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$L(3,2,1)$-LABELING FOR THE PRODUCT OF A COMPLETE GRAPH AND A CYCLE

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Abstract. Given a graph $G = (V, E)$, a function f on V is an $L(3, 2, 1)$-labeling if for each pair of vertices u, v of G, it holds that $|f(u) - f(v)| \geq 4 - \text{dist}(u, v)$. $L(3, 2, 1)$-labeling number for G, denoted by $\lambda_{3,2,1}(G)$, is the minimum span of all $L(3, 2, 1)$-labeling f for G. In this paper, when $G = K_m \square C_n$ is the Cartesian product of the complete graph K_m and the cycle C_n, we show that the lower bound of $\lambda_{3,2,1}(G)$ is $5m - 1$ for $m \geq 3$, and the equality holds if and only if n is a multiple of 5. Moreover, we show that $\lambda_{3,2,1}(K_3 \square C_n) = 15$ when $n \geq 28$ and $n \not\equiv 0 \pmod 5$.

1. INTRODUCTION

A distance labeling problem for a graph $G = (V, E)$ is an effective assignment of labels on V in such a way that the labels assigned to each vertex satisfy certain distance constraints. A channel assignment problem is a motivation of the distance labeling problem of graphs. It is to find a proper assignment of channels to transmitters in a wireless network. If two transmitters in a broadcasting network, are located close to each other and the channels assigned on them are equal or have a very small difference, then there can be interference between them. The channels assigned to transmitters must satisfy certain distance constraints to avoid the existing interference between nearby transmitters. There should be a large difference between two channels assigned to two transmitters which are located very close. And two channels assigned to two transmitters which are closely located have a little difference. Because of the tremendous increases of the demand calls in the wireless networks, it is necessary to find an efficient assignment of channels to the network with minimum span.

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Hale [10] proposed a mathematical modeling for the distance constrained channel assignment problem. He considered a wireless communication network as a graph in such a way that the vertices are the transmitters and two vertices are adjacent if the corresponding transmitters are very closely located. After his work, Griggs and Yeh [9] introduced the distance two labeling, or $L(j,k)$-labeling, of graphs G.

A natural variant for distance two labeling problem is to expand the distance condition between transmitters. In some wireless network system, we need the constraints concerning the distance of not only one or two but also three or more. For a graph $G=(V,E)$ and nonnegative integers $j_0, j_1, \cdots, j_{d-1}$ with $d \le \text{diam}(G)$, an $L(j_0, j_1, \cdots, j_{d-1})$-*labeling* (or *distance* d *labeling*) is a function $f: V \to \{0,1,2\cdots\}$ such that for $u, v \in V$ with $\delta = \text{dist}(u,v)$, the labeling condition $|f(u)-f(v)| \ge j_{\delta-1}$ is satisfied. The *labeling number* $\lambda_{j_0,j_1,\cdots,j_{d-1}}(G)$ for G is the smallest integer N such that there is an $L(j_0, j_1, \cdots, j_{d-1})$-labeling $f: V \to [0,N]$. The *radio labeling* for G is the $L(d, d-1, \cdots, 2, 1)$-labeling f where $d = \text{diam}(G)$. In this case, the labeling condition between two vertices u and v is $|f(u)-f(v)| \ge d - \text{dist}(u,v)+1$. The *radio number* for G, denoted by $\text{rn}(G)$, is the minimum span of radio labelings for G. The classical work of the $L(j,k)$-labeling problems is the $L(2,1)$-labeling problem. In particular, $\lambda_{2,1}(G)$ is known as the λ-*number* of G and is denoted by $\lambda(G)$. For surveys of the $\lambda_{j,k}(G)$-labeling problem, including λ-number of graphs, see [4, 5, 7, 9, 18]. The most natural concepts of distance three labeling problem is the $L(3,2,1)$-labeling problem [11]. If $\text{diam}(G)=3$, then $\lambda_{3,2,1}(G) = \text{rn}(G)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* $G = G_1 \Box G_2 = (V,E)$ of G_1 and G_2 is the graph such that $V = V_1 \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 = v_1$ and $\{u_2, v_2\} \in E_2$, or $u_2 = v_2$ and $\{u_1, v_1\} \in E_1$. The *direct product* $G' = G_1 \times G_2 = (V', E')$ of G_1 and G_2 is the graph such that $V' = V = V_1 \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $\{u_1, v_1\} \in E_1$ and $\{u_2, v_2\} \in E_2$. In Figure 1, two examples of the Cartesian and direct product of paths P_3 and P_4 are given.

