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MAXIMUM PACKINGS AND MINIMUM COVERINGS OF MULTIGRAPHS WITH PATHS AND STARS

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Abstract. Let F, G, and H be multigraphs. An (F, G)-decomposition of H is an edge decomposition of H into copies of F and G using at least one of each. For subgraphs L and R of H, an (F, G)-packing of H with leave L is an (F, G)-decomposition of $H - E(L)$, and an (F, G)-covering of H with padding R is an (F, G)-decomposition of $H + E(R)$. A maximum (F, G)-packing of H is an (F, G)-packing of H with a minimum leave. A minimum (F, G)-covering of H is an (F, G)-covering of H with a minimum padding. Let k be a positive integer. A k-path, denoted by P_k, is a path on k vertices. A k-star, denoted by S_k, is a star with k edges. In this paper, we obtain a maximum (P_{k+1}, S_k)-packing of λK_n, which has a leave of size $< k$, and a minimum (P_{k+1}, S_k)-covering of λK_n, which has a padding of size $< k$. A similar result for $\lambda K_{n,n}$ is also obtained. As corollaries, necessary and sufficient conditions for the existence of (P_{k+1}, S_k)-decompositions of both λK_n and $\lambda K_{n,n}$ are given.

1. INTRODUCTION

For positive integers m and n, K_n denotes the complete graph with n vertices, and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. If $m = n$, the complete bipartite graph is referred to as *balanced*. Let k be a positive integer. A k-$star$, denoted by S_k, is the complete bipartite graph $K_{1,k}$. A *k-path*, denoted by P_k, is a path on k vertices. A k-$cycle$, denoted by C_k, is a cycle of length k. For a graph H and a positive integer λ, we use λH to denote the multigraph obtained from H by replacing each edge e by λ edges each having the same endpoints as e.

Let F, G, and H be multigraphs. A *decomposition* of H is a set of edge-disjoint subgraphs of H whose union is H. A *G-decomposition* of H is a decomposition of H in which each subgraph is isomorphic to G. If H has a G-decomposition, we say that H is *G-decomposable*. An (F, G)-*decomposition* of H is a decomposition of H

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with members isomorphic to F or G such that at least one of each occurs. If H has an (F, G)-decomposition, we say that H is (F, G)-*decomposable*.

Recently, decomposition of a graph into a pair of graphs has attracted a fair share of interest. Abueida and Daven [3] investigated the problem of (K_k, S_k)-decomposition of the complete graph K_n. Abueida and Daven [4] investigated the problem of the (C_4, E_2)-decomposition of several graph products where E_2 denotes two vertex disjoint edges. Abueida and O'Neil [7] studied the existence problem for (C_k, S_{k-1})-decomposition of the complete multigraph λK_n for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [16, 17] investigated the existence of (G, H)-decompositions of λK_n and $\lambda K_{n,n}$ where $G, H \in \{C_n, P_n, S_{n-1}\}$. A *graph-pair* (G, H) *of order* m is a pair of non-isomorphic graphs G and H with $V(G) = V(H)$ such that both G and H contain no isolated vertices and $G \cup H$ is isomorphic to K_m. Abueida and Daven [2] and Abueida, Daven and Roblee [5] completely determined the values of n for which λK_n admits a (G, H)-decomposition where (G, H) is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of $K_n - F$ into the graph-pairs of order 4 and 5, respectively, where F is a Hamiltonian cycle, a 1-factor, or an almost 1-factor. Lee [12], Lee and Lin [13], and Lin [14] established necessary and sufficient conditions for the existence of (C_k, S_k)-decompositions of the complete bipartite graph, the complete bipartite graph with a 1-factor removed, and the multicrown, respectively. Shyu studied the problem of decomposing a graph into copies of a graph G and copies of a graph H where the number of copies of G and the number of copies of H are essential. He gave necessary and sufficient conditions for the decomposition of K_n into paths and stars (both with 3 edges) [18], paths and cycles (both with k edges where $k = 3, 4$) [19, 20], and cycles and stars (both with 4 edges) [21]. He [22] also gave necessary and sufficient conditions for the decomposition of $K_{m,n}$ into paths and stars both with 3 edges.

Let F, G, and H be multigraphs. For subgraphs L and R of H, an (F, G)-*packing* of H with *leave* L is an (F, G)-decomposition of $H - E(L)$, and an (F, G)-*covering* of H with *padding* R is an (F, G)-decomposition of $H + E(R)$. A maximum (F, G)-packing of H is an (F, G)-packing of H with a minimum leave (i.e. a leave with the minimum number of edges). A minimum (F, G)-covering of H is an (F, G)-covering of H with a minimum padding. Clearly, an (F, G)-decomposition of H is an (F, G)-packing of H with an empty graph as its leave, and is an (F, G)-covering of H with an empty graph as its padding.

Abueida and Daven [3] obtained the maximum (K_k, S_k)-packing and the minimum (K_k, S_k)-covering of the complete graph K_n. Abueida and Daven [2] and Abueida, Daven and Roblee [5] gave the maximum (F, G)-packing and the minimum (F, G)-covering of K_n and λK_n, respectively, where (F, G) is a graph-pair of order 4 or 5. In this paper, we obtain a maximum (P_{k+1}, S_k)-packing of λK_n, which has a leave of size $< k$, and a minimum (P_{k+1}, S_k)-covering of λK_n, which has a padding of size $< k$.

