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SOME NORMAL CRITERIA FOR FAMILIES OF MEROMORPHIC FUNCTIONS

Bing Xiao, Weiling Xiong and Wenjun Yuan*

Abstract. In the paper, we study the normality of families of meromorphic functions related a Hayman Conjecture. We consider whether a family meromorphic functions $\mathcal{F}$ is normal in D, if for each function f in $\mathcal{F}$, $f' + af^n = b$ has at most one zero, where n is a positive integer, a and $b \neq 0$ are two finite complex numbers. Some examples show that the conditions in our results are best possible.

1. Introduction and Main Results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in a domain $D \subseteq \mathbb{C}$, and let a be a finite complex value. We say that f and g share a CM (or IM) in D provided that $f-a$ and $g-a$ have the same zeros counting (or ignoring) multiplicity in D. When $a=\infty$ the zeros of $f-a$ means the poles of f (see [21]). It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory ([8, 9, 20] or [21]).

Bloch's principle [1] states that every condition which reduces a meromorphic function in the plane $\mathbb{C}$ to be a constant forces a family of meromorphic functions in a domain D normal. Although the principle is false in general (see [17]), many authors proved normality criterion for families of meromorphic functions corresponding to Liouville-Picard type theorem (see [6] or [20]).

It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schwick [18] first proved an interesting result that a family of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivative. And later, more

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*Corresponding author.

results about normality criteria concerning shared values have emerged, for instance, (see [15, 16, 24]). In recent years, this subject has attracted the attention of many researchers worldwide.

We now first introduce a normality criterion related to a Hayman normal conjecture [10].

Theorem 1.1. *Let $\mathcal{F}$ be a family of holomorphic (meromorphic) functions defined in a domain D , $n \in \mathbb{N}, a \neq 0, b \in \mathbb{C}$. If $f'(z) + af^n(z) - b \neq 0$ for each function $f(z) \in \mathcal{F}$ and $n \geq 2(n \geq 3)$, then $\mathcal{F}$ is normal in D.*

The results for the holomorphic case are due to Drasin [6] for $n \geq 3$, Pang [14] for $n = 3$, Chen and Fang [4] for $n = 2$, Ye [22] for $n = 2$, Chen and Gu [5] for the generalized result with a and b replaced by meromorphic functions. The results for the meromorphic case are due to Li [12], Li [13] and Langley [11] for $n \geq 5$, Pang [14] for $n = 4$, Chen and Fang [4] for $n = 3$, Zalcman [26] for $n = 3$, obtained independently.

When $n = 2$ and $\mathcal{F}$ is meromorphic, Theorem 1.1 is not valid in general. Fang and Yuan [7] gave an example to show this, and got a special result below.

Example 1.1. The family of meromorphic functions $\mathcal{F} = \{f_j(z) = \frac{jz}{(\sqrt{j}z-1)^2} : j = 1, 2, \cdots, \}$ is not normal in $D = \{z : |z| < 1\}$. This is deduced by $f_j^{\#}(0) = j \to \infty$, as $j \to \infty$ and Marty's criterion [8], although for any $f_j(z) \in \mathcal{F}, f_j' + f_j^2 = j(\sqrt{j}z - 1)^{-4} \neq 0$.

Here $f^{\#}(\xi)$ denotes the spherical derivative

$$f^{\#}(\xi) = \frac{|f'(\xi)|}{1 + |f(\xi)|^2}.$$

Theorem 1.2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and $, a \neq 0, b \in \mathbb{C}$. If $f'(z) + a(f(z))^2 - b \neq 0$ and the poles of $f(z)$ are of multiplicity ≥ 3 for each $f(z) \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

It is nature to ask whether the conditions in above theorems that $f'(z)+af^n(z)-b \neq 0$ can be relaxed. In this paper, we answer above question and prove the following results.