D. Liu and X. Zhu [16] completely determine the radio numbers for paths and cycles. D. Liu [14] treated the radio number for trees. D. Liu and M. Xie [15] determine the radio number for the square of paths and partially square of cycles. There are some results on the distance 3 labelings for graphs. Especially $\lambda_{1,1,1}(G)$ and $\lambda_{2,1,1}(G)$ are computed when G is a path, a cycle, a grid, a complete binary tree or a cube[1, 2, 3, 14, 16, 19]. One of the important problems on distance three labeling is to find the values of $\lambda_{3,2,1}(G)$ for classes of graphs G [6, 8, 12, 17].

Let $G = K_m \Box C_n$ be the Cartesian product of the complete graph K_m of order m and the cycle C_n of order n. The λ-number for $G = K_m \Box C_n$ was investigated in[13]. In this paper, we investigate $\lambda_{3,2,1}(K_m \Box C_n)$ when $m, n \ge 3$. We show that the lower bound of $\lambda_{3,2,1}(K_m \Box C_n)$ is $5m-1$ for $m \ge 3$ and it is optimal only when n is a multiple of 5. Moreover we show that $\lambda_{3,2,1}(K_3 \Box C_n) = 15$ when $n \ge 28$ and $n \not\equiv 0 \pmod 5$.

2. Some Lemmas and Main Theorems

From now on, we assume that $m, n \geq 3$. Let $G = K_m \Box C_n = (V, E)$ be the Cartesian product of a complete graph K_m of order $m \geq 3$ and a path C_n of order n. Let $V = \{(i,j) | 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ and two vertices (i,j), (i',j') are adjacent if $i' = i$ and $j' \equiv j \pm 1 \pmod{n}$, or $i' \neq i$ and $j' = j$.

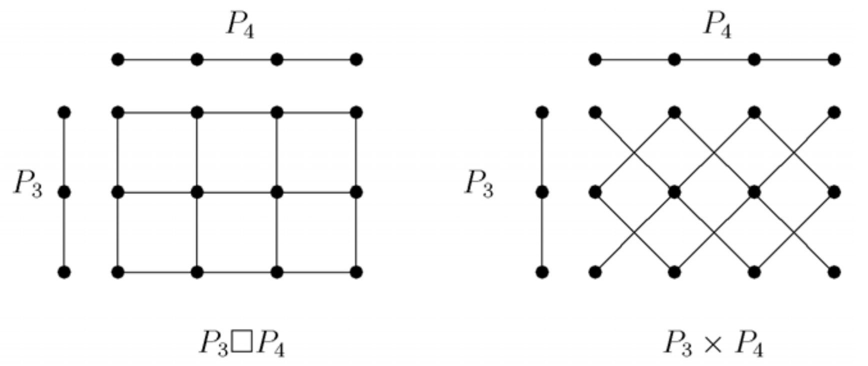

Fig. 1. $P_3 \Box P_4$ and $P_3 \times P_4$

Define $T_j = \{(i,j) | 0 \leq i \leq m-1\}$ for $j \in [0, n-1]$. For convenience we define T_j for all $j \in \mathbb{Z}$ such that $T_j = T_{j'}$ where j' is the residue of j modulo n. For examples $T_n = T_0$ and $T_{2n+1} = T_1$. Figure 2 represents the graph $K_4 \Box C_6$ and the vertex set T_2.

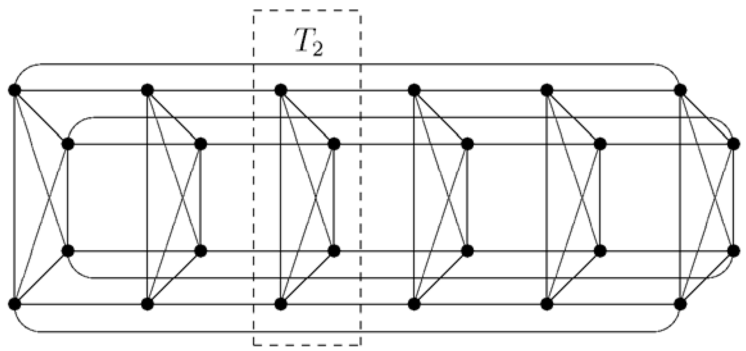

Fig. 2. $K_4 \Box C_6$ and T_2

Let $f : V \to [0, N]$ be an $L(3,2,1)$-labeling of G. Since any two vertices in T_j are adjacent the following lemma is obvious.