A similar result for $\lambda K_{n,n}$ is also obtained. As corollaries, necessary and sufficient conditions for the existence of (P_{k+1}, S_k)-decompositions of both λK_n and $\lambda K_{n,n}$ are given. Since P_{k+1} is isomorphic to S_k for $k = 1, 2$, we restrict the discussions to $k \geq 3$.

2. Packing and Covering of λK_n

In this section the problems of the maximum (P_{k+1}, S_k)-packing and the minimum (P_{k+1}, S_k)-covering of λK_n are investigated. We first collect some needed terminology and notation.

Let G be a multigraph. The *degree* of a vertex x of G, denoted by $\deg_G x$, is the number of edges incident with x. For $k \geq 2$, the vertex of degree k in S_k is the *center* of S_k and any vertex of degree 1 is an *endvertex* of S_k. In addition, $v_1 v_2 \dots v_k$ denotes the k-path through vertices $v_1, v_2, \dots, v_k$ in order, and the vertices v_1 and v_k are referred to as its *origin* and *terminus*. If $P = x_1 x_2 \dots x_t$, $Q = y_1 y_2 \dots y_s$ and $x_t = y_1$, then $P + Q$ denotes the walk $x_1 x_2 \dots x_t y_2 \dots y_s$. Moreover, we use $P_k(v_1, v_k)$ to denote a k-path with origin v_1 and terminus v_k. For $U, W \subseteq V(G)$ with $U \cap W = \phi$, we use $G[U]$ and $G[U, W]$ to denote the subgraph of G induced by U, and the maximal bipartite subgraph of G with bipartition (U, W), respectively. When $G_1, G_2, \dots, G_t$ are edge disjoint subgraphs of a graph, we use $G_1 \cup G_2 \cup \cdots \cup G_t$ to denote the graph with vertex set $\bigcup_{i=1}^{t} V(G_i)$ and edge set $\bigcup_{i=1}^{t} E(G_i)$.

Before going into more details, we present some results which are useful for our discussions.

Proposition 2.1. (Bryant [9]). *For positive integers λ, n, and t, and any sequence $m_1, m_2, \dots, m_t$ of positive integers, the complete multigraph λK_n can be decomposed into paths of lengths $m_1, m_2, \dots, m_t$ if and only if each $m_i \leq n - 1$ and $m_1 + m_2 + \cdots + m_t = |E(\lambda K_n)|$.*

Proposition 2.2. (Bosák [8], Hell and Rosa [10]). *For an even integer n and $V(K_n) = \{x_0, x_1, \dots, x_{n-1}\}$, the complete graph K_n can be decomposed into the following $n/2$ copies of n-paths : $P_n(x_0, x_{n/2}), P_n(x_1, x_{1+n/2}), \dots, P_n(x_{n/2-1}, x_{n-1})$.*

The following lemma is trivial.

Lemma 2.3. *For an odd integer n with $n \geq 3$, the complete graph K_n can be decomposed into n copies of $(n+1)/2$-paths whose origins are all distinct.*

Proof. Let $V(K_n) = \{x_0, x_1, \dots, x_{n-1}\}$. We define $(n+1)/2$-paths as follows. For $0 \leq i \leq n-1$,

$$P^i = \begin{cases} x_i x_{i-1} x_{i+1} x_{i-2} \dots x_{i-\frac{n-1}{4}} x_{i+\frac{n-1}{4}} & \text{if } n \equiv 1 \pmod 4, \\ x_i x_{i-1} x_{i+1} x_{i-2} \dots x_{i+\frac{n-3}{4}} x_{i-\frac{n+1}{4}} & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where the subscripts of x's are taken modulo n. It is easy to check that $\{P^0, P^1, \ldots, P^{n-1}\}$ is a $P_{(n+1)/2}$-decomposition of K_n as required. ■

Lemma 2.4. *Let k, m, p, and s be positive integers and let t be a nonnegative integer with $\max\{m,t\} \leq k-1$. If $pk+t = m(m-1)/2+(sk+1)sk/2+m(sk+1)$, then $p-(s+1)m > 0$.*

Proof. Note that

$$\begin{aligned} pk-(s+1)mk &= m(m-1)/2+(sk+1)sk/2+m-t-mk \\ &\geq m(m-1)/2+(k+1)k/2+m-t-mk \\ &= (k-1-m)(k-m)/2+k-t \\ &> 0. \end{aligned}$$

Thus $p-(s+1)m > 0$. ■

Theorem 2.5. *Let n and k be positive integers with $k \geq 3$ and $n \geq k+2$. If $|E(K_n)| \equiv t \pmod{k}$ where $0 \leq t \leq k-1$, then K_n has a (P_{k+1}, S_k)-packing with leave P_{t+1}.*

Proof. Let $n = qk+r$ where $q \in \mathbb{N}$, $0 \leq r \leq k-1$ and let $|E(K_n)| = pk+t$ where $p \in \mathbb{N}$. We can see that $p \geq q+1$. Suppose that $r = 1$. Then K_n can be decomposed into K_{qk} and $K_{1,qk}$. Note that $|E(K_{qk})| = (p-q)k+t$. Thus by Proposition 2.1, K_{qk} can be decomposed into $p-q$ copies of $(k+1)$-paths and one $(t+1)$-path. Obviously, $K_{1,qk}$ is S_k-decomposable. Hence K_n has a (P_{k+1}, S_k)-packing with leave P_{t+1} for $r = 1$.

