3]. This question was studied by Saejung [12] as well.

Corollary 5.2. *Let (λ_n) be a sequence in $(0, \infty)$ such that $\liminf_n \lambda_n > 0$ and $A \subset H \times H$ be a maximal monotone operator such that $A^{-1}(0) \neq \emptyset$. Suppose $(\alpha_n), (\beta_n)$, and (γ_n) are sequences in $(0, 1)$ which satisfy conditions (5.1) and (d_n) is a sequence in H such that $\lim_n \|d_n\| = 0$. If $x_1 \in H$ and (x_n) is generated by:*

$$x_{n+1} = \beta_n x_n + \gamma_n J_{\lambda_n} x_n + \alpha_n d_n,$$

then (x_n) converges strongly to $z = P_{A^{-1}(0)}0$.

Proof. By taking $u = 0$ and $e_n = \alpha_n d_n$ in the iteration $x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{\lambda_n} x_n + e_n$, the proof is an immediate consequence of Theorem 5.1. ■

In the following corollary, we verify the problem:

$$(5.2) \quad \text{Find } x^* \in S := \bigcap_{k \geq 0} A_k^{-1}(0),$$

where $(A_k)_{k \geq 0} : H \rightarrow 2^H$ is an infinite countable family of maximal monotone operators with $\bigcap_{k \geq 0} A_k^{-1}(0) \neq \emptyset$.

Corollary 5.3. *Suppose $(\theta_k) \subset (0, \infty)$ and $(\lambda_n) \subset (0, \infty)$ such that $\sum_{n=1}^\infty \lambda_n < \infty$ and $(\alpha_n), (\beta_n)$, and (γ_n) are sequences in $(0, 1)$, which satisfy (5.1) and (e_n) is a sequence in H such that $\sum_{n=1}^\infty \|e_n\| < \infty$ or $\lim_n \frac{\|e_n\|}{\alpha_n} = 0$. If $(u, x_1) \in H^2$ and (x_n) is generated by:*

$$(5.3) \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \left(\left(\sum_{i=1}^n \lambda_i \right)^{-1} \sum_{k=1}^n \lambda_k J_{\theta_k}^{A_k} x_n \right) + e_n,$$

then (x_n) converges strongly to a solution of (5.2).

Proof. Set $T_n = \sum_{k=1}^n \eta_n^k J_{\theta_k}^{A_k}$, $\forall n \in \mathbb{N}$, where $\eta_n^k = \begin{cases} \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}, & k \leq n, \\ 0, & k > n. \end{cases}$

Since for each $k \in \mathbb{N}$, $J_{\theta_k}^{A_k}$ with $\theta_k \in (0, \infty)$ is a firmly nonexpansive mapping, with a similar proof of part (3) of Proposition 4.1, (T_n) is a strongly nonexpansive sequence. Thus, Theorem 3.3 and Lemma 5.2 complete the proof. ■

6. OTHER APPLICATIONS

In this section, we give some applications in minimization and subgradient projection.

6.1. Constrained minimization and convex feasibility

Let $(D_k)_{k \geq 0}$ be an infinite (or finite, with $D_k = H$, for k large enough) countable family of convex and closed subset of H such that $\bigcap_{k \geq 0} D_k \neq \emptyset$, and consider the following convex feasibility problem

$$(6.1) \quad \text{Find } x^* \in \bigcap_{k \geq 0} D_k.$$

Corollary 6.1. *Suppose $(\lambda_n) \subset (0, \infty)$ such that $\sum_{n=1}^\infty \lambda_n < \infty$ and $(\alpha_n), (\beta_n)$, and (γ_n) are sequences in $(0, 1)$ such that satisfy (5.1) and (e_n) is a sequence in H such that $\sum_{n=1}^\infty \|e_n\| < \infty$ or $\lim_n \frac{\|e_n\|}{\alpha_n} = 0$. If $(u, x_1) \in H^2$ and (x_n) is generated by:*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \left(\left(\sum_{i=1}^n \lambda_i \right)^{-1} \sum_{k=1}^n \lambda_k P_{D_k} \right) x_n + e_n,$$

where P is the metric projection onto D_k , then (x_n) converges strongly to a solution of (6.1).

Proof. Set $T_n = \sum_{k=1}^n \theta_n^k P_{D_k}$, $\forall n \in \mathbb{N}$, where $\theta_n^k = \begin{cases} \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}, & k \leq n, \\ 0, & k > n. \end{cases}$

Since for each $k \in \mathbb{N}$, P_{D_k} is a firmly nonexpansive mapping with $F(P_{D_k}) = D_k$, thus with a similar proof of part (3) of Proposition 4.1 we can get that (T_n) is a strongly nonexpansive sequence. Hence, the result is deduced from Theorem 3.3 and Lemma 5.2. ■