Lemma 1. *For each j, $|f(T_j)| = m$.*

For $S \subset [0,N]$, we define $\widehat{S} = \{x \in [0,N] | |x-a| = 1 \text{ for some } a \in S\}$. Note that if $a \in f(T_j)$ for some j, then

$$a \notin (f(T_{j-2}) \cup f(T_{j-1}) \cup f(T_{j+1}) \cup f(T_{j+2}))$$

and

$$a \pm 1 \notin (f(T_{j-1}) \cup f(T_{j+1})).$$

Also if $a_1, a_2 \in f(T_j)$ and $a_1 \neq a_2$, then $|a_1 - a_2| \geq 3$.

Lemma 2. *For each j, $f(T_{j-1}),\ f(T_j),\ f(T_{j+1})$ and $\widehat{f(T_j)}$ are mutually disjoint.*

Proof. If $a, b \in T_j$ and $a \neq b$, then since a and b are adjacent $f(a) \neq f(b)$. If $a \in T_j$ and b is a vertices in T_{j-1} or T_{j+1}, then they are within distance two and $|f(a) - f(b)| \geq 2$. The two vertices a and b choosing from each T_{j-1} and T_{j+1} respectively, are of distance two or three and so $f(a) \neq f(b)$. Hence $f(T_{j-1}),\ f(T_j)$ and $f(T_{j+1})$ are mutually disjoint. If $x \in \widehat{f(T_j)}$, then there is $a \in T_j$ such that $|x - f(a)| = 1$. If $b \in T_j$ and $a \neq b$, then $|f(a) - f(b)| \geq 3$. And we have $|x - f(b)| \geq |f(a) - f(b)| - |x - f(a)| = 2$. If $b \in T_{j-1} \cup T_{j+1}$, then dist$(a, b)$ is one or two. Since $|f(a) - f(b)| \geq 2$, we have $|x - f(b)| \geq |f(a) - f(b)| - |x - f(a)| \geq 1$. So $f(T_{j-1}) \cup f(T_j) \cup f(T_{j+1})$ and $\widehat{f(T_j)}$ are disjoint. ∎

Thus we have the following lemma.

Lemma 3. *If h is the number of elements of $f(T_j) \cap \{0, N\}$, then $|\widehat{f(T_j)}| = 2m - h$.*

Proof. Let $A = \{x \in [0,N] | x = a + 1, a \in f(T_j)\}$ and $B = \{x \in [0,N] | x = a - 1, a \in f(T_j)\}$. If $N \in f(T_j)$, then $|A| = m - 1$. And if $N \notin f(T_j)$, then $|A| = m$. We also have if $0 \in f(T_j)$, then $|B| = m - 1$, and if $0 \notin f(T_j)$, then $|B| = m$. Since $A \cap B = \emptyset$, we have

$$\widehat{f(T_j)} = |A \cup B| = |A| + |B| = \begin{cases} 2m & \text{if } 0, N \notin f(T_j) \\ 2m-2 & \text{if } 0, N \in f(T_j) \\ 2m-1 & \text{otherwise.} \end{cases}$$

Thus we have the result. ∎

Corollary 1. *If $0, N \notin f(T_j)$, then $|\widehat{f(T_j)}| = 2m$.*

Proposition 1. $\lambda_{3,2,1}(K_m \square C_n) \geq 5m - 1$ *for* $m, n \geq 3$.

Proof. Let $f : V \to [0, N]$ be an $L(3,2,1)$-labeling of $G = K_m \square C_n$ with span N. Since $f(T_{j-1}),\ f(T_j)$ and $f(T_{j+1})$ are mutually disjoint, there is j_0 such that

$0, N \notin f(T_{j_0})$ and $0 \leq j_0 \leq 2$. By Lemma 1 and Corollary 1, we have

$$\begin{aligned}\lambda_{3,2,1}(G) = N &\geq |f(T_{j_0-1}) \cup f(T_{j_0}) \cup f(T_{j_0+1}) \cup \widehat{f(T_{j_0})}| - 1 \\ &= |f(T_{j_0-1})| + |f(T_{j_0}))| + |f(T_{j_0+1}))| + |\widehat{f(T_{j_0})}| - 1 \\ &= 5m - 1.\end{aligned}$$

■

In the next theorem when n is a multiple of 5, by providing an $L(3,2,1)$-labeling for $K_m \Box C_n$ we show that $\lambda_{3,2,1}(K_m \Box C_n) = 5m - 1$.