Now we consider the case $r \neq 1$. If $r = 0$, then $n = qk$ where $q \geq 2$ for $n \geq k+2$; write $n = (k-1)+(q-1)k+1$ where $k-1 \geq 1$, $q-1 \geq 1$. If $r \geq 2$, then $n = qk+r = (r-1)+qk+1$ where $1 \leq r-1 \leq k-1$, $q \geq 1$. Thus for $r \neq 1$ we can set $n = m+sk+1$ where m and s are positive integers with $m \leq k-1$. Let $A = \{x_0, x_1, \ldots, x_{m-1}\}$, $B = \{y_0, y_1, \ldots, y_{sk}\}$ and $V(K_n) = A \cup B$. Note that $K_n = K_{m+sk+1} = K_{m+sk+1}[A] \cup K_{m+sk+1}[B] \cup K_{m+sk+1}[A,B]$, and $K_{m+sk+1}[A] \cong K_m$, $K_{m+sk+1}[B] \cong K_{sk+1}$, and $K_{m+sk+1}[A,B] \cong K_{m,sk+1}$. Thus $pk+t = m(m-1)/2+(sk+1)sk/2+m(sk+1)$. Hence by Lemma 2.4,

$$p-(s+1)m > 0. \tag{1}$$

Also

$$|E(K_{sk+1})| = (sk+1)sk/2 = pk+t-m(m-1)/2-m(sk+1). \tag{2}$$

Case 1. m is odd.

By (2), $|E(K_{sk+1})| = (k-(m+1)/2)m+k(p-(s+1)m)+t$. By (1) and Proposition 2.1, K_{sk+1} has a path decomposition $\mathscr{D}$, which consists of m copies of $(k-(m-1)/2)$-paths, $p-(s+1)m$ copies of $(k+1)$-paths and one $(t+1)$-path.

If $m = 1$, then $n = sk + 2$, $A = \{x_0\}$, and as mentioned above, K_{sk+1} has a decomposition consisting of one P_k, $p - s - 1$ copies of P_{k+1} and one P_{t+1}. Assume that in the above decomposition, $P_k = P_k(y_{w_0}, y_{w_1})$ (i.e. y_{w_0}, y_{w_1} are the endvertices of this P_k). Let $P = P_k(y_{w_0}, y_{w_1}) + x_0 y_{w_0}$. Then P is a $(k+1)$-path. Moreover, $K_{sk+2}[A, B] - \{x_0 y_{w_0}\} \cong K_{1,sk}$, which is S_k-decomposable. Hence K_{sk+2} has a (P_{k+1}, S_k)-packing with leave P_{t+1}.

If $m \geq 3$, then Lemma 2.3 implies that there exists a $P_{(m+1)/2}$-decomposition $\mathscr{D}'$ of K_m where $\mathscr{D}' = \{P_{(m+1)/2}(x_i, x_{j_i}) \mid 0 \leq i \leq m-1\}$ and $x_1, x_2, \ldots, x_{m-1}$ are distinct. Suppose that the m copies of $(k - (m-1)/2)$-paths in $\mathscr{D}$ are $P_{k-(m-1)/2}(y_{w_0}, y_{w'_0}), P_{k-(m-1)/2}(y_{w_1}, y_{w'_1}), \ldots, P_{k-(m-1)/2}(y_{w_{m-1}}, y_{w'_{m-1}})$. For $i \in \{0, 1, \ldots, m-1\}$, let

$$P^i = P_{\frac{m+1}{2}}(x_i, x_{j_i}) + x_i y_{w_i} + P_{k-\frac{m-1}{2}}(y_{w_i}, y_{w'_i}).$$

Then P^i is a $(k+1)$-path for each i. Moreover, let $G = K_{m+sk+1}[A, B] - \{x_i y_{w_i} \mid 0 \leq i \leq m-1\}$. Then G is a bipartite graph with $\deg_G x_i = sk$ for $0 \leq i \leq m-1$. Thus G is S_k-decomposable, and in turn, K_{m+sk+1} has a (P_{k+1}, S_k)-packing with leave P_{t+1}.

Case 2. m is even.

By Proposition 2.2, there exists a P_m-decomposition $\{P_m(x_i, x_{i+m/2}) \mid 0 \leq i \leq m/2 - 1\}$ of K_m. By (2), $|E(K_{sk+1})| = (k-m-1)m/2 + k(p - sm - m/2) + t$. By (1), $p - sm - m/2 > 0$. Hence by Proposition 2.1, K_{sk+1} has a path decomposition $\mathscr{D}''$, which consists of $m/2$ copies of $(k-m)$-paths, $p - sm - m/2$ copies of $(k+1)$-paths and one $(t+1)$-path. Suppose that the $m/2$ copies of $(k-m)$-paths in $\mathscr{D}''$ are $P_{k-m}(y_{w_0}, y_{w'_0}), P_{k-m}(y_{w_1}, y_{w'_1}), \ldots, P_{k-m}(y_{w_{m/2-1}}, y_{w'_{m/2-1}})$. For $i \in \{0, 1, \ldots, m/2 - 1\}$, let

$$Q^i = P_m(x_i, x_{i+\frac{m}{2}}) + x_i y_{w_i} + P_{k-m}(y_{w_i}, y_{w'_i}) + x_{i+\frac{m}{2}} y_{v_i}$$

where $y_{v_i} \notin V(P_{k-m}(y_{w_i}, y_{w'_i}))$. Then Q^i is a $(k+1)$-path for each i. Moreover, let $H = K_{m+sk+1}[A, B] - \{x_i y_{w_i}, x_{i+m/2} y_{v_i} \mid 0 \leq i \leq m/2 - 1\}$. Then H is a bipartite graph with $\deg_H x_i = sk$ for $0 \leq i \leq m-1$. Thus H is S_k-decomposable, and in turn K_{m+sk+1} has a (P_{k+1}, S_k)-packing with leave P_{t+1}. ■

It is trivial that the leave P_{t+1} in Theorem 2.5 can be chosen arbitrarily.