6.2. Subgradient projection methods

Assume that $lev_{\leq 0}(\phi) := \{x \in H \mid \phi(x) \leq 0\} \neq \emptyset$, where $\phi : H \rightarrow \mathbb{R}$ is a continuous convex function. Recall that a subgradient projection relative to ϕ is a map $T_{(\phi)} : H \rightarrow H$ with

$$T_{(\phi)}(x) := \begin{cases} x - \frac{\phi(x)}{|\phi'(x)|^2} \phi'(x) & \text{if } \phi(x) > 0, \\ x & \text{otherwise,} \end{cases}$$

where $\phi' : H \rightarrow H$ is a selection of $\partial\phi : H \rightarrow 2^H$ (the Fenchel subdifferential of ϕ) in the sense that $\phi'(x) \in \partial\phi(x)$, for all $x \in H$. Clearly, $T_{(\phi)}$ is a firmly quasi-nonexpansive map and $F(T_{(\phi)}) = \text{lev}_{\leq 0}(\phi)$. We can see some applications of subgradient projection techniques in [2, 19].

The following corollary is finding x^* in nonempty, closed, and convex subset $S = \{x \in H \mid \sup_{k \geq 0} \phi_k(x) \leq 0\} = \bigcap_{k \geq 0} F(T_{(\phi_k)})$ of H , where $(\phi_k)_{k \geq 0} : H \rightarrow \mathbb{R}$ is an infinite (or finite) countable family of continuous and convex functions such that, for each $k \geq 0$, $\text{lev}_{\leq 0}(\phi_k) \neq \emptyset$.

Corollary 6.2. *Suppose $(\lambda_n) \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $(\alpha_n), (\beta_n)$, and (γ_n) are sequences in $(0, 1)$ which satisfy (5.1). If $(u, x_1) \in H^2$ and (x_n) is generated by:*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \left(\left(\sum_{i=1}^n \lambda_i \right)^{-1} \sum_{k=1}^n \lambda_k T_{(\phi_k)} \right) x_n,$$

then (x_n) converges strongly to $x^* \in \bigcap_{k \geq 0} F(T_{(\phi_k)})$.

Proof. Set $T_n = \sum_{k=1}^n \theta_n^k T_{(\phi_k)}$, $\forall n \in \mathbb{N}$, where $\theta_n^k = \begin{cases} \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}, & k \leq n, \\ 0, & k > n. \end{cases}$

Since for each $k \in \mathbb{N}$, $T_{(\phi_k)}$ is a firmly quasi-nonexpansive mapping, by part (3) of Proposition 4.1, (T_n) is a strongly quasi-nonexpansive sequence. Hence, by setting $R_k = T_{(\phi_k)}$, Theorem 3.1 and Lemma 5.2 imply the interested result. ■

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REFERENCES

1. K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, *Nonlinear Anal.*, **67** (2007), 2350-2360.
2. H. H. Bauschke and P. L. Combettes, A weak-to-strong convergence principle for Fejer monotone methods in Hilbert space, *Math. Oper. Res.*, **26** (2001), 248-264.
3. O. A. Boikanyo and G. Morosanu, A proximal point algorithm converging strongly for general errors, *Optim. Lett.*, **4** (2010), 635-641.
4. R. E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston J. Math.*, **3** (1977), 459-470.

5. C. E. Chidume and C. O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, *J. Math. Anal. Appl.*, **318** (2006), 288-295.
6. K. Goebel and W. A. Kirk, Topics in metric fixed point theory, *Cambridge Studies in Advanced Mathematics*, vol. 28, Cambridge University Press, Cambridge, 1990.
7. B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.*, **73** (1967), 957-961.
8. P. E. Maingé, Convergence theorems for inertial KM-type algorithms, *J. Comput. Appl. Math.*, **219** (2008), 223-236.
9. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.
10. G. Morosanu, Nonlinear evolution equations and applications, *Mathematics and Its Applications*, Vol. 26, D. Reidel, Dordrecht, The Netherlands (1988).
11. S. Saejung, Halpern's iteration in Banach spaces, *Nonlinear Anal.*, **73** (2010), 3431-3439.
12. S. Saejung, A supplement to a regularization method for the proximal point algorithm, *J. Global Optim.*, **56** (2013), 121-129.
13. S. Saejung and P. Yotkaew, Approximation of zeros of inverse strongly mono-tone operators in Banach spaces, *Nonlinear Anal.*, **75** (2012), 742-750.
14. T. Suzuki, A sufficient and necessary condition for Halpern-type strong convergence to the fixed point of nonexpansive mapping, *Proc. Amer. Math. Soc.*, **135** (2007), 99-106.
15. F. Wang and H. Cui, On the contraction proximal point algorithms with multi parameters, *J. Glob. Optim.*, **54** (2012), 485-491.
16. R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.*, **58** (1992), 486-491.
17. H. K. Xu, Another control condition in an iterative method for nonexpansive mappings, *Bull. Austral. Math. Soc.*, **65** (2002), 109-113.
18. H. K. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.*, **66** (2002), 240-256.
19. I. Yamada and N. Ogura, Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings, *Numer. Funct. Anal. Optim.*, **25** (2004), 619-655.
20. Y. Yao and N. Shazad, Strong convergence of a proximal point algorithm with general errors, *Optim. lett.*, **6** (2012), 621-628.

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