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REPRESENTATIONS FOR THE PSEUDO DRAZIN INVERSE OF ELEMENTS IN A BANACH ALGEBRA

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Abstract. In this paper, we investigate the pseudo Drazin invertibility of the sum and the product of elements in a Banach algebra $\mathscr{A}$. Given pseudo Drazin invertible elements a and b such that $a^2b = aba$ and $b^2a = bab$, it is shown that ab is pseudo Drazin invertible and $a + b$ is pseudo Drazin invertible if and only if so is $1 + a^{\ddagger}b$, and the related formulae are provided.

1. Introduction

Representations for the *pseudo Drazin inverse* (abbr. *p-Drazin inverse*) of the sums and the products of two elements in certain algebras have attracted wide interest. In general, it is a challenging task to characterize the p-Drazin inverses of $a + b$ and ab without additional hypothesis. For instance, given a and b in a Banach algebra $\mathscr{A}$ with p-Drazin inverses $a^{\ddagger}$ and $b^{\ddagger}$, respectively. If $ab = ba$=0 then it follows that $a + b$ is p-Drazin invertible with $(a + b)^{\ddagger} = a^{\ddagger} + b^{\ddagger}$ (see [8, Theorem 2.5]). Further, if ab=ba, the authors proved that $a + b$ is p-Drazin invertible if and only if $1 + a^{\ddagger}b$ is p-Drazin invertible. In this case, the representations of $(a + b)^{\ddagger}$ and $1 + a^{\ddagger}b$ are given (see [8, Theorem 2.7]). Note that the condition that $ab = ba$ implies $a^2b = aba$ and $b^2a = bab$. However, the reverse statement may not be true, such as, take $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in the ring of 2×2 matrices.

Motivated by the aforementioned work, we investigate the representations for the p-Drazin inverse of the sum and product of two elements under the conditions $a^2b = aba$ and $b^2a = bab$ in a Banach algebra. Some results on p-Drazin inverses in [8] are extended. Representations on *Drazin inverse* (see [1]) of the sum and the product of elements in various sets can be referred to mathematical literature [2, 3, 4, 6, 7, 9].

Recently, the authors [5] introduced a new kind of generalized inverse, that is p-Drazin inverse, whose properties and related expressions are obtained in associative

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rings and Banach algebras. Throughout this paper, let $\mathscr{A}$ be a Banach algebra with unity 1 and $J(\mathscr{A})$ denote the Jacobson radical of $\mathscr{A}$. For any $a \in \mathscr{A}$, the commutant and double commutant of a are defined by $\mathrm{comm}(a) = \{x \in \mathscr{A} : ax = xa\}$ and $\mathrm{comm}^2(a) = \{x \in \mathscr{A} : xy = yx \ for\ all\ y \in \mathrm{comm}(a)\}$. An element $a \in \mathscr{A}$ is said to have a *p-Drazin inverse* if there exists $b \in \mathscr{A}$ such that the following conditions hold [5]:

$$b \in \mathrm{comm}(a),\ bab = b,\ a^k - a^{k+1}b \in J(\mathscr{A})$$

for some integer $k \geqslant 1$. If such a b exists, it is unique and is denoted by $a^{\ddagger}$. In a Banach algebra, the condition $b \in \mathrm{comm}(a)$ in the above definition is equivalent to $b \in \mathrm{comm}^2(a)$. According to [5], $a^{\ddagger} \in \mathrm{comm}(a)$. By $a^{\Pi} = 1 - aa^{\ddagger}$ and $\mathscr{A}^{pD}$ we denote the strongly spectral idempotent of a and set of p-Drazin invertible elements of $\mathscr{A}$, respectively. Note that some techniques such as matrix decompositions, orthogonal decomposition of Hilbert space, and spectral theory are not used in this article. The results in this paper are proved by a purely ring theoretical method.

2. Main Results

In this section, we start with some lemmas which play an important role in the sequel. We will freely use the fact that $a^{\ddagger}a^{\Pi} = a^{\Pi}a^{\ddagger} = 0$ and $a^{\ddagger}x = xa^{\ddagger}$ for every $x \in \mathrm{comm}(a)$ in the context (see [5]).

Lemma 2.1. *Let $a, b \in \mathscr{A}$ with $a^2b = aba$ and $b^2a = bab$. If $a \in \mathscr{A}^{pD}$, then*

$$(1) \quad (a^{\ddagger})^2 b = a^{\ddagger} b a^{\ddagger}, \tag{2.1}$$

$$(2) \quad b^2 a^{\ddagger} = b a^{\ddagger} b. \tag{2.2}$$

Proof. (1) As $a^2b = aba$, that is $a(ab) = (ab)a$, then $ab \in \mathrm{comm}(a^{\ddagger})$.
Hence, $(a^{\ddagger})^2 b = (a^{\ddagger})^2 a^{\ddagger} ab = (a^{\ddagger})^2 aba^{\ddagger} = a^{\ddagger}ba^{\ddagger}$.