Theorem 2.6. *Let λ, n and k be positive integers with $k \geq 3$ and $n \geq k+2$. If $|E(\lambda K_n)| \equiv t \pmod{k}$ where $0 \leq t \leq k-1$, then* (1) *λK_n has a (P_{k+1}, S_k)-packing with leave P_{t+1},* (2) *λK_n has a (P_{k+1}, S_k)-covering with padding P_{k-t+1}.*

Proof. (1) Let $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$ and $|E(K_n)| \equiv r \pmod{k}$ with $0 \leq r \leq k-1$. From the assumption $|E(\lambda K_n)| \equiv t \pmod{k}$, we have $\lambda r \equiv t \pmod{k}$. Clearly, λK_n can be decomposed into λ copies of K_n, say $G_0, G_1, \ldots, G_{\lambda-1}$.

Theorem 2.5 implies that each G_i has a (P_{k+1}, S_k)-packing with leave $(r+1)$-path, say P^i. By the note following Theorem 2.5, we can assume that for $0 \leq i \leq \lambda - 1$, $P^i = x_{ir}x_{ir+1}\ldots x_{ir+r}$ where the subscripts of x's are taken modulo n. By cutting method, we can see that $P^0 + P^1 + \cdots + P^{\lambda-1}$ which is a walk with λr edges can be decomposed into several copies of $(k+1)$-paths and one $(t+1)$-path. This completes the proof of (1).

(2) follows from (1) directly. ■

In the proof of the following corollary, we use $S(u; v_1, v_2, \ldots, v_k)$ to denote a star with center u and endvertices $v_1, v_2, \ldots, v_k$.

Corollary 2.7. *For positive integers λ, k, and n with $k \geq 3$, the complete multigraph λK_n is (P_{k+1}, S_k)-decomposable if and only if $n \geq k+1$, $\lambda n(n-1)/2 \equiv 0 \pmod{k}$ and $(\lambda, n) \neq (1, k+1)$.*

Proof. (Necessity) Since $|V(K_n)|$ and $|V(P_{k+1})|$ are n and $k+1$, respectively, $n \geq k+1$ is necessary. Since λK_n has $\lambda n(n-1)/2$ edges and each subgraph in a decomposition has k edges, k must divide $\lambda n(n-1)/2$. Since $K_{k+1} - E(S_k)$ contains no $(k+1)$-path, K_{k+1} is not (P_{k+1}, S_k)-decomposable. Hence $(\lambda, n) \neq (1, k+1)$.

(Sufficiency) When $n \geq k+2$ and k divides $\lambda n(n-1)/2$, the existence of (P_{k+1}, S_k)-decomposition of λK_n follows from Theorem 2.6. So it remains to consider $n = k+1$. Then $\lambda \geq 2$ by the assumption. We distinguish two cases according to the parity of λ.

Case 1. λ is even.

Let $V(2K_{k+1}) = \{x_0, x_1, \ldots, x_{k-1}, x_\infty\}$. For $i = 0, 1, \ldots, k-1$, let

$$P^i = x_i x_{i-1} x_{i+1} x_{i-2} \ldots x_{\lfloor i+k/2 \rfloor} x_\infty$$

where the subscripts of x's are taken modulo k. It is easy to see that $\{S(x_\infty; x_0, x_1, \ldots, x_{k-1}), P^0, P^1, \ldots, P^{k-1}\}$ is a (P_{k+1}, S_k)-decomposition of $2K_{k+1}$. Since λ is even, λK_{k+1} can be decomposed into $\lambda/2$ copies of $2K_{k+1}$. Hence λK_{k+1} is (P_{k+1}, S_k)-decomposable.

Case 2. $\lambda \geq 3$ is odd.

The condition $\lambda n(n-1)/2 \equiv 0 \pmod{k}$ with $n = k+1$ and odd λ implies that $k+1$ is even. Note that $\lambda K_{k+1} = (\lambda - 1)K_{k+1} \cup K_{k+1}$. By Case 1, $(\lambda-1)K_{k+1}$ is (P_{k+1}, S_k)-decomposable. By Proposition 2.2, K_{k+1} is P_{k+1}-decomposable. Thus λK_{k+1} is (P_{k+1}, S_k)-decomposable. ■

3. Packing and Covering of $\lambda K_{n,n}$

In this section the (P_{k+1}, S_k)-packing, (P_{k+1}, S_k)-covering and (P_{k+1}, S_k)-decomposition of $\lambda K_{n,n}$ are investigated. Before moving on, we need more terminology and notation, and some useful results.

Let G be a multigraph. We use $(v_1, v_2, \ldots, v_k)$ to denote the k-cycle of G, which is through vertices $v_1, v_2, \ldots, v_k$ in order, and $\mu(uv)$ to denote the number of edges of G joining the vertices u and v. Given an S_k-decomposition of G, a *central function* c from $V(G)$ to the set of nonnegative integers is defined as follows: for each $v \in V(G)$, $c(v)$ is the number of k-stars in the decomposition whose center is v.

