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ON THE UNIQUENESS PROBLEMS OF ENTIRE FUNCTIONS AND THEIR DIFFERENCE OPERATORS

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Abstract. In this paper, the uniqueness problems of entire functions and their difference operators are investigated. It is shown that if a finite order entire function f shares $0, \alpha$ CM with its difference operator $\Delta_\eta f(z) = f(z+\eta) - f(z)$, then $\Delta_\eta f \equiv f$, where α is an entire function with order less than f. The research results also include a difference analogue of Brück conjecture, and extend some results in Chen-Yi *Results Math.*, **63** (2013), 557-565).

1. Introduction and Main Results

Let $f(z)$ be a non-constant meromorphic function in the complex plane. We adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [7, 11, 16]. In addition, we use notations $\sigma(f),\ \lambda(f)$ to denote the order and the exponent of convergence of the sequence of zeros of f respectively. It will be convenient to let E denote any set of finite logarithmic measure, not necessarily the same at each occurrence.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let a be a complex number in the extended plane. We say that f and g share a CM, provided that f and g have the same a-points with the same multiplicities. Similarly, we say that f and g share a IM, provided that f and g have the same a-points ignoring multiplicities.

Mues and Steinmetz [14] proved that if a non-constant entire function f shares two distinct finite values IM with its derivative f', then $f \equiv f'$. In general, this theorem is false, if f and f' share only one value CM (see [16], p. 386). Especially, Brück posed the well-known conjecture.

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Conjecture. [1]. Let f be a non-constant entire function of hyper-order $\sigma_2(f) < \infty$, where $\sigma_2(f)$ is not a positive integer. If f and f' share one finite value a CM, then $f - a \equiv c(f' - a)$ for some nonzero constant.

The conjecture has been verified in the special cases when $a = 0$ or $N(r, f' = 0) = S(r, f)$ (see [1]), or when f is of finite order (see [5], [15]). But the conjecture is still an open question until now.

Recently, many authors [8, 9, 12] started to consider the uniqueness of meromorphic functions sharing values with their shifts or their difference operators. Heittokangas et al. proved the following result.

Theorem A. [8]. Let f be a meromorphic function of $\sigma(f) < 2$, and η be a non-zero constant. If $f(z)$ and $f(z+\eta)$ share the finite value a and ∞ CM, then

$$\frac{f(z+\eta) - a}{f(z) - a} = \tau$$

for some constant τ .

In [8], Heittokangas et al. gave the example $f(z) = e^{z^2} + 1$ which shows that $\sigma(f) < 2$ can't be relaxed to $\sigma(f) \leq 2$.

It is known that $\Delta_\eta f(z) = f(z+\eta) - f(z)$ is regarded as the difference counterpart of $f'(z)$. Considering the difference analogue of the Brück conjecture, Chen and Yi [2] obtained the following result.

Theorem B. [2]. Let f be a finite order transcendental entire function which has a finite Borel exceptional value a, and let η be a constant such that $f(z+\eta) \not\equiv f(z)$. If f and $\Delta_\eta f$ share a CM, then

$$a = 0 \quad and \quad \frac{f(z+\eta) - f(z)}{f(z)} = c$$

for some non-zero constant c.

When the condition "f has a finite Borel exceptional value " is omitted, They also obtained the following result.

Theorem C. [2]. Let f be a transcendental entire function such that its order $\sigma(f)$ is not an integer or infinite, and let η be a constant such that $f(z+\eta) \not\equiv f(z)$. If f and $\Delta_\eta f$ share two distinct finite values a, b CM, then $f \equiv \Delta_\eta f$.

Regarding Theorems B and C, it is natural to ask, what can be said if a non-constant entire function f shares a small and finite order entire function α with $\Delta_\eta f$? For the case $\sigma(\alpha) < 1$, Li and Yi obtained the following result.