Theorem 1.3. *Let $\mathcal{F}$ be a family of meromorphic (holomorphic) functions in D, n be a positive integer and a, b be two finite complex numbers such that $a \neq 0$. If $n \geq 4$ $(n \geq 2)$ and for each function f in $\mathcal{F}$, $f' + af^n - b$ has at most one zero in D, ignoring multiplicity, then $\mathcal{F}$ is normal in D.*

Example 1.2. The family of meromorphic functions $\mathcal{F} = \{f_j(z) = \frac{1}{\sqrt{j}(z-\frac{1}{j})} : j = 1, 2, \cdots, \}$ is not normal in $D = \{z : |z| < 1\}$. Obviously $f_j' - f_j^3 = -\frac{z}{\sqrt{j}(z-\frac{1}{j})^3}$. So for each $j, f_j' - f_j^3$ takes the value 0 in D, but $\mathcal{F}$ is not normal at the point $z = 0$, since $f_j^{\#}(0) = \frac{2(\sqrt{j})^3}{1+j} \to \infty$, as $j \to \infty$.

Remark 1.4. Example 1.2 shows that Theorem 1.3 is not valid when $n = 3$, and the condition $n = 4$ is best possible for meromorphic case.

Theorem 1.5. *Let $\mathcal{F}$ be a family of meromorphic functions in D, a and b be two finite complex numbers such that $a \neq 0$. Suppose that each $f(z) \in \mathcal{F}$ has no simple pole. If for each function f in $\mathcal{F}$, $f' + af^3 - b$ has at most one zero in D, ignoring multiplicity, then $\mathcal{F}$ is normal in D.*

Remark 1.6. Example 1.2 shows that the condition added in Theorem 1.5 about the multiplicity of poles of $f(z)$ is best possible.

Theorem 1.7. *Let $\mathcal{F}$ be a family of meromorphic functions in D, a and b be two finite complex numbers such that $a \neq 0$. Suppose that $f(z)$ admits the zeros of multiple and the poles of multiplicity ≥ 3 for each $f(z) \in \mathcal{F}$. If for each function f in $\mathcal{F}$, $f' + af^2 - b$ has at most one zero in D, ignoring multiplicity, then $\mathcal{F}$ is normal in D.*

Remark 1.8. Example 1.1 shows that the condition added in Theorem 1.7 about the multiplicity of poles and zeros of $f(z)$ is best possible.

Theorem 1.9. *Let $\mathcal{F}$ be a family of meromorphic functions in D, a and b be two non-zero finite complex numbers. Suppose that $f(z) \neq 0$, its poles are multiple and $f' + af - b$ has at most one zero in D for each $f(z) \in \mathcal{F}$, ignoring multiplicity, then $\mathcal{F}$ is normal in D.*

Corollary 1.10. *Let $\mathcal{F}$ be a family of holomorphic functions in D, a and b be two finite complex numbers such that $b \neq 0$. Suppose that $f(z) \neq 0$ for each $f(z) \in \mathcal{F}$. If for each function f in $\mathcal{F}$, $f' + af - b$ has at most one zero in D, ignoring multiplicity, then $\mathcal{F}$ is normal in D.*

Example 1.3. *The family of holomorphic functions $\mathcal{F} = \{f_j(z) = jze^z - je^z + j - b : j = 1, 2, \cdots, \}$ is not normal in $D = \{z : |z| < 1\}$. Obviously $f_j' - f_j = j(e^z - 1) + b$. So for each $j, f_j' - f_j$ takes the value b in D. On the other hand, $f_j(0) = -b,\ f_j(\frac{1}{\sqrt{j}}) = \sqrt{j}(1 + \frac{1}{\sqrt{j}} + o(1)) \to \infty$, as $j \to \infty$. This implies that the family $\mathcal{F}$ fails to be equicontinuous at 0, and thus $\mathcal{F}$ is not normal at 0.*

In 2011, Yuan et al. [23] proved the following theorem.

Theorem 1.11. *Let $\mathcal{F}$ be a family of meromorphic functions in D, a and b be two finite complex numbers such that $b \neq 0$. Suppose that $f(z) \neq 0$ and $f'(z) - af(z) \neq b$ for each $f(z) \in \mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

Example 1.4. The family of holomorphic functions $\mathcal{F} = \{f_j(z) = j(z+1) - 1 : j = 1, 2, \cdots, \}$ is normal in $D = \{z : |z| < 1\}$. Obviously $f_j(z) \neq 0$ and $f_j' - f_j = -jz + 1$. So for each $j, f_j' - f_j$ takes the value 1 in D. Corollary 1.10 implies that the family $\mathcal{F}$ is normal in D.