Proposition 3.1. (Hoffman [11]). *Let H be a multigraph and c be a function from $V(H)$ to the set of nonnegative integers. Then c is a central function for some S_k-decomposition of H if and only if*

(i) $k\sum_{v\in V(H)} c(v) = |E(H)|$,

(ii) *for all* $x, y \in V(H)$ *with* $x \neq y$, $\mu(xy) \leq c(x) + c(y)$,

(iii) *for all* $S \subseteq V(H)$, $k\sum_{v\in S} c(v) \leq \varepsilon(S) + \sum_{x\in S, y\in V(H)-S} \min\{c(x), \mu(xy)\}$,

where $\varepsilon(S)$ denotes the number of edges of H with both ends in S.

Proposition 3.2. (Parker [15]). *There exists a P_{k+1}-decomposition of $K_{m,n}$ if and only if $mn \equiv 0 \pmod{k}$ and one of the following cases occurs.*

Case	*k*	*m*	*n*	*Conditions*
1	*even*	*even*	*even*	$k \leq 2m, k \leq 2n$, *not both equalities*
2	*even*	*even*	*odd*	$k \leq 2m-2, k \leq 2n$
3	*even*	*odd*	*even*	$k \leq 2m, k \leq 2n-2$
4	*odd*	*even*	*even*	$k \leq 2m-1, k \leq 2n-1$
5	*odd*	*even*	*odd*	$k \leq 2m-1, k \leq n$
6	*odd*	*odd*	*even*	$k \leq m, k \leq 2n-1$
7	*odd*	*odd*	*odd*	$k \leq m, k \leq n$

Lemma 3.3. *If λ and k are positive integers with $k \geq 2$, then $\lambda K_{k,k}$ is (P_{k+1}, S_k)-decomposable.*

Proof. Note that $K_{k,k} = K_{k,k-1} \cup K_{k,1}$ for $k \geq 2$. It is easy to see that $K_{k,k-1}$ is P_{k+1}-decomposable by cases 2 and 6 of Proposition 3.2. Clearly $K_{k,1}$ is S_k-decomposable. Thus $K_{k,k}$ is (P_{k+1}, S_k)-decomposable, and so is $\lambda K_{k,k}$. ■

In the sequel of the paper, we use (A, B) to denote the bipartition of $\lambda K_{n,n}$ where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$.

Lemma 3.4. *Let λ, k, and n be positive integers with $3 \leq k < n < 2k$. If $\lambda(n-k)^2 < k$, then $\lambda K_{n,n}$ has a (P_{k+1}, S_k)-packing $\mathscr{P}$ with $|\mathscr{P}| = \lfloor \lambda n^2/k \rfloor$ and a (P_{k+1}, S_k)-covering $\mathscr{C}$ with $|\mathscr{C}| = \lceil \lambda n^2/k \rceil$.*

Proof. Let $n = k + r$. The assumption $k < n < 2k$ implies $0 < r < k$.

We first give a required packing. Note that

$$\lambda K_{n,n} = \lambda K_{k,k} \cup \lambda K_{k,r} \cup \lambda K_{r,k} \cup \lambda K_{r,r}.$$

By Lemma 3.3, $\lambda K_{k,k}$ has a (P_{k+1}, S_k)-decomposition $\mathscr{D}_1$ with $|\mathscr{D}_1| = \lambda k$. Trivially, $\lambda K_{k,r}$ and $\lambda K_{r,k}$ have S_k-decompositions $\mathscr{D}_2$ and $\mathscr{D}_3$ with $|\mathscr{D}_2| = |\mathscr{D}_3| = \lambda r$, respectively. Let $\mathscr{P} = \bigcup_{i=1}^{3} \mathscr{D}_i$. Then $\mathscr{P}$ is a (P_{k+1}, S_k)-packing of $\lambda K_{n,n}$ with $|\mathscr{P}| = \lambda(k+2r)$. Since $\lambda r^2 = \lambda(n-k)^2 < k$, we have $|\mathscr{P}| = \lfloor \lambda(k+r)^2/k \rfloor = \lfloor \lambda n^2/k \rfloor$. Thus $\mathscr{P}$ is a required packing.

Now we give a required covering. Let $s = \lambda r^2$. Note $s < k < n$. Let $A_0 = \{a_0, a_1, \ldots, a_{\lfloor (s-1)/2 \rfloor}\}$, $A_1 = A - A_0$, $B_0 = \{b_0, b_1, \ldots, b_{k-1}\}$ and $B_1 = B - B_0$. Define a $(k+1)$-path P as follows:

$$P = \begin{cases} b_0 a_0 b_1 a_1 \ldots b_{\frac{k-1}{2}} a_{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \\ b_0 a_0 b_1 a_1 \ldots b_{\frac{k}{2}-1} a_{\frac{k}{2}-1} b_{\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

Let P' be the $(k-s+1)$-subpath of P with end vertices $a_{\frac{k-1}{2}}, a_{\frac{s-1}{2}}$, when k is odd, s is odd, end vertices $a_{\frac{k-1}{2}}, b_{\frac{s}{2}}$, when k is odd, s is even, end vertices $b_{\frac{k}{2}}, a_{\frac{s-1}{2}}$, when k is even, s is odd, and end vertices $b_{\frac{k}{2}}, b_{\frac{s}{2}}$, when k is even, s is even. Note that since $s > 0$, P' is a proper subgraph of P.

