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SOME GENERALIZATIONS OF PETRYSHYN'S THEOREM AND ALTMAN'S THEOREM

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Abstract. In this paper, some new fixed point theorems for semi-closed 1-set-contractive operators of functional type are obtained, which generalize the famous Petryshyn's theorem and Altman's theorem by replacing the norm with some classes of functionals.

1. Introduction and Preliminaries

Topological degree theory is an important tool in studying nonlinear functional analysis. It can be used to estimate the number of solutions of an equation and is closely connected to the existence of fixed points of nonlinear operators. It has applications in many other problems, such as complementarity problems, differential equations, differential inclusions, and so on ([1-3]).

On the other hand, Li first introduced the concept of semi-closed 1-set-contractive mappings ([4]). The class of 1-set-contractive operators is a wide class of operators that includes completely continuous operators, strict set-contractive operators, condensing operators, non-expansive maps, semi-contractive maps, LANE maps and others ([5]). For the relevant literatures on 1-set-contractive operators and its generalizations, please refer to [6-14].

In this paper, some new fixed point theorems for semi-closed 1-set-contractive operators of functional type are obtained, which generalize the famous Petryshyn's theorem and Altman's theorem by replacing the norm with some classes of functionals. Our results remain valid for many classes of operators mentioned above.

We will now give some preliminaries that will be used in the sequel.

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Let E be a real Banach space with norm $\|\cdot\|$, Ω be a nonempty bounded open subset of E and P be a cone in E. Denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and boundary of Ω in E, respectively, and θ the zero element of E. Now we state the well-known Petryshyn's theorem and Altman's theorem as follows.

Theorem 1.1. ([1-3]). *Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $\theta \in \Omega$. Suppose that $A : \Omega \to P$ is completely continuous and one of the following conditions is satisfied:*

(i) $(Rothe)$ $\|Ax\| \leq \|x\|, \forall x \in \partial\Omega;$

(ii) $(Petryshyn)$ $\|Ax\| \leq \|Ax - x\|, \forall x \in \partial\Omega;$

(iii) $(Altman)$ $\|Ax - x\|^2 \leq \|Ax\|^2 - \|x\|^2, \forall x \in \partial\Omega.$

Then $deg(I - A, \Omega, \theta) = 1$, that is, A has at least one fixed point in $\overline{\Omega}$.

Definition 1.1. ([4]). Let $A : \Omega \to E$ be a 1-set-contractive operator. Then A is said to be a semi-closed 1-set-contractive operator, if $I - A$ is a closed operator.

Definition 1.2. ([9]). $\rho : P \to R$ is said to be a convex functional if $\rho(tx + (1 - t)y) \leq t\rho(x) + (1 - t)\rho(y)$ for all $x, y \in P$ and $t \in [0, 1]$.

Definition 1.3. ([10]). $\varphi : P \to R$ is said to be a sub-linear functional if $\varphi(tx) \leq t\varphi(x)$, for all $x \in P$ and $t \in [0, 1]$.

Definition 1.4. ([13]). $\rho : P \to R$ is said to satisfy the Property (A), if $\rho(tx) < \rho(x)$ for all $x \neq \theta$ and $t \in [0, 1)$.

It is obvious that the Property (A) is equivalent to the following one: $\rho(tx) > \rho(x)$ for all $x \neq \theta$ and $t \in (1, +\infty)$.

It is easy to see that if $\rho(x) = ||x||$, then ρ is a convex functional, and in this case, since $\rho(tx) = \rho(tx + (1 - t)\theta) \leq t\rho(x) + (1 - t)\rho(\theta) = t\rho(x)$, ρ is also a sub-linear functional. Furthermore, ρ satisfies the Property (A) .

2. Main Results

Theorem 2.1. *Let P be a cone in a real Banach space E, Ω be a bounded open subset of E with $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ be a semi-closed 1-set-contractive operator, and $\rho : E \to [0, +\infty)$ be a continuous functional satisfying the Property (A). Suppose that there exist $\alpha \geq 0$, $\lambda \geq 1$ and $\mu \geq 1$, such that for all $x \in P \cap \partial\Omega$, the following holds*

$$(H_1)\ (\rho(Ax - \mu x))^\alpha \geq \lambda(\rho(Ax))^\alpha.$$

Then $Ax = \mu x (\mu \geq 1)$ has a solution in $P \cap \overline{\Omega}$.

Proof. Without loss of generality, suppose that $Ax = \mu x (\mu \geq 1)$ has no solution on $P \cap \partial\Omega$ (otherwise, the conclusion holds).

Let $H(t, x) = \dfrac{1-t}{\mu} Ax$, for $(t, x) \in [0, 1] \times (P \cap \overline{\Omega})$. Then it is easy to see that $H : [0, 1] \times (P \cap \overline{\Omega}) \to P$ is a semi-closed 1-set-contractive operator. Suppose that there exist $t_0 \in [0, 1]$ and $x_0 \in P \cap \partial\Omega$, such that $x_0 = H(t_0, x_0) = \dfrac{1-t_0}{\mu} Ax_0$. Note that $t_0 \neq 0$ (otherwise, $Ax_0 = \mu x_0$, which is in contradiction to our assumption), and $t_0 \neq 1$ (otherwise, $x_0 = \theta$, which is in contradiction to $\theta \in \Omega$). Hence $Ax_0 = \dfrac{Ax_0 - \mu x_0}{t_0}$, where $t_0 \in (0, 1)$.