Theorem D. [13]. Let f be a non-constant entire function of finite order, η be a non-zero constant, and let $\alpha(\not\equiv 0)$ be an entire function such that $\sigma(\alpha) < 1$ and $\lambda(f - \alpha) < \sigma(f)$. Then $f - \alpha$ and $\Delta_\eta^n f - \alpha$ share 0 CM, if and only if

$$f(z) = \alpha(z) + B(\Delta_\eta^n \alpha(z) - \alpha(z))e^{Az} \quad and \quad \Delta_\eta^{2n}\alpha(z) - \Delta_\eta^n \alpha(z) \equiv 0,$$

where A, B are nonzero constants and $e^{A\eta} = 1$.

In this paper, we continue to investigate the above question and obtain the following results, which extend Theorems B–D.

Theorem 1.1. *Let f be a non-constant entire function of finite order, η be a non-zero constant, and let $\alpha(\not\equiv 0)$ be an entire function such that $\sigma(\alpha) < \sigma(f)$ and $\lambda(f-\alpha) < \sigma(f)$. If f and $\Delta_\eta f$ share α CM, then $\sigma(f) = 1$.*

From Theorem 1.1 and Theorem D, we can obtain the following corollary.

Corollary 1.1. *Let f, α satisfy the hypothesis of Theorem 1.1. If f and $\Delta_\eta f$ share α CM, then*

$$f(z) = \alpha(z) + B(\Delta_\eta \alpha(z) - \alpha(z))e^{Az} \quad \textit{and} \quad \Delta_\eta^2 \alpha(z) - \Delta_\eta \alpha(z) \equiv 0,$$

where A, B are non-zero constants and $e^{A\eta} = 1$.

Theorem 1.2. *Let f be a non-constant entire function of finite order, η be a non-zero constant, and let $\alpha(\not\equiv 0)$ be an entire function of $\sigma(\alpha) < \sigma(f)$. If f and $\Delta_\eta f$ share $0, \alpha$ CM, then $f \equiv \Delta_\eta f$.*

By Lemma 2.4, we know that if a finite order non-constant entire function f shares 0 CM with its difference operator $\Delta_\eta f$, then $\sigma(f) \geq 1$. This deduces $\sigma(z) < \sigma(f)$. Hence by Theorem 1.2, we obtain the following result.

Corollary 1.2. *Let f be a non-constant entire function of finite order, and let η be a non-zero constant. If f and $\Delta_\eta f$ share $0, z$ CM, then $f \equiv \Delta_\eta f$.*

2. Lemmas

Lemma 2.1. [3]. *Let f be a meromorphic function of finite order σ, η be a non-zero constant. Let $\varepsilon > 0$ be given, then there exists a set $E \subset (1, \infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin E \bigcup [0,1]$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.2. [16]. *Let $f_j (j = 1, \cdots, n+1)$ and $g_j (j = 1, \cdots, n)$ be entire functions such that*

(*i*) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$,

(*ii*) *The order of f_j is less than the order of e^{g_k} for $1 \leq j \leq n+1, 1 \leq k \leq n$; And furthermore, the order of f_j is less than the order of $e^{g_h - g_k}$ for $n \geq 2$ and $1 \leq j \leq n+1, 1 \leq h < k \leq n$.*

Then $f_j(z) \equiv 0 (j = 1, \cdots n+1)$.

Lemma 2.3. [4]. *Let f be a meromorphic function with $\sigma(f) < 1$, η be a non-zero constant. Then for any given $\varepsilon > 0$, and integers $0 \le j < k$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0, 1]$, we have*

$$\Big|\frac{\Delta_\eta^k f(z)}{\Delta_\eta^j f(z)}\Big| \le |z|^{(k-j)(\sigma(f)-1)+\varepsilon}.$$

Lemma 2.4. *Let f be a non-constant entire function of finite order and η be a non-zero constant. If f and $\Delta_\eta f$ share 0 CM, then $\sigma(f) \ge 1$.*

Proof. Since f and $\Delta_\eta f$ share 0 CM, we have

$$\frac{\Delta_\eta f}{f} = e^P, \tag{2.1}$$

where P is a polynomial. If $\sigma(f) < 1$, by (2.1) and Lemma 2.3, for any given $\varepsilon (0 < \varepsilon < 1 - \sigma(f))$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0, 1]$, we have

$$|e^{P(z)}| \le \Big|\frac{\Delta_\eta f(z)}{f(z)}\Big| \le r^{\sigma(f)-1+\varepsilon} \to 0, \ (r \to \infty).$$