(2) Again $ab \in \mathrm{comm}(a^{\ddagger})$ implies $b^2a^{\ddagger} = b^2a(a^{\ddagger})^2 = bab(a^{\ddagger})^2 = b(a^{\ddagger})^2ab = ba^{\ddagger}b$. ■

Lemma 2.2. *Let $a, b \in \mathscr{A}^{pD}$ with $a^2b = aba$ and $b^2a = bab$. Then*

$$(1) \quad \{ab, a^{\ddagger}b, ab^{\ddagger}, a^{\ddagger}b^{\ddagger}\} \subseteq \mathrm{comm(a)}, \tag{2.3}$$

$$(2) \quad \{ba, b^{\ddagger}a, ba^{\ddagger}, b^{\ddagger}a^{\ddagger}\} \subseteq \mathrm{comm}(b). \tag{2.4}$$

Proof. (1) As $ab \in \mathrm{comm}(a^{\ddagger})$, then

$$aa^{\ddagger}b = (a^{\ddagger})^2a^2b = (a^{\ddagger})^2aba = a^{\ddagger}ba.$$

Similarly, $ba \in \mathrm{comm}(b^{\ddagger})$ guarantees that

$$ab^{\ddagger}a = a(b^{\ddagger})^2ba = aba(b^{\ddagger})^2 = a^2b(b^{\ddagger})^2 = a^2b^{\ddagger}.$$

By $ab^{\ddagger} \in \mathrm{comm}(a)$, we obtain

$$aa^{\ddagger}b^{\ddagger} = (a^{\ddagger})^2a^2b^{\ddagger} = (a^{\ddagger})^2ab^{\ddagger}a = a^{\ddagger}b^{\ddagger}a.$$

(2) can be obtained in a similar way of (1). ■

By Lemmas 2.1 and 2.2, we can get the following result.

Lemma 2.3. *Let* $a, b \in \mathscr{A}^{pD}$ *with* $a^2b = aba$ *and* $b^2a = bab$. *If* $\xi = 1 + a^{\ddagger}b$, *then*

$$\{a, ab, a^{\ddagger}b, ab^{\ddagger}, a^{\ddagger}b^{\ddagger}\} \subseteq \mathrm{comm}(\xi). \tag{2.5}$$

Lemma 2.4. *Let* $a, b \in \mathscr{A}^{pD}$ *with* $a^2b = aba$ *and* $b^2a = bab$. *Then*

$$(1) \quad ab^{\ddagger}b^{\ddagger}a = (ab^{\ddagger})^2 = a^2(b^{\ddagger})^2, \tag{2.6}$$

$$(2) \quad (ab^{\ddagger})^{i+1} = ab^{\ddagger}(b^{\ddagger}a)^i = a^{i+1}(b^{\ddagger})^{i+1} \ for\ any\ positive\ integer\ i, \tag{2.7}$$

$$(3) \quad (a^{\ddagger}b)^{i+1} = a^{\ddagger}b(ba^{\ddagger})^i = (a^{\ddagger})^{i+1}b^{i+1} \ for\ any\ positive\ integer\ i \tag{2.8}$$

Proof. (1) It follows from Lemma 2.2 that $ab^{\ddagger}(b^{\ddagger}a) = a(b^{\ddagger}a)b^{\ddagger} = (ab^{\ddagger})ab^{\ddagger} = a(ab^{\ddagger})b^{\ddagger}$.

(2) It suffices to show the inductive step. Suppose $(ab^{\ddagger})^{i+1} = ab^{\ddagger}(b^{\ddagger}a)^i = a^{i+1}(b^{\ddagger})^{i+1}$ then one can see that

$$(ab^{\ddagger})^{i+2} = ab^{\ddagger}(ab^{\ddagger})^{i+1} = ab^{\ddagger}ab^{\ddagger}(b^{\ddagger}a)^i \overset{(2.6)}{=} ab^{\ddagger}b^{\ddagger}a(b^{\ddagger}a)^i = ab^{\ddagger}(b^{\ddagger}a)^{i+1}$$

and

$$(ab^{\ddagger})^{i+2} = ab^{\ddagger}(ab^{\ddagger})^{i+1} = ab^{\ddagger}a^{i+1}(b^{\ddagger})^{i+1} = a^{i+1}ab^{\ddagger}(b^{\ddagger})^{i+1} = a^{i+2}(b^{\ddagger})^{i+2}.$$

(3) Use a similar way of (2). ■

In Lemma 2.4, one can substitute $(a, b, a^{\ddagger}, b^{\ddagger})$ for $(b, a, b^{\ddagger}, a^{\ddagger})$ and get the next result.