Example 1.5. *The family of meromorphic functions $\mathcal{F} = \{f_j(z) = \frac{z}{j} - 1 : j = 1, 2, \cdots, \}$ is normal in $D = \{z : |z| < 1\}$. The reason is the conditions of Theorem 1.11 hold that $f_j(z) \neq 0$ and $f_j' - f_j = \frac{1-z}{j} + 1 \neq 1$ in $D = \{z : |z| < 1\}$.*

Remark 1.12. Example 1.3 shows that Theorem 1.3 is not valid when $n = 1$ and holomorphic case, and the condition $f(z) \neq 0$ is necessary in Theorem 1.9, Corollary 1.10. Both Example 1.4 and Example 1.5 tell us that Corollary 1.10 and Theorem 1.11 occur.

2. Preliminary Lemmas

In order to prove our result, we need the following lemmas. The first is the extended version Zalcman's [25] concerning normal families.

Lemma 2.1. [27]. *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc satisfying all zeros of functions in $\mathcal{F}$ which have multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ which have multiplicity $\geq q$. Let α be a real number satisfying $-q < \alpha < p$. Then $\mathcal{F}$ is not normal at 0 if and only if there exist*

(a) *a number* $0 < r < 1$;

(b) *points* z_n *with* $|z_n| < r$;

(c) *functions* $f_n \in \mathcal{F}$;

(d) *positive numbers* $\rho_n \to 0$

such that $g_n(\zeta) := \rho^{-\alpha} f_n(z_n + \rho_n\zeta)$ converges spherically uniformly on each compact subset of $\mathbb{C}$ to a non-constant meromorphic function $g(\zeta)$, whose all zeros have multiplicity $\geq p$ and all poles have multiplicity $\geq q$ and order is at most 2.

Remark 2.2. If $\mathcal{F}$ is a family of holomorphic functions on the unit disc in Lemma 2.1, then $g(\zeta)$ is a nonconstant entire function whose order is at most 1.

The order of g is defined by using the Nevanlinna's characteristic function $T(r, g)$:

$$\rho(g) = \lim_{r\to\infty} \sup \frac{\log T(r, g)}{\log r}.$$

Lemma 2.3. [3] or [19]. *Let $f(z)$ be a meromorphic function and $c \in \mathbb{C}\backslash\{0\}$. If $f(z)$ has neither simple zero nor simple pole, and $f'(z) \neq c$, then $f(z)$ is constant.*

Lemma 2.4. [2]. *Let $f(z)$ be a transcendental meromorphic function of finite order in $\mathbb{C}$, and have no simple zero, then $f'(z)$ assumes every non-zero finite value infinitely often.*

Lemma 2.5. [9]. *Let $f(z)$ be a meromorphic function in $\mathbb{C}$, then*

$$T(r,f) \leq (2+\frac{1}{k})N(r,\frac{1}{f}) + (2+\frac{2}{k})\overline{N}(r,\frac{1}{f^{(k)}-1}) + S(r,f). \tag{2.1}$$

and

$$T(r,f) \leq \overline{N}(r,f) + N(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f^{(k)}-1}) + S(r,f). \tag{2.2}$$

Remark 2.6. Both (2.1) and (2.2) are called as Hayman inequality and Milloux inequality, respectively.

3. Proof of the Results

Proof of Theorem 1.3. Suppose that $\mathcal{F}$ is a family meromorphic and not normal in D. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0. Without loss of generality we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$ and functions $f_j \in \mathcal{F}$ such that

$$g_j(\xi) = \rho_j^{\frac{1}{n-1}} f_j(z_j + \rho_j\xi) \Rightarrow g(\xi) \tag{3.1}$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function in $\mathbb{C}$. Moreover, the order of g is less than 2.

From (3.1) we know

$$g_j'(\xi) = \rho_j^{\frac{n}{n-1}} f_j'(z_j + \rho_j\xi) \Rightarrow g'(\xi)$$

and

$$\begin{aligned} \rho_j^{\frac{n}{n-1}}(f_j'(z_j+\rho_j\xi) + af_j^n(z_j+\rho_j\xi) - b) &= g_j'(\xi) + ag_j^n(\xi) - \rho_j^{\frac{n}{n-1}}b \\ &\Rightarrow g'(\xi) - ag^n(\xi) \end{aligned} \tag{3.2}$$

in $\mathbb{C}\backslash\mathbf{S}$ locally uniformly with respect to the spherical metric, where $\mathbf{S}$ is the set of all poles of $g(\xi)$.