This is a contradiction. So $\sigma(f) \ge 1$. ∎

Remark 2.1. *The following examples show that the result in Lemma 2.4 is the best possible.*

Example 2.1. *Let $f(z) = e^z, \eta = \log 2$, then f and $\Delta_\eta f$ share 0 CM. Here $\sigma(f) = 1$.*

Example 2.2. *Let $f(z) = \sin z, \eta = \pi$, then f and $\Delta_\eta f$ share 0 CM. Here $\sigma(f) = 1$.*

Lemma 2.5. [10]. *Let $\varphi(r)$ be a nondecreasing, continuous function on $\mathbb{R}^+$, and let $0 < \rho < \overline{\lim}_{r\to\infty} \frac{\log \varphi(r)}{\log r}$ and $H = \{r \in \mathbb{R}^+ : |\varphi(r)| \ge r^\rho\}$. Then*

$$\overline{\log dens} H = \overline{\lim_{r\to\infty}} \frac{\int_{H\bigcap[1,r]} \frac{1}{t} dt}{\log r} > 0.$$

Lemma 2.6. [3]. *Let f be a transcendental meromorphic function of finite order, and let η be a non-zero constant. Then*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r)$$

as $r \to \infty$, where ε is any given positive number.

Lemma 2.7. [6]. *Let $f(z)$ be a transcendental meromorphic function of finite order, k, j ($k > j \geq 0$) be integers. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, +\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0, 1]$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma(f)-1+\varepsilon)}.$$

3. Proofs of the Results

Proof of Theorem 1.1. By the Hadamard factorization theorem and $\lambda(f - \alpha) < \sigma(f)$, we get

$$f(z) = \alpha(z) + h(z)e^{P(z)}, \tag{3.1}$$

where $h(z)(\not\equiv 0)$ is an entire function, $P(z)$ is a polynomial such that

$$\sigma(h) = \lambda(h) = \lambda(f - \alpha) < \sigma(f) = \deg P. \tag{3.2}$$

Since $\Delta_\eta f$ and f share α CM, we have

$$\frac{\Delta_\eta f(z) - \alpha(z)}{f(z) - \alpha(z)} = e^{Q(z)}, \tag{3.3}$$

where $Q(z)$ is a polynomial. By (3.2) and (3.3), we get

$$\deg Q \leq \deg P. \tag{3.4}$$

Substituting (3.1) into (3.3), we have

$$h(z + \eta)e^{P(z+\eta)-P(z)} - h(z)e^{Q(z)} - h(z) = \Big(2\alpha(z) - \alpha(z + \eta)\Big)e^{-P(z)}. \tag{3.5}$$

Now we discuss the following two cases.

Case 1. $2\alpha(z) - \alpha(z + \eta) \equiv 0$. If $\sigma(\alpha) < 1$, then by Lemma 2.1, for any given $\varepsilon(0 < 2\varepsilon < 1 - \sigma(\alpha))$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0, 1]$, we have

$$2 = \left|\frac{\alpha(z + \eta)}{\alpha(z)}\right| \leq \exp\{r^{\sigma(\alpha)-1+\varepsilon}\} \to 0, \ (r \to \infty).$$

This is a contradiction. Hence we have

$$\sigma(\alpha) \geq 1. \tag{3.6}$$

Next we discuss the following three subcases.