Corollary 2.5. *Let* $a, b \in \mathscr{A}^{pD}$ *with* $a^2b = aba$ *and* $b^2a = bab$. *Then*

$$(1) \quad ba^{\ddagger}a^{\ddagger}b = (ba^{\ddagger})^2 = b^2(a^{\ddagger})^2, \tag{2.9}$$

$$(2) \quad ba^{\ddagger})^{i+1} = ba^{\ddagger}(a^{\ddagger}b)^i = b^{i+1}(a^{\ddagger})^{i+1} \ for\ any\ positive\ integer\ i, \tag{2.10}$$

$$(3) \quad (b^{\ddagger}a)^{i+1} = b^{\ddagger}a(ab^{\ddagger})^i = (b^{\ddagger})^{i+1}a^{i+1} \ for\ any\ positive\ integer\ i. \tag{2.11}$$

Lemma 2.6. *Let $a, b \in \mathscr{A}$ with $a^2b = aba$ and $b^2a = bab$. Then following hold for any integer $k \geq 0$, we have*
(1) $(ab)^k = a^k b^k$,
(2) $(a+b)^k = \sum_{i=0}^{k-1}(a^{k-i}b^i + b^{k-i}a^i)$.

Proof. (1) It is obvious for $k = 1$.
Assume $(ab)^n = a^n b^n$. For $n+1$ case, we have

$$(ab)^{n+1} = ab(ab)^n = aba^n b^n = a^n(ab)n^n = a^{n+1}b^{n+1}.$$

(2) By induction. ∎

Lemma 2.7. *Let $a, b \in \mathscr{A}$. Then*
(1) *If $a \in J(\mathscr{A})$ or $b \in J(\mathscr{A})$, then ab, $ba \in J(\mathscr{A})$,*
(2) *If $a \in J(\mathscr{A})$ and $b \in J(\mathscr{A})$, then $(a+b)^k \in J(\mathscr{A})$ for integer $k \geqslant 1$.*

Let $\mathscr{A}$ be a Banach algebra. Given p-Drazin invertible elements $a, b \in \mathscr{A}$ such that $ab = ba$, the authors [5] proved that ab is p-Drazin invertible and $(ab)^{\ddagger} = b^{\ddagger}a^{\ddagger} = a^{\ddagger}b^{\ddagger}$. The following theorem extends the result in [5, Proposition 5.2].

Theorem 2.8. *Let $a, b \in \mathscr{A}^{pD}$. If $a^2b = aba$ and $b^2a = bab$, then $ab \in \mathscr{A}^{pD}$ and*

$$(ab)^{\ddagger} = a^{\ddagger}b^{\ddagger}.$$

Proof. We prove that $x = a^{\ddagger}b^{\ddagger}$ is the p-Drazin inverse of ab, i.e., the following conditions hold: (1) $(ab)x = x(ab)$; (2) $x(ab)x = x$; (3) $(ab)^k - (ab)^{k+1}x \in J(\mathscr{A})$.
(1) Since $(ab)a^{\ddagger} = a^{\ddagger}(ab)$, we have

$$\begin{aligned}(ab)x &= (ab)a^{\ddagger}b^{\ddagger} = a^{\ddagger}abb^{\ddagger} \\ &= a(a^{\ddagger}b^{\ddagger})b = a^{\ddagger}b^{\ddagger}ab \\ &= x(ab).\end{aligned}$$

(2) We have

$$\begin{aligned}x(ab)x &= a^{\ddagger}b^{\ddagger}aba^{\ddagger}b^{\ddagger} = a(a^{\ddagger}b^{\ddagger})ba^{\ddagger}b^{\ddagger} = aa^{\ddagger}b(b^{\ddagger}a^{\ddagger})b^{\ddagger} \\ &= aa^{\ddagger}(b^{\ddagger}a^{\ddagger})bb^{\ddagger} = a(a^{\ddagger}b^{\ddagger})a^{\ddagger}bb^{\ddagger} = aa^{\ddagger}(a^{\ddagger}b^{\ddagger})bb^{\ddagger} \\ &= x.\end{aligned}$$

(3) Note that equality $(ab)^k = a^k b^k$ in Lemma 2.6 (1). We can prove $a^{k+1}a^{\ddagger}b^{k+1}b^{\ddagger}$ $= (ab)^{k+1}a^{\ddagger}b^{\ddagger}$ by induction.
For $k = 0$, we have $aa^{\ddagger}bb^{\ddagger} = a^{\ddagger}(ab)b^{\ddagger} = aba^{\ddagger}b^{\ddagger}$.

For the case of $k+1$, we obtain

$$\begin{aligned} a^{k+1}a^{\ddagger}b^{k+1}b^{\ddagger} &= aa^{k}a^{\ddagger}b^{k}b^{\ddagger}b = a(ab)^{k}a^{\ddagger}b^{\ddagger}b \\ &= (ab)^{k}a^{\ddagger}abb^{\ddagger} = (ab)^{k}aba^{\ddagger}b^{\ddagger} \\ &= (ab)^{k+1}a^{\ddagger}b^{\ddagger}. \end{aligned}$$

Since a, $b \in \mathscr{A}^{pD}$, there exist integers k_1, k_2 such that $a^{k_1} - a^{k_1+1}a^{\ddagger} \in J(\mathscr{A})$ and $b^{k_2} - b^{k_2+1}a^{\ddagger} \in J(\mathscr{A})$.