Proposition 2. *If n is a multiple of 5, then $\lambda_{3,2,1}(K_m \Box C_n) = 5m-1$ for $m, n \geq 3$.*

Proof. By Proposition 1, it is enough to show that $\lambda_{3,2,1}(K_m \Box C_n) \leq 5m - 1$. Let $f : V \to [0, 5m-1]$ such that

$$f(i,j) = \begin{cases} 5i - 3t, & \text{if } \ i \geq \frac{3}{5}t, \\ 5m + 5i - 3t, & \text{if } i < \frac{3}{5}t, \end{cases}$$

where t is the residue of j modulo 5. Then f is an $L(3,2,1)$-labeling of $K_m \Box C_n$ with span $5m - 1$. Table 1 represents f.

Thus $\lambda_{3,2,1}(K_m \Box C_n) = 5m - 1$. ■

In what follows we assume that there is an $L(3,2,1)$-labeling f of $K_m \Box C_n$ with span $5m-1$. Let $f(T_j) = \{a_{1,j}, a_{2,j}, \cdots, a_{m,j}\}$ such that $a_{1,j} > a_{2,j} > \cdots > a_{m,j}$.

In the following we will show that the lower bound $5m - 1$ for $\lambda_{3,,2,1}(K_m \Box C_n)$ arises only when n is a multiple of 5.

Claim 1. If $0, 5m-1 \notin f(T_j)$, then $[0, 5m-1] = f(T_{j-1}) \cup f(T_j) \cup f(T_{j+1}) \cup \widehat{f(T_j)}$.

Proof. By Lemmas 1, 2 and Corollary 1, we have

$$|f(T_{j-1}) \cup f(T_j) \cup f(T_{j+1}) \cup \widehat{f(T_j)}| = |f(T_{j-1})| + |f(T_j))| + |f(T_{j+1}))| + |\widehat{f(T_j)}| = 5m.$$

Thus $[0, 5m-1] = f(T_{j-1}) \cup f(T_j) \cup f(T_{j+1}) \cup \widehat{f(T_j)}$. ■

By a similar method as above, we have

$$[0, 5m-1] = f(T_{j-1}) \cup f(T_j) \cup f(T_{j+1}) \cup \widehat{f(T_j)} \cup \{x\}$$

for some x if exactly one of 0 and $5m-1$ belongs to $f(T_j)$ and

$$[0, 5m-1] = f(T_{j-1}) \cup f(T_j) \cup f(T_{j+1}) \cup \widehat{f(T_j)} \cup \{x_1, x_2\}$$

for some x_1, x_2 if both of 0 and $5m-1$ belong to $f(T_j)$.

Claim 2. If $0, 5m-1 \notin f(T_j)$ for some j, then for any t with $0 \le t \le 5m-3$ we have $[t, t+2] \cap (f(T_j) \cup \widehat{f(T_j)}) \ne \emptyset$. Hence for any t with $0 \le t \le 5m-5$ we have $[t, t+4] \cap f(T_j) \ne \emptyset$.

Table 1. $L(3,2,1)$- labeling for $K_m \Box C_{5t}$

0	$5m-3$	$5m-6$	$5m-9$	$5m-12$	0	$5m-3$	·	·	·	$5m-12$
5	2	$5m-1$	$5m-4$	$5m-7$	5	2	·	·	·	$5m-7$
10	7	4	1	$5m-2$	10	7	·	·	·	$5m-2$
15	12	9	6	3	15	12	·	·	·	3
20	17	14	11	8	20	17	·	·	·	8
·	·	·	·	·	·	·	·			·
·	·	·	·	·	·	·		·		·
·	·	·	·	·	·	·			·	·
$5m-5$	$5m-8$	$5m-11$	$5m-14$	$5m-17$	$5m-5$	$5m-8$	·	·	·	$5m-17$

Proof. If there is $t \in [0, 5m-3]$ such that $[t, t+2] \cap (f(T_j) \cup \widehat{f(T_j)}) = \emptyset$. By Claim 1 we have $[t, t+2] \subset f(T_{j-1}) \cup f(T_{j+1})$. Since any two vertices in T_{j+1} are adjacent, the labeling difference between them is at least 3. So $f(T_{j+1})$ contains at most one number in $[t, t+2]$ and $f(T_{j-1})$ also contains at most one number in $[t, t+2]$. This contradicts that $[t, t+2] \subset f(T_{j-1}) \cup f(T_{j+1})$.

For $t \in [0, 5m-5]$, if $[t, t+4] \cap f(T_j) = \emptyset$, then $1 \le t+1 \le 5m-4$ and we have $[t+1, t+3] \cap \widehat{f(T_j)} = \emptyset$. This is a contradiction. ■

Claim 3. For any j and t with $0 \le t \le 5m-6$, we have $[t, t+5] \cap f(T_j) \ne \emptyset$.