Subcase 1.1. $1 \le \deg Q < \deg P$. By (3.5), we get

$$\frac{h(z+\eta)}{h(z)} e^{P(z+\eta)-P(z)} - 1 = e^{Q(z)}. \tag{3.7}$$

By (3.7), we know that $\frac{h(z+\eta)}{h(z)}$ is a non-zero entire function. Then by Lemma 2.1, for any given $\varepsilon(0 < 2\varepsilon < \deg P - \sigma(h))$, there exists a set $E \subset (1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0,1]$, we have

$$\Big|\frac{h(z+\eta)}{h(z)}\Big| \le \exp\{r^{\sigma(h)-1+\varepsilon}\}. \tag{3.8}$$

Since $\frac{h(z+\eta)}{h(z)}$ is an entire function, by (3.8), we get for all z satisfying $|z| = r \notin E_2 \bigcup [0,1]$,

$$T\Big(r, \frac{h(z+\eta)}{h(z)}\Big) = m\Big(r, \frac{h(z+\eta)}{h(z)}\Big) \le r^{\sigma(h)-1+\varepsilon}.$$

Hence we get

$$\sigma\Big(\frac{h(z+\eta)}{h(z)}\Big) \le \sigma(h) - 1 + \varepsilon < \deg P - 1. \tag{3.9}$$

If $\deg Q < \deg P - 1$, since $\deg(P(z+\eta) - P(z)) = \deg P - 1$, by (3.8), we obtain that the order of the left side of (3.7) is $\deg P - 1$, and the order of the right side of (3.7) is $\deg Q$, which is less than $\deg P - 1$. This is a contradiction. If $\deg Q = \deg P - 1$, by (3.9), we get

$$\begin{aligned} \lambda\Big(\frac{h(z+\eta)}{h(z)} e^{P(z+\eta)-P(z)}\Big) &= \lambda\Big(\frac{h(z+\eta)}{h(z)}\Big) \le \sigma\Big(\frac{h(z+\eta)}{h(z)}\Big) \\ &< \deg P - 1 = \sigma\Big(\frac{h(z+\eta)}{h(z)} e^{P(z+\eta)-P(z)}\Big). \end{aligned} \tag{3.10}$$

By (3.10), we know that 0 is a Borel exceptional value of $\frac{h(z+\eta)}{h(z)} e^{P(z+\eta)-P(z)}$. But by (3.7), we know that 1 is also a Borel exceptional value of $\frac{h(z+\eta)}{h(z)} e^{P(z+\eta)-P(z)}$. This contradicts that $\frac{h(z+\eta)}{h(z)} e^{P(z+\eta)-P(z)}$ is an entire function.

Subcase 1.2. $\deg Q = \deg P \ge 1$. By (3.7) and (3.9), we obtain that the order of the left side of (3.7) is $\deg P - 1$, and the order of the right side of (3.7) is $\deg P$. This is a contradiction.

Subcase 1.3. Q is a constant. Then by (3.7) we get

(3.11) $$\frac{h(z+\eta)}{h(z)}e^{P(z+\eta)-P(z)} = c+1,$$

where $c(=e^Q)$ is a non-zero constant. Since $\frac{h(z+\eta)}{h(z)}$ is a non-zero entire function, we get $c \neq -1$. If $\deg P > 1$, then by (3.9) and $\deg(P(z+\eta)-P(z)) = \deg P - 1 \geq 1$, we know that $\sigma\Big(\frac{h(z+\eta)}{h(z)}e^{P(z+\eta)-P(z)}\Big) \geq 1$, but $\sigma(c+1) = 0$. This is a contradiction. So $\deg P \leq 1$. Then combining (3.2) and (3.6), we get $\sigma(f) \leq \sigma(\alpha)$. This contradicts the hypothesis of Theorem 1.1.

Case 2. $2\alpha(z)-\alpha(z+\eta) \not\equiv 0$. If $\deg Q < \deg P$, then by (3.2) we obtain that the order of the left side of (3.5) is less than $\deg P$, and the order of the right side of (3.5) is $\deg P$. This is a contradiction. Hence by (3.4) and (3.2), we get $\deg Q = \deg P \geq 1$. Set

$$P(z) = a_m z^m + \cdots + a_0, \quad Q(z) = b_m z^m + \cdots + b_0,$$

where $a_m(\neq 0), \cdots, a_0, b_m(\neq 0), \cdots, b_0$ are constants, $m \geq 1$ is an integer. Next we discuss the following two subcases.