Take $k = \max\{k_1, k_2\}$, it follows that

$$\begin{aligned} (ab)^{k} - (ab)^{k+1}x &= a^{k}b^{k} - a^{k+1}b^{k+1}a^{\ddagger}b^{\ddagger} = a^{k}b^{k} - a^{k+1}a^{\ddagger}b^{k+1}b^{\ddagger} \\ &= (a^{k} - a^{k+1}a^{\ddagger})(b^{k} - b^{k+1}b^{\ddagger}) + a^{k+1}a^{\ddagger}(b^{k} - b^{k+1}b^{\ddagger}) \\ &\quad +(a^{k} - a^{k+1}a^{\ddagger})b^{k+1}b^{\ddagger}. \end{aligned}$$

It follows from Lemma 2.7 (2) that $(ab)^{k} - (ab)^{k+1}x \in J(\mathscr{A})$.

Therefore, $ab \in \mathscr{A}^{pD}$ and $(ab)^{\ddagger} = a^{\ddagger}b^{\ddagger}$. ∎

Remark 2.9. It is well known that the reverse-order law holds for commutative p-Drazin invertible elements a, b in a Banach algebra $\mathscr{A}$ with identity 1. More precisely, $(ab)^{\ddagger} = b^{\ddagger}a^{\ddagger} = a^{\ddagger}b^{\ddagger}$ for commutative p-Drazin invertible elements a and b in $\mathscr{A}$. However, Under the conditions $a^2b = aba$ and $b^2a = bab$, $(ab)^{\ddagger}$ may not be equal to $b^{\ddagger}a^{\ddagger}$. For instance, take $a = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in the ring of 2×2 matrices. It follows that $(ab)^{\ddagger} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ while $b^{\ddagger}a^{\ddagger} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

The following theorem presents a necessary and sufficient condition for the existence of $(a+b)^{\ddagger}$ in a Banach algebra.

Theorem 2.10. *Let $a, b \in \mathscr{A}^{pD}$ with $a^2b = aba$ and $b^2a = bab$. Then $a+b \in \mathscr{A}^{pD}$ and only if $1 + a^{\ddagger}b \in \mathscr{A}^{pD}$. In this case, we have*

$$\begin{aligned} (a+b)^{\ddagger} &= a^{\ddagger}(1+a^{\ddagger}b)^{\ddagger} + a^{\Pi}b[a^{\ddagger}(1+a^{\ddagger}b)^{\ddagger}]^{2} + \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi} \\ &\quad +b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^{i}a^{\Pi}, \end{aligned}$$

and

$$(1+a^{\ddagger}b)^{\ddagger} = a^{\Pi} + a^{2}a^{\ddagger}(a+b)^{\ddagger}.$$

Proof. Assume $a+b \in \mathscr{A}^{pD}$. Write $1 + a^{\ddagger}b = a_1 + b_1$ with $a_1 = a^{\Pi}$ and $b_1 = a^{\ddagger}(a+b)$.

Lemma 2.2 implies $(a^{\ddagger})^2(a+b) = a^{\ddagger}(a+b)a^{\ddagger}$, $(a+b)^2a^{\ddagger} = (a+b)a^{\ddagger}(a+b)$ and $a_1b_1 = b_1a_1 = 0$. Since $(a^{\ddagger})^{\ddagger} = a^2a^{\ddagger}$, it follows from Theorem 2.8 that b_1 is p-Drazin invertible and

$$(b_1)^{\ddagger} = [a^{\ddagger}(a+b)]^{\ddagger} = a^2a^{\ddagger}(a+b)^{\ddagger}.$$

According to [8, Theorem 2.5], it follows that $(1+a^{\ddagger}b)^{\ddagger} = a^{\Pi} + a^2a^{\ddagger}(a+b)^{\ddagger}$.

Conversely, let $\xi = 1 + a^{\ddagger}b$ be p-Drazin invertible and

$$\begin{aligned} x &= a^{\ddagger}\xi^{\ddagger} + a^{\Pi}b(a^{\ddagger}\xi^{\ddagger})^2 + \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} + b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi} \\ &= x_1 + x_2, \end{aligned}$$

where $x_1 = a^{\ddagger}\xi^{\ddagger} + a^{\Pi}b(a^{\ddagger}\xi^{\ddagger})^2$ and $x_2 = \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} + b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}$ $(-a)^i a^{\Pi}$.

We prove x is the p-Drazin inverse of $a+b$ by three steps.

Step 1. We prove that $(a+b)x = x(a+b)$.

First, we give the following equalities.

By Lemma 2.3 and Corollary 2.5, it follows that $(a+b)a^{\Pi}b(a^{\ddagger})^2 = 0$ and hence $(a+b)a^{\Pi}b(a^{\ddagger}\xi^{\ddagger})^2 = 0$.

Similarly, $(a+b)b^{\Pi}a(b^{\ddagger})^2 = 0$, so $(a+b)b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi} = 0$.

Let $y_1 = (a+b)a^{\ddagger}\xi^{\ddagger}$ and $y_2 = (a+b)\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi}$. Then

$$\begin{aligned} (a+b)x &= (a+b)[a^{\ddagger}\xi^{\ddagger} + \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi}] \\ &= y_1 + y_2. \end{aligned}$$

To prove $x(a+b) = y_1 + y_2$, we check that $x_1(a+b) = y_1$ and $x_2(a+b) = y_2$.