If $g' + ag^n \equiv 0$ then $\frac{1}{n-1}\frac{1}{g^{n-1}} \equiv a\xi + c$ where c is a constant. This contradicts with g being a meromorphic function and $n \geq 4$. So $g' + ag^n \not\equiv 0$.

If $g' + ag^n \neq 0$, then $\frac{g'}{g^n} \neq -a$. Set $g = \frac{1}{\varphi}$, then $\varphi^{n-2}\varphi' \neq a$. By Lemma 2.3 then φ is a constant, so g is also a constant which is a contradiction with g being a non-constant. Hence, $g' + ag^n$ is a non-constant meromorphic function and has at least one zero.

Next we prove that $g' + ag^n$ has just a unique zero. By contraries, let ξ_0 and ξ_0^* be two distinct zeros of $g' + ag^n$, and choose $\delta(> 0)$ small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$ where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From(3.2), by $Hurwitz's$ theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$f_j'(z_j + \rho_j\xi_j) + af_j^n(z_j + \rho_j\xi_j) - b = 0,$$

$$f_j'(z_j + \rho_j\xi_j^*) + af_j^n(z_j + \rho_j\xi_j^*) - b = 0.$$

Since $z_j \to 0$, positive numbers $\rho_j \to 0$, we have $z_j + \rho_j\xi_j \in D(\xi_0, \delta)$, $z_j + \rho_j\xi_j^* \in D(\xi_0^*, \delta)$ for sufficiently large j. Thus each $f_j'(z) + af_j^n(z) - b$ has two distinct zeros, which contradicts with our hypothesis. So $g' + ag^n$ has just a unique zero, which can be denoted by ξ_0.

Set $g = \frac{1}{\varphi}$ again, then $g' + ag^n = -\frac{\varphi'\varphi^{n-2}-a}{\varphi^n}$. So $\frac{\varphi'\varphi^{n-2}-a}{\varphi^n}$ has only a unique zero ξ_0. Therefore ξ_0 is a multiple pole of φ, or else a zero of $\varphi'\varphi^{n-2} - a$. If ξ_0 is a multiple pole of φ, since $\frac{\varphi'\varphi^{n-2}-a}{\varphi^n}$ has only one zero ξ_0, then $\varphi'\varphi^{n-2} - a \neq 0$. By Lemma 2.3 again, φ is a constant which contradicts with the idea that g is a non-constant.

So φ has no multiple pole and $\varphi'\varphi^{n-2} - a$ has just a unique zero ξ_0. By Lemma 2.3, φ is not any transcendental function.

If φ is a non-constant polynomial, then $\varphi'\varphi^{n-2} - a = A(\xi - \xi_0)^l$, where A is a non-zero constant, l is a positive integer, $l \geq n - 2 \geq 2$. Set $\psi = \frac{1}{n-1}\varphi^{n-1}$, then $\psi' = A(\xi - \xi_0)^l + a$, and $\psi'' = Al(\xi - \xi_0)^{l-1}$. Note that $n \geq 4$, we see that the zeros of ψ are of multiplicities $\geq n - 1 \geq 3$. But ψ'' has only one zero ξ_0, so ψ has only the same zero ξ_0 too. Hence $\psi'(\xi_0) = 0$ which contradicts with $\psi'(\xi_0) = a \neq 0$. Therefore φ and ψ are rational functions which are not polynomials, and $\psi' - a$ has just a unique zero ξ_0.

Next we prove that there exists no rational function such as ψ. Noting that $\psi = \frac{1}{n-1}\varphi^{n-1}$ and φ has no multiple pole, we consider two Csaes.

Case 1. $\psi(\xi)$ has zero.