Let

$$H = \lambda K_{n,n} - E(P) + E(P').$$

Note that H is a proper subgraph of $\lambda K_{n,n}$.

We will show that H has an S_k-decomposition.

Note that $V(H) = V(\lambda K_{n,n})$, $|E(H)| = \lambda n^2 - k + (k-s) = \lambda n^2 - \lambda r^2 = \lambda k(k+2r)$, and $\mu(uv) \le \lambda$ for all $u, v \in V(H)$. Define a function $c : V(H) \to \mathbb{N}$ as follows:

$$c(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ \lambda & \text{otherwise.} \end{cases}$$

We will show that the function c satisfies (i), (ii), and (iii) in Proposition 3.1. First, $k \sum_{v \in V(H)} c(v) = k\lambda(k+2r) = |E(H)|$. This proves (i). Next, if $u, v \in B_0$, then $c(u) + c(v) = 0 = \mu(uv)$; otherwise, $c(u) + c(v) \ge \lambda \ge \mu(uv)$. This proves (ii). To prove (iii), let $S \subseteq V(H)$. For $i \in \{0, 1\}$, let $X_i = S \cap A_i$ and $Y_i = S \cap B_i$. Moreover, let $X = X_0 \cup X_1$ and $Y = Y_0 \cup Y_1$. Define a set T of ordered pairs of vertices as follows:

$$T = \{(u, v) \mid (u \in X, v \in B_1 - Y_1) \text{ or } (u \in X_1, v \in B_0 - Y_0) \text{ or } (u \in Y_1, v \in A - X)\}.$$

Note that

$$k \sum_{w \in S} c(w) = k\lambda(|X| + |Y_1|), \tag{1}$$

(2) $$\varepsilon(S) = \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u\in X_0, v\in Y_0} \mu(uv),$$

and for $u \in S$ and $v \in V(H) - S$

(3) $$\min\{c(u), \mu(uv)\} = \begin{cases} \lambda & \text{if } (u,v) \in T, \\ \mu(uv) & \text{if } u \in X_0, v \in B_0 - Y_0, \\ 0 & \text{otherwise.} \end{cases}$$

For $S \subseteq V(H)$, let

$$g(S) = \varepsilon(S) + \sum_{u\in S, v\in V(H)-S} \min\{c(u), \mu(uv)\} - k\sum_{w\in S} c(w).$$

To show (iii), we need show $g(S) \geq 0$.
Note that, if $v \in A_0 - \{a_{\lfloor \frac{s-1}{2} \rfloor}\}$, there are $\lambda k - 2$ edges in H joining v and all vertices in B_0; if $v = a_{\lfloor \frac{s-1}{2} \rfloor}$, there are $\lambda k - \rho$ edges in H joining v and all vertices in B_0 where $\rho = 1$ if s is odd, and $\rho = 2$ if s is even.

Hence

$$\sum_{u\in X_0, v\in B_0} \mu(uv) = \begin{cases} |X_0|(\lambda k - 2) & \text{if } a_{\lfloor (s-1)/2 \rfloor} \notin X_0, \\ |X_0|(\lambda k - 2) + 2 - \rho & \text{if } a_{\lfloor (s-1)/2 \rfloor} \in X_0. \end{cases}$$

By (1)–(3) and $|X_0| + |X_1| = |X|$, we have

$$\begin{aligned} g(S) &= \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u\in X_0, v\in Y_0} \mu(uv) \\ &\quad + \lambda(|X|(r - |Y_1|) + |X_1|(k - |Y_0|) + |Y_1|(k + r - |X|)) \\ &\quad + \sum_{u\in X_0, v\in B_0 - Y_0} \mu(uv) - k\lambda(|X| + |Y_1|) \\ &= \lambda r|X| + \lambda|Y_1|r - \lambda|Y_1||X| - \lambda k|X_0| + \sum_{u\in X_0, v\in B_0} \mu(uv) \\ &= \begin{cases} \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| & \text{if } a_{\lfloor (s-1)/2 \rfloor} \notin X_0, \\ \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| + 2 - \rho & \text{if } a_{\lfloor (s-1)/2 \rfloor} \in X_0. \end{cases} \end{aligned}$$

If $a_{\lfloor (s-1)/2 \rfloor} \notin X_0$, then $|X_0| \leq \lfloor (s-1)/2 \rfloor$, which implies $-2|X_0| \geq -s$. If $a_{\lfloor (s-1)/2 \rfloor} \in X_0$, then $|X_0| \leq \lfloor (s-1)/2 \rfloor + 1$, which implies $-2|X_0| + 2 - \rho \geq$

$-2\lfloor (s-1)/2 \rfloor - \rho = -2(s-\rho)/2 - \rho = -s$. Suppose $|X| \geq r$. We have

$$\begin{aligned} g(S) &\geq \lambda(r|X| - |Y_1|(|X| - r)) - s \\ &= \lambda(r|X| - |Y_1|(|X| - r)) - \lambda r^2 \\ &= \lambda(|X| - r)(r - |Y_1|) \\ &\geq 0. \end{aligned}$$

Suppose $|X| < r$.