By Lemma 2.3, we have $a^{\ddagger}\xi^{\ddagger} = \xi^{\ddagger}a^{\ddagger}$ and hence

$$\begin{aligned} x_1(a+b) &= [a^{\ddagger}\xi^{\ddagger} + a^{\Pi}b(a^{\ddagger}(\xi^{\ddagger})^2](a+b) \\ &\overset{(2.5)}{=} a^{\ddagger}(a+b)\xi^{\ddagger} + a^{\Pi}b(a^{\ddagger})^2(a+b)(\xi^{\ddagger})^2 \\ &= a^{\ddagger}(a+b)\xi^{\ddagger} + (a^{\Pi}ba^{\ddagger} + a^{\Pi}ba^{\ddagger}a^{\ddagger}b)(\xi^{\ddagger})^2 \\ &= a^{\ddagger}(a+b)\xi^{\ddagger} + a^{\Pi}ba^{\ddagger}\xi(\xi^{\ddagger})^2 \\ &= a^{\ddagger}(a+b)\xi^{\ddagger} + a^{\Pi}ba^{\ddagger}\xi^{\ddagger} \\ &= y_1. \end{aligned}$$

According to $b^{\ddagger}a^{\Pi}b = b^{\ddagger}(1-aa^{\ddagger})b = bb^{\ddagger} - (b^{\ddagger}a^{\ddagger})ba = bb^{\ddagger} - bb^{\ddagger}a^{\ddagger}a = bb^{\ddagger}a^{\Pi}$, (2.12)

we obtain

$$\begin{aligned}
x_2(a+b) &= [\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} + b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi}](a+b) \\
&= -\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i+1}a^{\Pi} + \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi}b \\
&\quad -b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^{i+1}a^{\Pi} + b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi}b \\
&= -\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi} + \sum_{i=0}^{\infty}b^{\ddagger}(-b^{\ddagger}a)^i a^{\Pi}b - b^{\Pi}ab^{\ddagger}\sum_{i=0}^{\infty}(i+1)(-b^{\ddagger}a)^{i+1}a^{\Pi} \\
&\quad +b^{\Pi}ab^{\ddagger}\sum_{i=0}^{\infty}(i+1)(-b^{\ddagger}a)^i b^{\ddagger}a^{\Pi}b \\
&= -\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i b^{\ddagger}a^{\Pi}b - b^{\Pi}ab^{\ddagger}\sum_{i=0}^{\infty}(i+1)(-b^{\ddagger}a)^{i+1}a^{\Pi} \\
&\quad +b^{\Pi}ab^{\ddagger}\sum_{i=0}^{\infty}(i+1)(-b^{\ddagger}a)^i bb^{\ddagger}a^{\Pi} \\
&= -\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i bb^{\ddagger}a^{\Pi} - b^{\Pi}ab^{\ddagger}\sum_{i=1}^{\infty}i(-b^{\ddagger}a)^i a^{\Pi} \\
&\quad +b^{\Pi}ab^{\ddagger}bb^{\ddagger}\sum_{i=0}^{\infty}(i+1)(-b^{\ddagger}a)^i a^{\Pi} \\
&= -\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i bb^{\ddagger}a^{\Pi} - b^{\Pi}ab^{\ddagger}\sum_{i=1}^{\infty}i(-b^{\ddagger}a)^i a^{\Pi} \\
&\quad +b^{\Pi}ab^{\ddagger}\sum_{i=1}^{\infty}i(-b^{\ddagger}a)^i a^{\Pi} + b^{\Pi}ab^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i a^{\Pi} \\
&= -\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i bb^{\ddagger}a^{\Pi} + ab^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i a^{\Pi} \\
&\quad -bb^{\ddagger}ab^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i a^{\Pi} \\
&= -\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i bb^{\ddagger}a^{\Pi} + ab^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i a^{\Pi} \\
&\quad -b^{\ddagger}a\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i a^{\Pi}
\end{aligned}$$

$$\begin{aligned}
&= ab^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i}a^{\Pi} + bb^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i}a^{\Pi}\\
&= a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi} + b\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&= y_2.
\end{aligned}$$

Hence, $x(a+b) = (a+b)x$.

Step 2. We have $x(a+b)x = x$. Indeed,

$$\begin{aligned}
x(a+b)x &= x(a+b)[a^{\ddagger}\xi^{\ddagger} + \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}]\\
&= (a+b)[a^{\ddagger}\xi^{\ddagger} + \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}][a^{\ddagger}\xi^{\ddagger} + \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}]\\
&= (a+b)(a^{\ddagger}\xi^{\ddagger})^2 + (a+b)a^{\ddagger}\xi^{\ddagger}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&\quad +(a+b)\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&= z_1 + z_2 + z_3,
\end{aligned}$$

where $z_1 = (a+b)(a^{\ddagger}\xi^{\ddagger})^2$, $z_2 = (a+b)a^{\ddagger}\xi^{\ddagger}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}$ and