We can set

$$\psi(\xi) = A\frac{(\xi - \xi_1)^{m_1}(\xi - \xi_2)^{m_{n-1}} \cdots (\xi - \xi_s)^{m_s}}{(\xi - \eta_1)^{n-1}(\xi - \eta_{n-1})^{n-1} \cdots (\xi - \eta_t)^{n-1}} \tag{3.3}$$

where A is a non-zero constant, $s \geq 1,\ t \geq 1,\ m_i \geq n - 1\ (i = 1, 2, \cdots, s)$. For stating briefly, denote

$$m = m_1 + m_2 + \cdots + m_s \geq (n - 1)s. \tag{3.4}$$

From (3.3) then

$$\psi'(\xi) = \frac{A(\xi-\xi_1)^{m_1-1}(\xi-\xi_2)^{m_2-1}\cdots(\xi-\xi_s)^{m_s-1}h(\xi)}{(\xi-\eta_1)^n(\xi-\eta_2)^n\cdots(\xi-\eta_t)^n} = \frac{p_1(\xi)}{q_1(\xi)}, \tag{3.5}$$

where

$$\begin{aligned} h(\xi) &= (m-t(n-1))\xi^{s+t-1} + a_{s+t-2}\xi^{s+t-2} + \cdots + a_0, \\ p_1(\xi) &= A(\xi-\xi_1)^{m_1-1}(\xi-\xi_2)^{m_2-1}\cdots(\xi-\xi_s)^{m_s-1}h(\xi), \\ q_1(\xi) &= (\xi-\eta_1)^n(\xi-\eta_2)^n\cdots(\xi-\eta_t)^n \end{aligned} \tag{3.6}$$

are polynomials. Since $\psi'(\xi)-a$ has only a unique zero ξ_0, set

$$\psi'(\xi) - a = \frac{B(\xi-\xi_0)^l}{(\xi-\eta_1)^n(\xi-\eta_2)^n\cdots(\xi-\eta_t)^n} \tag{3.7}$$

where B is a non-zero constant, so

$$\psi''(\xi) = \frac{(\xi-\xi_0)^{l-1}p_2(\xi)}{(\xi-\eta_1)^{n+1}(\xi-\eta_2)^{n+1}\cdots(\xi-\eta_t)^{n+1}} \tag{3.8}$$

where $p_2(\xi) = B(l-nt)\xi^t + b_{t-1}\xi^{t-1} + \cdots + b_0$ is a polynomial. From (3.5) we also have

$$\psi''(\xi) = \frac{(\xi-\xi_1)^{m_1-2}(\xi-\xi_2)^{m_2-2}\cdots(\xi-\xi_s)^{m_s-2}p_3(\xi)}{(\xi-\eta_1)^{n+1}(\xi-\eta_2)^{n+1}\cdots(\xi-\eta_t)^{n+1}} \tag{3.9}$$

where $p_3(\xi)$ is also a polynomial.

Let $\deg(p)$ denote the degree of a polynomial $p(\xi)$.

From (3.5), (3.6) then

$$\deg(h) \le s+t-1, \deg(p_1) \le m+t-1, \deg(q_1) = nt. \tag{3.10}$$

Similarly from (3.8), (3.9) and noting (3.10) then

$$\deg(p_2) \le t, \tag{3.11}$$

$$\deg(p_3) \le \deg(p_1) + t - 1 - (m-2s) \le 2t + 2s - 2, \tag{3.12}$$

Note that $m_i \ge n-1$ $(i = 1, 2, \cdots, s)$, it follows from (3.5) and (3.7) that $\psi'(\xi_i) = 0$ $(i = 1, 2, \cdots, s)$ and $\psi'(\xi_0) = a \ne 0$. Thus $\xi_0 \ne \xi_i$ $(i = 1, 2, \cdots, s)$, and then $(\xi-\xi_0)^{l-1}$ is a factor of $p_3(\xi)$. Hence we get that $l-1 \le \deg(p_3)$. Combining (3.8) and (3.9) we also have $m-2s = \deg(p_2) + l - 1 - \deg(p_3) \le \deg(p_2)$. By (3.11) we obtain

$$m - 2s \le \deg(p_2) \le t. \tag{3.13}$$

Since $m \geq (n-1)s$, we know by (3.13) and $n \geq 4$ that

$$s \leq t. \tag{3.14}$$

If $l \geq nt$, from (3.8) and (3.9), we have $l-1 \leq \deg(p_3)$. By (3.12), then $nt-1 \leq l-1 \leq \deg(p_3) \leq 2t+2s-2$. Noting (3.14), we obtain $(n-4)t+1 \leq 0$, a contradiction with $n \geq 4$.