If $\lambda r = 1$, then $|X_0| = |X| = 0$, which implies $-2|X_0| = -\lambda r|X_0|$. If $\lambda r \geq 2$, then $-2|X_0| \geq -\lambda r|X_0|$. Note that $2 - \rho \geq 0$. Hence we have

$$\begin{aligned} g(S) &\geq \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| \\ &\geq \lambda(r|X| + |Y_1|(r - |X|)) - \lambda r|X_0| \\ &= \lambda(r|X_1| + |Y_1|(r - |X|)) \\ &\geq 0. \end{aligned}$$

This settles (iii). By Proposition 3.1, H has an S_k-decomposition, say $\mathscr{D}$. Since $H = \lambda K_{n,n} - E(P) + E(P')$, $\mathscr{D} \cup \{P\}$ is an (P_{k+1}, S_k)-covering of $\lambda K_{n,n}$. Also since $\mathscr{D}$ is a packing but not a decomposition of $\lambda K_{n,n}$ and $\mathscr{D} \cup \{P\}$ is a covering of $\lambda K_{n,n}$, we have $|\mathscr{D} \cup \{P\}| = \lceil \lambda n^2/k \rceil$. ■

The following result is needed in the proof of Lemma 3.6.

Lemma 3.5. *Let λ, k, and r be positive integers with $k \geq 3$, $r < k$ and $\lambda r^2 \geq k$. If k is odd or $(\lambda, r) \neq (1, k-1)$, then $\lfloor \lfloor \lambda r^2/k \rfloor /2 \rfloor \leq \lambda r/2 - 1$.*

Proof. We first consider the case $\lambda = 1$. If $r = k-1$, then $\lfloor \lambda r^2/k \rfloor = \lfloor r^2/(r+1) \rfloor = r - 1$, and k is odd by assumption. Thus r is even. Hence $\lfloor \lfloor \lambda r^2/k \rfloor /2 \rfloor = \lfloor (r-1)/2 \rfloor = r/2 - 1$. If $r \leq k-2$, then $r^2 \geq r+2$ from the assumption $\lambda r^2 \geq k$. Thus $r \geq 2$. Note that $\lfloor \lambda r^2/k \rfloor \leq \lfloor r^2/(r+2) \rfloor$. In the case $r = 2$, $\lfloor \lfloor r^2/(r+2) \rfloor /2 \rfloor = 0 = r/2 - 1$, which implies $\lfloor \lfloor \lambda r^2/k \rfloor /2 \rfloor \leq r/2 - 1$. In the case $r \geq 3$, $\lfloor r^2/(r+2) \rfloor = r - 2$, which implies $\lfloor \lfloor \lambda r^2/k \rfloor /2 \rfloor \leq \lfloor (r-2)/2 \rfloor \leq r/2 - 1$.

Now we consider the case $\lambda \geq 2$. Since $\lambda r^2 \geq k \geq 3$, we have $r \geq \sqrt{3/\lambda}$. Thus

$$\frac{\lambda r^2}{k} \leq \frac{\lambda r^2}{r+1} = \lambda r - \frac{\lambda}{1 + 1/r} \leq \lambda r - \frac{\lambda}{1 + \sqrt{\lambda/3}}.$$

Note that $\lambda/(1 + \sqrt{\lambda/3})$ is increasing with respect to λ. Thus $\lambda r^2/k \leq \lambda r - 2/(1 + \sqrt{2/3})$. Therefore, $\lfloor \lambda r^2/k \rfloor \leq \lambda r - 2$. In turn, $\lfloor \lfloor \lambda r^2/k \rfloor /2 \rfloor \leq \lambda r/2 - 1$. This completes the proof. ■

Lemma 3.6. *Let λ, k, and n be positive integers with $3 \le k < n < 2k$. If $\lambda(n-k)^2 \ge k$, then $\lambda K_{n,n}$ has a (P_{k+1}, S_k)-packing $\mathscr{P}$ with $|\mathscr{P}| = \lfloor \lambda n^2/k \rfloor$ and a (P_{k+1}, S_k)-covering $\mathscr{C}$ with $|\mathscr{C}| = \lceil \lambda n^2/k \rceil$.*

Proof. Let $n = k + r$. From the assumption $k < n < 2k$, we have $0 < r < k$. Let $\lambda r^2 = tk + s$ where s and t are integers with $1 \le t$, $0 \le s < k$. Note that $t = \lfloor \lambda r^2/k \rfloor$. Hence $\lfloor \lambda n^2/k \rfloor = \lfloor \lambda(k+r)^2/k \rfloor = \lambda(k+2r) + t$ and

$$\left\lceil \frac{\lambda n^2}{k} \right\rceil = \left\lceil \frac{\lambda(k+r)^2}{k} \right\rceil = \begin{cases} \lambda(k+2r) + t & \text{if } s = 0, \\ \lambda(k+2r) + t + 1 & \text{if } s > 0. \end{cases}$$

In the sequel of the proof, we will show that $\lambda K_{n,n}$ has a packing $\mathscr{P}$ consisting of t copies of $(k+1)$-paths and $\lambda(k+2r)$ copies of k-stars with leave P_{s+1}.