$$z_3 = (a+b)\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}.$$

Further, we have

$$\begin{aligned}
z_1 &= (a+b)(a^{\ddagger}\xi^{\ddagger})^2 = (a\xi + a^{\Pi}b)(a^{\ddagger}\xi^{\ddagger})^2\\
&= a\xi(a^{\ddagger}\xi^{\ddagger})^2 + a^{\Pi}b(a^{\ddagger}\xi^{\ddagger})^2\\
&\overset{(2.5)}{=} a^{\ddagger}\xi^{\ddagger} + a^{\Pi}b(a^{\ddagger}\xi^{\ddagger})^2
\end{aligned}$$

and

$$\begin{aligned}
z_2 &= (a+b)a^{\ddagger}\xi^{\ddagger}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&= \xi^{\ddagger}aa^{\ddagger}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi} + b\xi^{\ddagger}a^{\ddagger}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}
\end{aligned}$$

$$\begin{aligned}
&= \xi^{\ddagger}\sum_{i=0}^{\infty}a^{\ddagger}ab^{\ddagger}(-b^{\ddagger}a)^{i}a^{\Pi}+b\xi^{\ddagger}\sum_{i=0}^{\infty}(a^{\ddagger})^{2}ab^{\ddagger}(-b^{\ddagger}a)^{i}a^{\Pi}\\
&\overset{(2.7)}{=} -\xi^{\ddagger}\sum_{i=0}^{\infty}a^{\ddagger}(-ab^{\ddagger})^{i+1}a^{\Pi}-b\xi^{\ddagger}\sum_{i=0}^{\infty}(a^{\ddagger})^{2}(-ab^{\ddagger})^{i+1}a^{\Pi}\\
&\overset{(2.3)}{=} -\xi^{\ddagger}\sum_{i=0}^{\infty}(-ab^{\ddagger})^{i+1}a^{\ddagger}a^{\Pi}-b\xi^{\ddagger}\sum_{i=0}^{\infty}(-ab^{\ddagger})^{i+1}(a^{\ddagger})^{2}a^{\Pi}\\
&= 0.
\end{aligned}$$

Next, we show that $z_3=\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}+b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^{i}a^{\Pi}$. (2.13)

One can see that

$$\begin{aligned}
z_3 &= (a+b)\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&= b\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}+a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&= [bb^{\ddagger}a^{\Pi}+\sum_{i=1}^{\infty}(-b^{\ddagger}a)^{i}a^{\Pi}]\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}+a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&= bb^{\ddagger}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}-bb^{\ddagger}aa^{\ddagger}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}+\sum_{i=1}^{\infty}(-b^{\ddagger}a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&\quad +a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&= \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}-bb^{\ddagger}aa^{\ddagger}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}+\sum_{i=1}^{\infty}(-b^{\ddagger}a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&\quad +a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&= \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}+bb^{\ddagger}\sum_{i=0}^{\infty}(-ab^{\ddagger})^{i+1}a^{\ddagger}a^{\Pi}+\sum_{i=1}^{\infty}(-b^{\ddagger}a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\\
&\quad +a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^{i}a^{\Pi}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} + \sum_{i=1}^{\infty}(-b^{\ddagger}a)^i a^{\Pi} \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} \\
&\quad + a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} \\
&= \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} + d_1 + d_2,
\end{aligned}
$$

where

$$d_1 = \sum_{i=1}^{\infty}(-b^{\ddagger}a)^i a^{\Pi} \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi}$$

and

$$d_2 = a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi}.$$

Noting equality (2.13). We only need to prove

$$d_1 + d_2 = b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi}.$$

Note that

$$
\begin{aligned}
& b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi} \\
&= a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi} - bb^{\ddagger}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi}.
\end{aligned}
$$

We next prove that

$$d_1 = -bb^{\ddagger}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi} \text{ and } d_2 = a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^i a^{\Pi}.$$

As $bb^{\ddagger}$ commutes with $b^{\ddagger}a$, then

$$
\begin{aligned}
d_1 &= \sum_{i=1}^{\infty}(-b^{\ddagger}a)^i a^{\Pi} \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^i a^{\Pi} \overset{(2.11)}{=} \sum_{i=1}^{\infty}(-bb^{\ddagger}b^{\ddagger}a)^i a^{\Pi} \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i b^{\ddagger}a^{\Pi} \\
&= bb^{\ddagger}\sum_{i=1}^{\infty}(-b^{\ddagger}a)^i a^{\Pi} \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i b^{\ddagger}a^{\Pi} = -bb^{\ddagger}\sum_{i=1}^{\infty}(-b^{\ddagger}a)^{i-1}b^{\ddagger}aa^{\Pi} \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i b^{\ddagger}a^{\Pi} \\
&\overset{(2.4)}{=} -bb^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i(b^{\ddagger}aa^{\Pi})b^{\ddagger}a^{\Pi} = -bb^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i b^{\ddagger}(b^{\ddagger}aa^{\Pi})a^{\Pi} \\
&= -bb^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i(b^{\ddagger})^2aa^{\Pi} \overset{(2.4)}{=} -bb^{\ddagger}(b^{\ddagger})^2a\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i \sum_{i=0}^{\infty}(-b^{\ddagger}a)^i a^{\Pi}
\end{aligned}
$$