If $l < 3t$, from (3.5) and (3.7), then $\deg(p_1) = \deg(q_1)$. Noting that $\deg(p_1) = m+t-i, 1 \leq i \leq s+t, \deg(p_2) = nt$, so $m+t-i = nt, m = (n-1)t+i \neq (n-1)t$. From (3.6), then $\deg(h) = s+t-1$, and then $\deg(p_1) = m+t-1$. Noting $\deg(q_1) = nt$, hence $m = (n-1)t+1$. By (3.13) then $(n-2)t \leq 2s-1$. From (3.14), we obtain $(n-4)t+1 \leq 0$ again, a contradiction with $n \geq 4$, too.

Case 2. $\psi(\xi)$ has no zero.

We can set

$$\psi(\xi) = \frac{A}{(\xi-\eta_1)^{n-1}(\xi-\eta_2)^{n-1}\cdots(\xi-\eta_t)^{n-1}}, \tag{3.15}$$

where A is a non-zero constant. In this case, $g(\xi)$ is an entire function. Then

$$\psi'(\xi) = \frac{Ap_1(\xi)}{(\xi-\eta_1)^{n}(\xi-\eta_2)^{n}\cdots(\xi-\eta_t)^{n}}, \tag{3.16}$$

where $p_1(\xi) = (l-n)t\xi^{t-1} + a_{t-2}\xi^{t-2} + \cdots + a_0$ is a polynomial. Since $\psi'(\xi) - a$ has only a unique zero ξ_0, set

$$\psi'(\xi) - a = \frac{B(\xi-\xi_0)^{l}}{(\xi-\eta_1)^{n}(\xi-\eta_2)^{n}\cdots(\xi-\eta_t)^{n}} \tag{3.7}$$

where B is a non-zero constant. Thus $l = nt$. Moreover, (3.7) gives

$$\psi''(\xi) = \frac{(\xi-\xi_0)^{l-1}p_2(\xi)}{(\xi-\eta_1)^{n+1}(\xi-\eta_2)^{n+1}\cdots(\xi-\eta_t)^{n+1}} \tag{3.17}$$

where $p_2(\xi)$ is a polynomial. From (3.16) we also have

$$\psi''(\xi) = \frac{p_3(\xi)}{(\xi-\eta_1)^{n+1}(\xi-\eta_2)^{n+1}\cdots(\xi-\eta_t)^{n+1}}, \tag{3.18}$$

where $p_3(\xi) = A((n-1)^2t^2 + (n-1)t)\xi^{2t-2} + b_{2t-3}\xi^{2t-3} + \cdots + b_0$ is also a polynomial.

Therefore, From (3.17) and (3.18), we deduce that $l-1 \leq \deg(p_3) = 2t-2$. Note that $l = nt$, we have $(n-2)t+1 \leq 0$, a contradiction with $n \geq 2$.

Suppose that $\mathcal{F}$ is a family holomorphic and not normal in D. As the same as the former arguments, noting that the $g(\xi)$ is a non-constant entire function, only Case 2 occurs. We omit the detail states.

The proof of Theorem 1.3 is complete. ■

Proof of Theorem 1.5. Suppose that $\mathcal{F}$ is not normal in D. As the similar as the arguments in the proof of Theorem 1.3 and take $n = 3$. Here, we state the different places from each other.

The poles of g are of multiplicity ≥ 2.

$m_i \geq 4 (i = 1, 2, \cdots, s)$.

$$(3.4)' \qquad m = m_1 + m_2 + \cdots + m_s \geq 4s.$$

Since $m \geq 4s$, we know by (3.13) that

$$(3.14)' \qquad 2s \leq t.$$

If $l \geq 3t$, by (3.12), then $3t - 1 \leq l - 1 \leq \deg(p_3) \leq 2t + 2s - 2$. Noting (3.14), we obtain $1 \leq 0$, a contradiction.