Proof. Suppose $[t, t+5] \cap f(T_j) = \emptyset$ for some j and t with $0 \le t \le 5m-6$. Consider the set

$$S = \{t \ge 0 |\ [t, t+5] \cap f(T_j) = \emptyset \text{ for some } j\}.$$

Since S is nonempty there is the minimum, say it h, of the set. Let $[h, h+5] \cap f(T_l) = \emptyset$ for some l.

If $0 < h < 5m-6$, then by the minimality of h, we have $h-1 \in f(T_l)$. By Claim 2, 0 or $5m-1$ belong to $f(T_l)$. Since $f(T_{l+1}) \cap f(T_{l-1}) = \emptyset$, at least one of $f(T_{l+1})$ and $f(T_{l-1})$ doesn't contain both 0 and $5m-1$. Thus without loss of generality we may assume that $0, 5m-1 \notin f(T_{l-1})$. If $h+1, h+2 \notin f(T_{l-1})$, then since $[h-1, h+2] \cap f(T_{l-1}) = \emptyset$, we have $h, h+1 \notin \widehat{f(T_{l-1})}$. Since $0, 5m-1 \notin f(T_{l-1})$ and $h, h+1 \notin f(T_{l-1}) \cup \widehat{f(T_{l-1})} \cup f(T_l)$, by Claim 1, we have $h, h+1 \in f(T_{l-2})$, which is impossible. Thus one of $h+1, h+2$ is an element of $f(T_{l-1})$. If $h+1 \in f(T_{l-1})$, then $h+2, h+3 \notin f(T_{l-1})$. In this case if $h+4 \in f(T_{l-1})$, then since $h-1 \in f(T_l)$

and $h+1, h+4 \in f(T_{l-1})$, we have $[h-1, h+4] \cap f(T_{l-2}) = \emptyset$, and which contradicts the minimality of h. Thus $h+4 \notin f(T_{l-1})$. If $h+5 \notin f(T_{l-1})$, then since $[h+2, h+5] \cap f(T_{l-1}) = \emptyset$, we have $h+3, h+4 \notin f(T_{l-1}) \cup \widehat{f(T_{l-1})} \cup f(T_l)$. Since $0, 5m-1 \notin f(T_{l-1})$, by Claim 1, $h+3, h+4 \in f(T_{l-2})$, which is also impossible. Thus $h+5 \in f(T_{l-1})$. It follows that $h+6 \notin f(T_l)$. Since $h+2, h+3, h+4 \notin \widehat{f(T_l)}$ and at most one of them belong to $f(T_{l+1})$, there are at least two numbers among $[0, 5m-1]$ which doesn't belong to $f(T_l) \cup \widehat{f(T_l)} \cup f(T_{l-1}) \cup f(T_{l+1})$. Thus $0, 5m-1 \in f(T_l)$. If $h+2 \notin f(T_{l+1})$, then since $0, 5m-1 \notin f(T_{l+1})$ and $h, h+1 \notin f(T_{l+1}) \cup \widehat{f(T_{l+1})} \cup f(T_l)$, we have $h, h+1 \notin f(T_{l+2})$, which is a contradiction. Thus $h+2 \in f(T_{l+1})$. If $h \notin f(T_{l+2})$, then $[h-1, h+3] \cap f(T_{l+2}) \neq \emptyset$, which contradicts Claim 2 since $0, 5m-1 \notin f(T_{l+2})$. Thus $h \in f(T_{l+2})$. Since $h+3, h+5 \notin f(T_{l+1})$, so $h+4 \notin f(T_{l+1}) \cup \widehat{f(T_{l+1})}$. This implies that $h+4 \in f(T_{l+2})$. Since $h+2 \in f(T_{l+1})$ and $h, h+4 \in f(T_{l+2})$, we obtain $[h-1, h+4] \cap f(T_{l+3}) = \emptyset$. This is a contradiction and thus we have $h+1 \notin f(T_{l-1})$. As a consequence $h+2 \in f(T_{l-1})$. By the similar method as above, we can show that $h+6 \in f(T_{l-1})$. Also we have $h, h+4 \in f(T_{l-2})$. It follows that $[h-1, h+4] \cap f(T_{l-3}) = \emptyset$, a contradiction.