$$\begin{aligned}
&\overset{(2.4)}{=} -b(b^{\ddagger})^2ab^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i\sum_{i=0}^{\infty}(-b^{\ddagger}a)^ia^{\Pi} \overset{(2.4)}{=} -bb^{\ddagger}a(b^{\ddagger})^2\sum_{i=0}^{\infty}(i+1)(-b^{\ddagger}a)^ia^{\Pi}\\
&\overset{(2.11)}{=} -bb^{\ddagger}a(b^{\ddagger})^2\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^i(-a)^ia^{\Pi}\\
&= -bb^{\ddagger}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^ia^{\Pi}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
d_2 &= a\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi}\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi} \overset{(2.11)}{=} a\sum_{i=0}^{\infty}(-b^{\ddagger}a)^ib^{\ddagger}a^{\Pi}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^ib^{\ddagger}a^{\Pi}\\
&\overset{(2.4)}{=} a\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i\sum_{i=0}^{\infty}(-b^{\ddagger}a)^ib^{\ddagger}a^{\Pi}b^{\ddagger}a^{\Pi} \overset{(2.4)}{=} a\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i\sum_{i=0}^{\infty}(-b^{\ddagger}a)^ib^{\ddagger}b^{\ddagger}a^{\Pi}a^{\Pi}\\
&= a\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i(b^{\ddagger})^2a^{\Pi} \overset{(2.4)}{=} a(b^{\ddagger})^2\sum_{i=0}^{\infty}(-b^{\ddagger}a)^i\sum_{i=0}^{\infty}(-b^{\ddagger}a)^ia^{\Pi}\\
&= a(b^{\ddagger})^2\sum_{i=0}^{\infty}(i+1)(-b^{\ddagger}a)^ia^{\Pi} \overset{(2.11)}{=} a(b^{\ddagger})^2\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^i(-a)^ia^{\Pi}\\
&= a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^ia^{\Pi}.
\end{aligned}$$

Hence $z_3 = \sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi} + b^{\Pi}a\sum_{i=0}^{\infty}(i+1)(b^{\ddagger})^{i+2}(-a)^ia^{\Pi}$.

Step 3. We show that $(a+b)^k-(a+b)^{k+1}x \in J(\mathscr{A})$ for some integer $k \geqslant 1$. We have

$$\begin{aligned}
&(a+b)-(a+b)^2x\\
&= (a+b)-[a^{\ddagger}\xi^{\ddagger}+\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi}](a+b)^2\\
&\overset{(2.5)}{=} (a+b)-\xi^{\ddagger}a^{\ddagger}(a+b)^2-\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi}(a+b)^2\\
&= (a+b)-\xi^{\ddagger}a(a^{\ddagger}(a+b))^2-\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi}(a^2+ab+ba+b^2)\\
&= a\xi+a^{\Pi}b-\xi^{\ddagger}a(\xi-a^{\Pi})^2-\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi}a^2-\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi}ab\\
&\quad -\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi}ba-\sum_{i=0}^{\infty}(b^{\ddagger})^{i+1}(-a)^ia^{\Pi}b^2
\end{aligned}$$

$$\begin{aligned}
&\overset{(2.11)}{=} a\xi + a^{\Pi}b - \xi^{\ddagger}a(\xi^2 - a^{\Pi}) + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}aa^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi}b \\
&\quad - \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i}bb^{\ddagger}aa^{\Pi} - \sum_{i=0}^{\infty} b^{\ddagger}(-b^{\ddagger}a)^{i}a^{\Pi}b^2 \\
&= a\xi + a^{\Pi}b - \xi^{\ddagger}a(\xi^2 - a^{\Pi}) - bb^{\ddagger}aa^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi}b \\
&\quad - \sum_{i=0}^{\infty} b^{\ddagger}(-b^{\ddagger}a)^{i}a^{\Pi}b^2 \\
&\overset{(2.4)}{=} a\xi + a^{\Pi}b - \xi^{\ddagger}a(\xi^2 - a^{\Pi}) - bb^{\ddagger}aa^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi}b \\
&\quad - \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i}b^{\ddagger}a^{\Pi}b^2 \\
&\overset{(2.12)}{=} a\xi + a^{\Pi}b - \xi^{\ddagger}a(\xi^2 - a^{\Pi}) - bb^{\ddagger}aa^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi}b \\
&\quad - \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i}bb^{\ddagger}a^{\Pi}b \\
&\overset{(2.4)}{=} a\xi + a^{\Pi}b - \xi^{\ddagger}a(\xi^2 - a^{\Pi}) - bb^{\ddagger}aa^{\Pi} + \sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i+1}a^{\Pi}b \\
&\quad - \sum_{i=0}^{\infty} bb^{\ddagger}(-b^{\ddagger}a)^{i}a^{\Pi}b \\
&= a\xi + a^{\Pi}b - \xi^{\ddagger}a\xi^2 + \xi^{\ddagger}aa^{\Pi} - bb^{\ddagger}aa^{\Pi} - bb^{\ddagger}a^{\Pi}b \\
&= (a\xi - \xi^{\ddagger}a\xi^2) + \xi^{\ddagger}aa^{\Pi} + (a^{\Pi}b - bb^{\ddagger}a^{\Pi}b) - bb^{\ddagger}aa^{\Pi} \\
&= a\xi\xi^{\Pi} + \xi^{\ddagger}aa^{\Pi} + b^{\Pi}a^{\Pi}b - bb^{\ddagger}aa^{\Pi}.
\end{aligned}$$

Next, we show $[(a+b)-(a+b)^2x]^k \in J(\mathscr{A})$.