Since ρ satisfies the Property (A), for $t_0 \in (0, 1)$, we obtain

$$\begin{aligned} \lambda(\rho(Ax_0))^\alpha &\geq (\rho(Ax_0))^\alpha = (\rho(\frac{Ax_0 - \mu x_0}{t_0}))^\alpha \\ &= (\rho(\frac{1}{t_0}(Ax_0 - \mu x_0)))^\alpha > (\rho(Ax_0 - \mu x_0))^\alpha, \end{aligned} \tag{2.1}$$

which contradicts with (H_1). Thus by the homotopy invariance property of the fixed point index, we have

$$\begin{aligned} i(\frac{A}{\mu}, P \cap \Omega, P) &= i(H(0, \cdot), P \cap \Omega, P) \\ &= i(H(1, \cdot), P \cap \Omega, P) = i(\theta, P \cap \Omega, P) = 1. \end{aligned} \tag{2.2}$$

By the solution property of the fixed point index, we obtain that $Ax = \mu x (\mu \geq 1)$ has a solution in $P \cap \Omega$. This completes the proof.

Remark 2.1. Theorem 2.1 generalizes the famous Petryshyn's theorem.

Remark 2.2. The conclusion of Theorem 2.1 still holds if we assume that ρ is a convex or sub-linear functional, while the other conditions remain the same.

Lemma 2.1. *The following inequality holds*

$$(x-1)^\alpha < \lambda x^\alpha - 1, \text{ for all } x > 1,$$

where $\alpha \geq 1, \lambda > 1$ (*or* $\alpha > 1, \lambda \geq 1$).

Proof. Consider the function defined by

$$f(x) = (x-1)^\alpha - \lambda x^\alpha + 1, \ x \in (1, +\infty).$$

Since $f'(x) = \alpha(x-1)^{\alpha-1} - \lambda\alpha x^{\alpha-1} = \alpha((x-1)^{\alpha-1} - \lambda x^{\alpha-1}) < 0$, $f(x)$ is a strictly monotone decreasing function in $(1, +\infty)$. Therefore, $f(x) < f(1) = 1 - \lambda \leq 0$, that is,

$$(x-1)^\alpha < \lambda x^\alpha - 1.$$

Theorem 2.2. *Let P be a cone in a real Banach space E, Ω be a bounded open subset of E with $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ be a semi-closed 1-set-contractive operator, and $\rho : E \to [0, +\infty)$ be a continuous sub-linear functional with $\rho(\theta) = 0$ and $\rho(x) > 0$ for all $x \neq \theta$. Suppose that there exist $\alpha \geq 1$, $\lambda > 1$ (or $\alpha > 1$, $\lambda \geq 1$) and $\mu \geq 1$, such that for all $x \in P \cap \partial\Omega$, the following holds*

$$(H_2)\ (\rho(Ax - \mu x))^\alpha \geq \lambda(\rho(Ax))^\alpha - (\rho(\mu x))^\alpha.$$

Then $Ax = \mu x (\mu \geq 1)$ has a solution in $P \cap \overline{\Omega}$.

Proof. Without loss of generality, suppose that $Ax = \mu x (\mu \geq 1)$ has no solution on $P \cap \partial\Omega$ (otherwise, the conclusion holds).

Let $H(t, x) = \dfrac{t}{\mu} Ax$, for $(t, x) \in [0, 1] \times (P \cap \overline{\Omega})$. Then it is easy to see that $H : [0, 1] \times (P \cap \overline{\Omega}) \to P$ is a semi-closed 1-set-contractive operator. Suppose that there exist $t_0 \in [0, 1]$ and $x_0 \in P \cap \partial\Omega$, such that $x_0 = H(t_0, x_0) = \dfrac{t_0}{\mu} Ax_0$. Note that $t_0 \neq 0$ (otherwise, $x_0 = \theta$, which is in contradiction to $\theta \in \Omega$), and $t_0 \neq 1$ (otherwise, $Ax_0 = \mu x_0$, which is in contradiction to our assumption). Hence $\mu x_0 = t_0 Ax_0$, where $t_0 \in (0, 1)$.