If $h = 0$, then $[0, 5] \cap f(T_l) = \emptyset$. By Claim 2, we have $5m-1 \in f(T_l)$. Thus

$$[0, 5m-1] = f(T_{l-1}) \cup f(T_l) \cup f(T_{l+1}) \cup \widehat{f(T_l)} \cup \{x\}$$

for some x. We may assume $5m-1 \notin f(T_{l-1})$. Since $[0, 4] \cap (f(T_l) \cup \widehat{f(T_l)}) = \emptyset$, we have $[0, 4] \subset (f(T_{l-1}) \cup f(T_{l+1}) \cup \{x\})$. Since each of $f(T_{l-1})$ and $f(T_{l+1})$ contains at most two elements of $[0, 4]$, they each contains two numbers in $[0, 4]$ respectively. We may assume that $0, 3 \in f(T_{l-1})$ and $1, 4 \in f(T_{l+1})$. As a consequence $[0, 4] \cap f(T_{l+2}) = \emptyset$. Since $5m-1 \in f(T_l)$, $5m-1 \notin f(T_{l+2})$, which contradict Lemma 5. Thus $[t, t+5] \cap f(T_j) \neq \emptyset$ for all j and t with $0 \leq t \leq 5m-6$.

Let $g : V \to [0, 5m-1]$ be defined as $g(v) = 5m-1-f(v)$. Since g is an $L(3,2,1)$-labeling of G and $g(T_l) \cap [0, 5] = \emptyset$, we also have a contradiction when $h = 5m-6$. ∎

By above it is obvious to see that there is no t such that $0 \leq t \leq 5m-8$ and $t, t+1, \cdots, t+7 \notin f(T_j)$. As a consequence we have the following observation: If $f(T_j) = \{a_{1,j}, a_{2,j}, \cdots, a_{m,j}\}$ such that $a_{1,j} > a_{2,j} > \cdots > a_{m,j}$, then for all $1 \leq i \leq m-1$ we have $a_{i+1,j} - a_{i,j} \leq 6$.

Lemma 4. *If f is an $L(3,2,1)$-labeling of $K_m \square C_n$ with span $5m-1$ for $m, n \geq 3$, then there is no j such that $f(T_j)$ contains both 0 and $5m-1$.*

Proof. Assume that $0, 5m-1 \in f(T_j)$ for some j. Since $|\widehat{f(T_j)}| = 2m-2$, we have $[0, 5m-1] = f(T_{j-1}) \cup f(T_j) \cup f(T_{j+1}) \cup \widehat{f(T_j)} \cup \{x_1, x_2\}$ for some x_1, x_2. Since $a_{m,j} - a_{1,j} = \sum_{i=1}^{m-1}(a_{i+1,j} - a_{i,j}) = 5m-1 = 5(m-1)+4$, and $a_{i+1,j} - a_{i,j} \leq 6$ by the above observation, there are at least three i such that $a_{i+1,j} - a_{i,j} = 6$. Thus among such three i's there is i_0 such that $a_{i_0+1,j} - a_{i_0,j} = 6$ and $x_1, x_2 \notin \{a_{i_0,j}+2, a_{i_0,j}+3, a_{i_0,j}+4\}$. Then $a_{i_0,j}+2, a_{i_0,j}+3, a_{i_0,j}+4 \in f(T_{j-1}) \cup f(T_{j+1})$. This is a contradiction. ■

In the next proposition we find the exact difference of $a_{i+1,j} - a_{i,j}$ and by using them we get the main result.

Proposition 3. *If $f(T_j) = \{a_{1,j}, a_{2,j}, \cdots, a_{m,j}\}$ such that $a_{1,j} > a_{2,j} > \cdots > a_{m,j}$, then for all $1 \leq i \leq m-1$ we have $a_{i+1,j} - a_{i,j} \neq 3, 4$.*

Proof. Suppose $a_{i+1,j} - a_{i,j} = 3$ for some j. We may assume that $a_{i,j}$ is the smallest among $a_{i'+1,j'} - a_{i',j'} = 3$. In other words, if $a_{i'+1,j'} - a_{i',j'} = 3$, then $a_{i',j'} \geq a_{i,j}$. If $f(T_{j-1})$ doesn't contain both 0 and $5m-1$, then since $a_{i,j}+1, a_{i,j}+2 \notin \widehat{f(T_{j-1})}$ and by Claim 1, $a_{i,j}+1$ and $a_{i,j}+2$ belong to $f(T_{j-2})$, which is a contradiction. Thus $f(T_{j-1})$ contains 0 or $5m-1$. Similarly $f(T_{j+1})$ also contains 0 or $5m-1$. As a consequence $f(T_j)$ doesn't contain both 0 and $5m-1$. Since $0 \in f(T_{j-1}) \cup f(T_{j+1})$, we have $a_{i,j} \geq 2$. By considering $g : V \to [0, 5m-1]$ such that $g(v) = 5m-1-f(v)$, we also have $a_{i+1,j} = a_{i,j}+3 \leq 5m-3$. It follows that $a_{i,j}-2, a_{i,j}-1, \cdots, a_{i,j}+4 \notin f(T_{j+1})$. Since $a_{i,j}-1, a_{i,j}, \cdots, a_{i,j}+3 \notin f(T_{j+1}) \cup \widehat{f(T_{j+1})}$, all but at most one of $a_{i,j}-1, a_{i,j}, \cdots, a_{i,j}+3$ belongs to $f(T_j) \cup f(T_{j+2})$. Since $a_{i,j}, a_{i,j}+3 \in f(T_j)$, at most two of $a_{i,j}, a_{i,j}+1, a_{i,j}+2$ belongs to $f(T_{j+2})$. Since $a_{i,j}-1, a_{i,j}+2 \in f(T_{j+2})$. This contradicts the minimality of $a_{i,j}$. So $a_{i+1,j} - a_{i,j} \neq 3$.