Subcase 2.1. $a_m + b_m \neq 0$. By (3.5), we get

(3.12) $$\Big(2\alpha(z)-\alpha(z+\eta)\Big)e^{-P(z)} + h(z)e^{Q(z)} = h(z+\eta)e^{P(z+\eta)-P(z)} - h(z).$$

Since

$$\deg(P(z+\eta)-P(z)) = m-1, \quad \sigma(h) < m, \quad \sigma(\alpha) < m,$$

we obtain that

$$\sigma(2\alpha(z)-\alpha(z+\eta)) < m, \quad \sigma(h(z+\eta)e^{P(z+\eta)-P(z)} - h(z)) < m.$$

Note that $e^{-P(z)}$, $e^{Q(z)}$ and $e^{Q(z)+P(z)}$ are of regular growth, by Lemma 2.2 and (3.12), we obtain that

$$2\alpha(z)-\alpha(z+\eta) \equiv 0, \quad h(z) \equiv 0.$$

This is absurd.

Subcase 2.2. $a_m + b_m = 0$. By (3.12), we get

(3.13) $$e^{-P(z)}\Big(2\alpha(z)-\alpha(z+\eta)+h(z)e^{Q(z)+P(z)}\Big) = h(z+\eta)e^{P(z+\eta)-P(z)} - h(z).$$

If $2\alpha(z)-\alpha(z+\eta)+h(z)e^{Q(z)+P(z)} \not\equiv 0$, then by

$$\sigma(\alpha) < m, \quad \sigma(h) < m, \quad \deg(P(z+\eta)-P(z)) = m-1,$$

we know that the order of the left side of (3.13) is m, and the order of the right side of (3.13) is less than m. This is a contradiction. If $2\alpha(z)-\alpha(z+\eta)+h(z)e^{Q(z)+P(z)} \equiv 0$, then by (3.13), we get

$$\frac{h(z+\eta)}{h(z)}e^{P(z+\eta)-P(z)} \equiv 1. \tag{3.14}$$

By (3.14), we know that $\frac{h(z+\eta)}{h(z)}$ is a non-zero entire function. Then using the same argument as that of subcase 1.1, we get

$$\sigma\Big(\frac{h(z+\eta)}{h(z)}\Big) < m-1.$$

Since $\deg(P(z+\eta)-P(z)) = m-1$, we get

$$\sigma\Big(\frac{h(z+\eta)}{h(z)}e^{P(z+\eta)-P(z)}\Big) = m-1.$$

Then by (3.14), we get $m = 1$. Hence by (3.2) we get $\sigma(f) = 1$. ■

Proof of Theorem 1.2. Since $\Delta_\eta f$ and f share $0, \alpha$ CM, we have

$$\frac{\Delta_\eta f(z)}{f(z)} = e^{P(z)}, \tag{3.15}$$

$$\frac{\Delta_\eta f(z)-\alpha(z)}{f(z)-\alpha(z)} = e^{Q(z)}, \tag{3.16}$$

where $P(z), Q(z)$ are polynomials of degree $\max\{\deg P, \deg Q\} \le \sigma(f)$. By (3.15), Lemma 2.1 and Lemma 2.4, for any given $\varepsilon > 0$, there exists a set $E \subset (1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin E\bigcup[0,1]$, we have

$$|e^{P(z)}| \le \Big|\frac{f(z+\eta)}{f(z)}\Big| + 1 \le 2\exp\{r^{\sigma(f)-1+\varepsilon}\}.$$

Hence we get

$$\deg P \le \sigma(f)-1 < \sigma(f). \tag{3.17}$$

By (3.15) and (3.16), we get

$$(e^P - e^Q)f = (1-e^Q)\alpha. \tag{3.18}$$

Now we discuss the following two cases.

Case 1. $\deg Q < \sigma(f)$. If $e^{P(z)} - e^{Q(z)} \not\equiv 0$, by (3.17), we get $\sigma(e^P - e^Q) < \sigma(f)$. So $\sigma((e^P - e^Q)f) = \sigma(f)$. But $\sigma((1 - e^Q)\alpha) < \sigma(f)$. This is a contradiction. If $e^{P(z)} - e^{Q(z)} \equiv 0$, by (3.18), we get $e^{Q(z)} \equiv 1$. Then by (3.16), we get $\Delta_\eta f \equiv f$.