By the sub-linearity of ρ, we have

$$\begin{aligned} (\rho(Ax_0 - \mu x_0))^\alpha &= (\rho(Ax_0 - t_0 Ax_0))^\alpha = (\rho((1 - t_0)Ax_0))^\alpha \\ &\leq ((1 - t_0)\rho(Ax_0))^\alpha = (1 - t_0)^\alpha (\rho(Ax_0))^\alpha, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \lambda(\rho(Ax_0))^\alpha - (\rho(\mu x_0))^\alpha &= \lambda(\rho(Ax_0))^\alpha - (\rho(t_0 Ax_0))^\alpha \\ &\geq \lambda(\rho(Ax_0))^\alpha - (t_0 \rho(Ax_0))^\alpha \\ &= (\lambda - {t_0}^\alpha)(\rho(Ax_0))^\alpha. \end{aligned} \tag{2.4}$$

Since $\theta \in \Omega$, we have $Ax_0 \neq \theta$ (otherwise, $x_0 = \dfrac{t_0}{\mu} Ax_0 = \theta \in P \cap \partial\Omega$, which is a contradiction). Noting that $\rho(x) > 0$ for all $x \neq \theta$, we have $\rho(Ax_0) > 0$. Hence, if follows from (H_2), (2.3) and (2.4) that $(1 - t_0)^\alpha \geq \lambda - {t_0}^\alpha$. Since $t_0 \in (0, 1)$, we obtain $(\dfrac{1}{t_0} - 1)^\alpha \geq \frac{\lambda}{{t_0}^\alpha} - 1$. This contradicts with Lemma 2.1. Thus by the homotopy invariance property of the fixed point index, we have

$$i(\frac{A}{\mu}, P \cap \Omega, P) = i(\theta, P \cap \Omega, P) = 1.$$

By the solution property of the fixed point index, we obtain that $Ax = \mu x (\mu \geq 1)$ has a solution in $P \cap \Omega$. This completes the proof.

Remark 2.3. Theorem 2.2 generalizes the famous Altman's theorem.

Remark 2.4. The conclusion of Theorem 2.2 still holds if we assume that ρ is a convex functional instead of a sub-linear functional, while the other conditions remain the same.

Lemma 2.2. ([13]). *Suppose that $\rho : P \to [0, +\infty)$ is a nonnegative continuous concave functional, and $\lim_{||x|| \mapsto +\infty} \rho(x) = +\infty$. Then ρ satisfies the Property (A).*

In [14], Sun and Zhang gave a fixed point theorem about cone expansion and compression of concave functional type. The following result presents a new theorem about concave functional which extends and complements the previous results.

Theorem 2.3. *Let P be a cone in a real Banach space E, Ω be a bounded open subset of E with $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ be a semi-closed 1-set-contractive operator, and $\rho : E \to [0, +\infty)$ be a continuous and concave functional with $\lim_{||x|| \to +\infty} \rho(x) = +\infty$. Suppose that there exist $\alpha \geq 0$, $\lambda \geq 1$ and $\mu \geq 1$, such that for all $x \in P \cap \partial\Omega$, the following holds*

$$(H_1{}') \ (\rho(Ax - \mu x))^{\alpha} \geq \lambda(\rho(Ax))^{\alpha}.$$

Then $Ax = \mu x (\mu \geq 1)$ has a solution in $P \cap \overline{\Omega}$.

Proof. We can easily prove the theorem by using Theorem 2.1 and Lemma 2.2.

REFERENCES

1. D. J. Guo and V. Lashmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Inc., Boston, New York, 1988.

2. A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, Berlin, 2003.

3. J. X. Sun, *Nonlinear Functional Analysis and its Application*, Science Press, 2008 (in Chinese).

4. G. Z. Li, The fixed-point index and the fixed-point theorems of 1-set-contraction mappings, *Proc. Amer. Math. Soc.*, **104** (1988), 1163-1170.

5. L. S. Liu, Approximation theorems and fixed point theorems for various class of 1-set-contractive mappings in Banach spaces, *Acta Math. Sinica (English Series)*, **17** (2001), 103-112.

6. C. X. Zhu, Generalizations of Krasnoselskii's theorem and Petryshyn's theorem, *Appl. Math. Lett.*, **19** (2006), 628-632.

7. C. X. Zhu and Z. B. Xu, Inequalities and solution of an operator equation, *Appl. Math. Lett.*, **21** (2008), 607-611.

8. C. X. Zhu, Research on some problems for nonlinear operators, *Nonlinear Anal. TMA*, **71** (2009), 4568-4571.

9. G. W. Zhang and J. X. Sun, A generalization of the cone expansion and compression fixed point theorem and applications, *Nonlinear Anal. TMA*, **67** (2007), 579-586.

10. D. R. Anderson and R. I. Avery, Fixed point theorem of cone expansion and compression of functional type, *Differ. Equa. Appl.*, **8** (2002), 1073-1083.

11. R. Avery, J. Henderson and D. O'Regan, Functional compression-expansion fixed point theorem, *Elec. J. Differ. Equa.*, **22** (2008), 1-12.

12. S. Y. Xu, New fixed point theorems for 1-set-contractive operators in Banach spaces, *Nonlinear Anal. TMA*, **67** (2007), 938-944.

13. X. L. Fei, F. Wang and Y. Chen, A cone expansion and compression fixed point theorem about a new functional and its applications (in Chinese), *Pure Appl. Math.*, **26** (2010), 546-553.

14. D. D. Sun, G. W. Zhang and T. Zhang, Fixed point theorem about cone expansion and compression of concave functional type, *Acta Math. Sinica (Chinese Series)*, **53** (2010), 847-852.

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