Assume that $a_{i+1,j} - a_{i,j} = 4$. we may assume that $a_{i,j}+2 \notin f(T_{j-1})$. If $a_{i,j}$ is 0 or 1, then since $0, 1, \cdots, 5 \notin f(T_{j-1})$, $0, 1, \cdots, 4 \notin f(T_{j-1}) \cup \widehat{f(T_{j-1})}$. Thus $f(T_{j-2}) \cup f(T_j)$ contains $[0, 4]$ except at most one of them. Since $0, 4 \in f(T_j)$ or $1, 5 \in f(T_j)$, $f(T_{j-2})$ contains at least two of $0, 1, 2, 3$. This is impossible. Thus $a_{i,j} \geq 2$. Similarly $a_{i+1,j} = a_{i,j}+4 \leq 5m-3$. Since $a_{i,j}-1, a_{i,j}, \cdots, a_{i,j}+5 \notin f(T_{j-1}) \cup \widehat{f(T_{j-1})}$, we have $a_{i,j}, \cdots, a_{i,j}+4 \notin f(T_{j-1})$. Since $a_{i,j}, a_{i,j}+4 \in f(T_j)$, at least two of $a_{i,j}+1, a_{i,j}+2, a_{i,j}+3$ belong to $f(T_{j-1})$. This is a contradiction. ■

Corollary 2. *If $0, 5m-1 \notin f(T_j)$ for some j, then $a_{i+1,j} - a_{i,j} = 5$ for all i.*

Corollary 3. *If $0, 5m-1 \notin f(T_j)$ for some j, then $f(T_j) = \{a_{i,j} + 5s | 0 \leq s \leq m-1\}$.*

Theorem 1. *If $\lambda_{3,2,1}(K_m \square C_n) = 5m-1$ for $m, n \geq 3$, then n is a multiple of 5.*

Proof. Without loss of generality we may assume $0 \in f(T_0)$ and $5m-1 \notin f(T_1)$. Let $U_t = \{5s+t |\ 0 \leq s \leq m-1\}$ for $t \in [0, 4]$. Let a be the minimum of $f(T_1)$. By Corollary 2, $f(T_1) = U_a$. Since $a_0 \geq 2$ and $a + 5(m-1) \leq 5m-2$, we have a is 2 or 3.

If $a = 2$, then $f(T_1) = U_2$ and $\widehat{f(T_1)} = U_1 \cup U_3$. Thus $f(T_0) \subset (U_0 \cup U_4)$. If there is $x \in f(T_0)$ such that $x \in U_4$, then there is i such that $a_{i+1,0} \in U_4$ and