Case 2. $\deg Q = \sigma(f)$. Then by (3.17), we get

$$\deg P \leq \deg Q - 1. \tag{3.19}$$

Differentiating (3.18) we get

$$e^P(P'f + f') - e^Q(Q'f + f' - Q'\alpha - \alpha') - \alpha' = 0. \tag{3.20}$$

Set $F = \Delta_\eta f$, then by (3.15), (3.16) and (3.20), we get

$$(P' - Q')Ff + \alpha Q'(F + f) + \alpha'(F - f) - \alpha FP' - \alpha F\frac{f'}{f} + \alpha f' - \alpha^2 Q' = 0. \tag{3.21}$$

By (3.15) we get

$$F'f - Ff' - fFP' = 0. \tag{3.22}$$

Then combining (3.21) and (3.22), we get

$$(P' - Q')Ff + \alpha Q'(F + f) + \alpha'(F - f) - \alpha(F' - f') - \alpha^2 Q' = 0. \tag{3.23}$$

For any given $\varepsilon(0 < 2\varepsilon < \min\{1, \sigma(f) - \sigma(\alpha)\})$, let

$$H = \{r : \log M(r, f) \geq r^{\sigma(f)-\varepsilon}\},$$

then by Lemma 2.5, we have $\overline{\log dens} H > 0$. Hence for the point z_r satisfying $|z_r| = r \in H$ and $|f(z_r)| = M(r, f)$, we have

$$|f(z_r)| \geq \exp\{r^{\sigma(f)-\varepsilon}\}. \tag{3.24}$$

By Lemma 2.6 and Lemma 2.7, for the above given $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0, 1]$, we have

$$\left|\frac{F'(z)}{F(z)}\right| \leq r^{\sigma(f)-1+\varepsilon}, \quad \left|\frac{f'(z)}{f(z)}\right| \leq r^{\sigma(f)-1+\varepsilon}. \tag{3.25}$$

On the other hand, for the above given $\varepsilon > 0$, there exists $r_1 > 0$, such that for all z satisfying $|z| = r > r_1$, we have

$$|\alpha(z)| \leq \exp\{r^{\sigma(\alpha)+\varepsilon}\}, \quad |\alpha'(z)| \leq \exp\{r^{\sigma(\alpha)+\varepsilon}\}, \quad |\alpha^2(z)| \leq \exp\{r^{\sigma(\alpha)+\varepsilon}\}, \tag{3.26}$$

$$|e^{-P(z)}| \leq \exp\{r^{\deg P+\varepsilon}\} \leq \exp\{r^{\sigma(f)-1+\varepsilon}\}, \tag{3.27}$$

$$|Q'(z)| \leq r^{\sigma(f)}. \tag{3.28}$$

By (3.19), (3.23)–(3.28), for the point z_r satisfying $|z_r| = r \in H - [0,1] - E$ and $|f(z_r)| = M(r,f)$, we have

$$\begin{aligned}
0 < |P'(z_r) - Q'(z_r)| &\leq \Big(|\alpha(z_r)||Q'(z_r)| + |\alpha'(z_r)|\Big)\Big(\frac{1}{|f(z_r)|} + \frac{1}{|F(z_r)|}\Big) \\
&\quad + |\alpha(z_r)|\Big(\Big|\frac{F'(z_r)}{F(z_r)}\Big|\frac{1}{|f(z_r)|} + \Big|\frac{f'(z_r)}{f(z_r)}\Big|\frac{1}{|F(z_r)|}\Big) \\
&\quad + |\alpha^2(z_r)||Q'(z_r)|\frac{1}{|F(z_r)||f(z_r)|} \\
&\leq Mr^{\sigma(f)}\exp\{r^{\sigma(\alpha)+\varepsilon} + r^{\sigma(f)-1+\varepsilon} - r^{\sigma(f)-\varepsilon}\} \\
&\to 0, (r \to \infty),
\end{aligned}$$

where $M > 0$ is a constant. This is a contradiction. ■

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