Firstly, We prove $(a\xi\xi^{\Pi} + \xi^{\ddagger}aa^{\Pi})^{m_1} \in J(\mathscr{A})$ for some integer $m_1 \geq 1$.

By Lemma 2.3, it follows that $(a\xi\xi^{\Pi})^{k_1} = a^{k_1}(\xi\xi^{\Pi})^{k_1} \in J(\mathscr{A})$ and $(\xi^{\ddagger}aa^{\Pi})^{k_2} = (\xi^{\ddagger})^{k_2}(aa^{\Pi})^{k_2} \in J(\mathscr{A})$.

Again, Lemma 2.3 guarantees that $a\xi\xi^{\Pi}$ commutes with $\xi^{\ddagger}aa^{\Pi}$. Take $m_1 = k_1+k_2$, it follows from Lemma 2.7 (2) that $(a\xi\xi^{\Pi}+\xi^{\ddagger}aa^{\Pi})^{m_1} = a^{k_1}(\xi\xi^{\Pi})^{k_1}+(\xi^{\ddagger})^{k_2}(aa^{\Pi})^{k_2} \in J(\mathscr{A})$.

Secondly, we present $(b^{\Pi}a^{\Pi}b - bb^{\ddagger}aa^{\Pi})^{m_2} \in J(\mathscr{A})$ for some $m_2 \geq 1$.

By induction, we obtain $(b^{\Pi}a^{\Pi}b)^{k_3} = b^{\Pi}a^{\Pi}(bb^{\Pi})^{k_3-1}a^{\Pi}b \in J(\mathscr{A})$ and $(bb^{\ddagger}aa^{\Pi})^{k_4} = bb^{\ddagger}(aa^{\Pi})^{k_4} \in J(\mathscr{A})$. One can see that

$$(bb^{\ddagger}aa^{\Pi})^2b^{\Pi}a^{\Pi}b = bb^{\ddagger}aa^{\Pi}b^{\Pi}a^{\Pi}bbb^{\ddagger}aa^{\Pi} = 0$$

and

$$(b^{\Pi}a^{\Pi}b\ bb^{\ddagger}aa^{\Pi})^2 = b^{\Pi}a^{\Pi}b(bb^{\ddagger}aa^{\Pi})b^{\Pi}a^{\Pi}b = 0.$$

It follows from Lemma 2.6 (2) that

$$\begin{aligned}&(b^{\Pi}a^{\Pi}b-bb^{\ddagger}aa^{\Pi})^{m_2}\\ =&\sum_{i=0}^{m_2-1}C_{m_2-1}^{i}[(b^{\Pi}a^{\Pi}b)^{m_2-i}(-bb^{\ddagger}aa^{\Pi})^{i}+(-bb^{\ddagger}aa^{\Pi})^{m_2-i}(b^{\Pi}a^{\Pi}b)^{i}]\in J(\mathscr{A})\end{aligned}$$

for $m_2=k_3+k_4$.

Pose $a_1=a\xi\xi^{\Pi}+\xi^{\ddagger}aa^{\Pi}$ and $a_2=b^{\Pi}a^{\Pi}b-bb^{\ddagger}aa^{\Pi}$. By virtue of Lemma 2.3 and Corollary 2.5, it is straight forward to check

$$(a_1)^2a_2=a_1a_2a_1 \text{ and } (a_2)^2a_1=a_2a_1a_2.$$

Hence, there exists $k=m_1+m_2$ such that

$$(a_1+a_2)^k=\sum_{i=0}^{k-1}C_{k-1}^{i}(a_1^{k-i}a_2^{i}+a_2^{k-i}a_1^{i})\in J(\mathscr{A}),$$

that is $(a+b)^k-(a+b)^{k+1}x=[(a+b)-(a+b)^2x]^k=(a_1+a_2)^k\in J(\mathscr{A})$

This completes the proof. ■

Corollary 2.11. [8, Theorem 2.7]. *If $a,b\in\mathscr{A}^{pD}$ and $ab=ba$, then $a+b\in\mathscr{A}^{pD}$ if and only if $1+a^{\ddagger}b\in\mathscr{A}^{pD}$. In this case, we have*

$$(a+b)^{\ddagger}=(1+a^{\ddagger}b)^{\ddagger}a^{\ddagger}+b^{\ddagger}\sum_{i=0}^{\infty}(-b^{\ddagger}a)^{i}a^{\Pi},$$

and

$$(1+a^{\ddagger}b)^{\ddagger}=a^{\Pi}+a^2a^{\ddagger}(a+b)^{\ddagger}.$$

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