$a_{i,0} \in U_0$. Since $a_{i+1,0} - a_{i,0} \equiv 4 \pmod 5$ and $a_{i+1,0} - a_{i,0} \geq 5$ by Proposition 3, $a_{i+1,0} - a_{i,0} \geq 9$, which contradicts Corollary 2. It follows that $f(T_0) = U_0$. Thus $f(T_2) = [0, 5m-1] \setminus (f(T_0) \cup f(T_1) \cup \widehat{f(T_1)}) = [0, 5m-1] \setminus (U_0 \cup U_1 \cup U_2 \cup U_3) = U_4$ by Lemma 3. Hence $\widehat{f(T_2)} = U_0 \cup U_3 \setminus \{0\}$. Therefore $f(T_3) \subset U_1 \cup \{0\}$. If $0 \in f(T_3)$, then $1 \notin f(T_3)$. As a consequence $f(T_3) = \{0\} \cup (U_1 \setminus \{1\})$. Since $1, 2, 3, 4, 5 \notin f(T_3)$, we have $2, 3, 4 \notin \widehat{f(T_3)}$. Since $4 \in f(T_2)$, 2 or 3 belongs to $f(T_4)$. Since $0 \in f(T_3)$ and $5m - 1 \in f(T_2)$, $f(T_4)$ doesn't contain both 0 and $5m-1$. Thus $f(T_4)$ is U_2 or U_3. Since $6 \in f(T_3)$, $7 \notin f(T_4)$ Hence $f(T_4) = U_3$. Since $\widehat{f(T_4)} = U_2 \cup U_4$, $5m - 1 \notin f(T_5)$. Since $1 \notin (f(T_3) \cup f(T_4) \cup \widehat{f(T_4)})$, $1 \in f(T_5)$. Thus $f(T_5) = U_1$ and contains 6, which contradicts the fact $6 \in f(T_3)$. Hence $0 \notin f(T_3)$. It follows that $f(T_3) = U_1$. Since $0, 5m - 1 \notin f(T_3)$, we have $f(T_4) = [0, 5m-1] \setminus (f(T_2) \cup f(T_3) \cup \widehat{f(T_3)}) = [0, 5m-1] \setminus (U_4 \cup U_1 \cup U_2 \cup U_0) = U_3$. Since $0, 5m - 1 \notin f(T_4)$, we have $f(T_5) = [0, 5m-1] \setminus (f(T_3) \cup f(T_4) \cup \widehat{f(T_4)}) = [0, 5m-1] \setminus (U_1 \cup U_3 \cup U_2 \cup U_4) = U_0$. Since $\widehat{f(T_5)} = \widehat{U_0} = (U_1 \cup U_4) \setminus \{5m-1\}$, $f(T_6) \subset U_2 \cup \{5m-1\}$. If $5m - 1 \in f(T_6)$, then $5m - 3 \notin f(T_6)$. Thus $f(T_6) = (U_2 \cup \{5m-1\}) \setminus \{5m-3\}$. Since $0 \in f(T_5)$ and $5m-1 \in f(T_6)$, $f(T_7)$ doesn't contain both 0 and $5m-1$. Since $0, 1, 2, 3 \in f(T_5) \cup f(T_6) \cup \widehat{f(T_6)}$, the minimum of $f(T_7)$ is at least 4. Thus the maximum of $f(T_7)$ is at least $4 + 5(m-1) = 5m-1$ by Proposition 3. This gives a contradiction since $5m - 1 \in f(T_6)$. Thus $5m - 1 \notin f(T_6)$. It follows that $f(T_6) = U_2$. By the same method we can prove $f(T_{5t}) = U_0$, $f(T_{5t+1}) = U_2$, $f(T_{5t+2}) = U_4$, $f(T_{5t+3}) = U_1$ and $f(T_{5t+4}) = U_3$ by induction.

If $a = 3$, then by the same method, we can prove that $f(T_{5t}) = U_0$, $f(T_{5t+1}) = U_3$, $f(T_{5t+2}) = U_1$, $f(T_{5t+3}) = U_4$ and $f(T_{5t+4}) = U_2$ by induction.

In any case, $f(T_n) = f(T_0) = U_0$. Thus n is a multiple of 5. ■

When $m = 3$, by providing two $L_{3,2,1}$-labelings of $K_3 \square C_5$ and $K_3 \square C_8$, we find the cases that $\lambda_{3,2,1}$-number equals to 15. For $4 \leq n \leq 9$, Shao and Vesel [17] computed $\lambda_{3,2,1}(K_3 \square C_m)$.

Theorem 2. *If* $n \geq 28$ *and* $n \not\equiv 0 \pmod 5$, *then* $\lambda_{3,2,1}(K_3 \square C_n) = 15$.

Proof. For any $n \geq 28$, n is a linear combination of 5 and 8. In other words, $n = 5s + 8t$ for some nonnegative s, t. Note that the leftmost two columns of the patterns in Table 2 coincide and they can be located to the right side of each pattern as an $L(3,2,1)$- labelings. By repeating Pattern A s times and Pattern B t times horizontally, we obtain $L(3,2,1)$-labeling for $K_3 \square C_n$ with span 15. By Theorem 1, $\lambda_{3,2,1}(K_3 \square C_n) > 14$ and we have $\lambda_{3,2,1}(K_3 \square C_n) = 15$. ■

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Table 2. $L(3, 2, 1)$-Labelings of $K_3\square C_5$ and $K_3\square C_8$

0	13	10	7	3
6	2	15	12	9
11	8	4	1	14

Pattern A

0	13	10	6	14	11	7	4
6	2	15	12	3	0	13	9
11	8	4	1	9	5	2	15

Pattern B

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