If $l < 3t$, from (3.5) and (3.7), then $\deg(p_1) = \deg(q_1)$. Noting that $\deg(p_1) = m + t - i, 1 \leq i \leq s + t, \deg(q_1) = 3t$, so $m + t - i = 3t, m = 2t + i \neq 2t$. From (3.6), then $\deg(h) = s + t - 1$, and then $\deg(p_1) = m + t - 1$. Noting $\deg(q_1) = 3t$, hence $m = 2t + 1$. By (3.13) then $t \leq 2s - 1$. From $(3.14)'$, we obtain $1 \leq 0$, a contradiction.

The proof of Theorem 1.5 is complete. ■

Proof of Theorem 1.7. Suppose that $\mathcal{F}$ is not normal in D. As the similar as the arguments in the proof of Theorem 1.3 and take $n = 2$. Here, we state the different places from each other.

All zeros and poles of $g(\xi)$ are multiple.

Hence φ is an entire function with no simple zero and growth order at most 2 and $\varphi' - a$ has just a unique zero ξ_0. By Lemma 2.4, φ is not any transcendental function. Therefore φ is a non-constant polynomial, and has the form that $\varphi' - a = C(\xi - \xi_0)^l$, where C is a non-zero constant, l is a positive integer, because the poles of g are of multiplicity ≥ 3. So the zeros of φ are of multiplicity ≥ 3, thus, $l \geq 2, \varphi'' = Cl(\xi - \xi_0)^{l-1}$. Note that φ'' has only one zero ξ_0, so φ has only the same zero ξ_0 too. Hence $\varphi'(\xi_0) = 0$ which contradicts with $\varphi'(\xi_0) = a \neq 0$.

The proof of Theorem 1.7 is complete. ■

Proof of Theorem 1.9. Suppose that $\mathcal{F}$ is not normal in D. As the similar as the arguments in the proof of Theorem 1.3. Here, we state the different places from each other.

$$g_j(\xi) = \rho_j^{-1} f_j(z_j + \rho_j \xi)$$

converges uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\xi)$ whose poles are multiple and $g(\xi) \neq 0$.

Thus

$$g_j'(\xi) = f_j'(z_j + \rho_j\xi) \Rightarrow g'(\xi)$$

and

$$\begin{aligned} g_j'(\xi) + a\rho_j g_j(\xi) - b &= f_j'(z_j + \rho_j\xi) + af_j(z_j + \rho_j\xi) - b \\ &\Rightarrow g'(\xi) - b \end{aligned}$$

also locally uniformly with respect to the spherical metric.

If $g' - b \equiv 0$, then $g = b\xi + c$ where c is a constant. This contradicts with $g(\xi) \neq 0$. So $g' - b \not\equiv 0$.

If $g' - b \neq 0$, then by Milloux inequality (2.2) of Lemma 2.5 we have

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g) + N(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g'-b}) + S(r,g) \\ &\leq \frac{1}{2}N(r,g) + S(r,g) \\ &\leq \frac{1}{2}T(r,g) + S(r,g). \end{aligned} \tag{3.19}$$

From (3.19) we know that g is a constant which contradicts with our conclusion. Hence, $g' - b$ is a non-constant meromorphic function and has at least one zero.

As the same argument in the proof of Theorem 1.3, we obtain that $g' - b$ has only one distinct zero denoted by ξ_0. Thus Hayman inequality (2.1) of Lemma 2.5 implies that g is a rational function of degree at most 4. Noting that $g \neq 0$ and has no simple pole, we obtain that g has at most two distinct poles. Using Milloux inequality (2.2) of Lemma 2.5 again we get that g has at most one distinct pole. Hence we can write $g(\xi) = \frac{1}{(d\xi+e)^m}$, $2 \leq m \leq 3$ where $d \neq 0$ and e are two finite complex numbers. Simple calculating shows that $g' - b$ has at least three distinct zeros. This is impossible.

The proof of Theorem 1.9 is complete. ■

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Bing Xiao
School of Mathematical Sciences
Xinjiang Normal University
Urumqi 830054
People's Republic of China
E-mail: xiaobing6101@163.com

Weiling Xiong
Department of Information and Computing Science
Guangxi University of Technology
Liuzhou 545006
People's Republic of China
E-mail: xiongwl@163.com

Wenjun Yuan
School of Mathematics and Information Science
Guangzhou University
Guangzhou 510006
People's Republic of China
E-mail: gzywj@tom.com
E-mail: wjyuan1957@126.com