Let

$$\delta = \begin{cases} 1 & \text{if } t \text{ is odd}, \\ 0 & \text{if } t \text{ is even}. \end{cases}$$

Let $A_0 = \{a_0, a_1, \ldots, a_{k-1}\}$, $A_1 = A - A_0$, $B_0 = \{b_0, b_1, \ldots, b_{k-1}\}$, and $B_1 = B - B_0$. Let $G = \lambda K_{n,n}[A_0 \cup B_0]$. Clearly, G is isomorphic to $\lambda K_{k,k}$. For $i \in \{0, 1, \ldots, \lfloor \lambda k/2 \rfloor - 1\}$, let $C(i) = (b_{2i}, a_0, b_{2i+1}, a_1, \ldots, b_{2i+k-1}, a_{k-1})$ where the subscripts of b's are taken modulo k. Trivially, $C(i)$ is a $2k$-cycle in G for each i. Note that

$$G = \begin{cases} \displaystyle\bigcup_{i=0}^{\lfloor \lambda k/2 \rfloor - 1} C(i) & \text{if } \lambda k \text{ is even}, \\ \displaystyle(\bigcup_{i=0}^{\lfloor \lambda k/2 \rfloor - 1} C(i)) \cup M & \text{if } \lambda k \text{ is odd}, \end{cases}$$

where M is a perfect matching in G. For $s > 0$ or odd t, define a $(\delta k + s + 1)$-path P, which is a subpath of $C(0)$, as follows:

$$P = \begin{cases} b_0 a_0 b_1 a_1 \ldots b_{\lceil (\delta k+s)/2 \rceil - 1} a_{\lceil (\delta k+s)/2 \rceil - 1} b_{\lceil (\delta k+s)/2 \rceil} & \text{if } \delta k + s \text{ is even}, \\ b_0 a_0 b_1 a_1 \ldots b_{\lceil (\delta k+s)/2 \rceil - 1} a_{\lceil (\delta k+s)/2 \rceil - 1} & \text{if } \delta k + s \text{ is odd}. \end{cases}$$

Since $\lambda(n-k)^2 \ge k > r$, we have $1 \le t < \lambda r$. Thus $t + 1 \le \lambda r$; in turn, $\lfloor \frac{t}{2} \rfloor \le \frac{t}{2} \le \frac{\lambda r - 1}{2} < \frac{\lambda k - 1}{2} \le \lfloor \frac{\lambda k}{2} \rfloor$. Hence $\lfloor \frac{t}{2} \rfloor \le \lfloor \frac{\lambda k}{2} \rfloor - 1$, which assures that the following F is well-defined. Define a subgraph F of G as follows :

$$F = \begin{cases} P & \text{if } t = 1, \\ \displaystyle\bigcup_{i=1}^{\lfloor t/2 \rfloor} C(i) & \text{if } s = 0 \text{ and } t \text{ is even}, \\ \displaystyle(\bigcup_{i=1}^{\lfloor t/2 \rfloor} C(i)) \cup P & \text{otherwise}. \end{cases}$$

Since C_{2k} can be decomposed into 2 copies of P_{k+1} and P can be decomposed into δ copies of P_{k+1} as well as one copy of P_{s+1}, there exists a decomposition of F into t copies of P_{k+1} and one copy of P_{s+1}. Thus F has a P_{k+1}-packing, say $\mathscr{P}_0$, with leave P_{s+1}. Let

$$H = \lambda K_{n,n} - E(F).$$

Note that $V(H) = V(\lambda K_{n,n})$, $|E(H)| = \lambda n^2 - (tk+s) = \lambda n^2 - \lambda r^2 = \lambda k(k+2r)$, and $\mu(uv) \le \lambda$ for all $u, v \in V(H)$ with $u \ne v$. Define a function $c : V(H) \to \mathbb{N}$ as follows:

for $v \in V(H)$,

$$c(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ \lambda & \text{otherwise.} \end{cases}$$

Now we show that the function c satisfies (i), (ii) and (iii) in Proposition 3.1.

First, $k\sum_{v\in V(H)} c(v) = k\lambda(k+2r) = |E(H)|$. This proves (i). Next, if $u, v \in B_0$, then $c(u)+c(v) = 0 = \mu(uv)$; otherwise, $c(u)+c(v) \ge \lambda \ge \mu(uv)$. This proves (ii). Finally, we prove (iii). For $S \subseteq V(H)$ and $i \in \{0,1\}$, let $X_i = S \cap A_i$, $Y_i = S \cap B_i$, $X = X_0 \cup X_1$, and $Y = Y_0 \cup Y_1$. Define a set T of ordered pairs of vertices as follows:

$$T = \{(u,v) \mid (u \in X, v \in B_1 - Y_1) \text{ or } (u \in X_1, v \in B_0 - Y_0) \text{ or } (u \in Y_1, v \in A - X)\}.$$

Note that

$$k\sum_{w\in S} c(w) = k\lambda(|X| + |Y_1|), \tag{4}$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u\in X_0, v\in Y_0} \mu(uv). \tag{5}$$

For $u \in S$ and $v \in V(H) - S$,

$$\min\{c(u), \mu(uv)\} = \begin{cases} \lambda & \text{if } (u,v) \in T, \\ \mu(uv) & \text{if } u \in X_0, v \in B_0 - Y_0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} &\sum_{u\in S, v\in V(H)-S} \min\{c(u), \mu(uv)\} \\ &= \sum_{(u,v)\in T} \min\{c(u), \mu(uv)\} + \sum_{u\in X_0, v\in B_0 - Y_0} \min\{c(u), \mu(uv)\} \\ &= \lambda(|X|(r - |Y_1|) + |X_1|(k - |Y_0|) + |Y_1|(k + r - |X|)) \\ &\quad + \sum_{u\in X_0, v\in B_0 - Y_0} \mu(uv). \end{aligned} \tag{6}$$

For $S \subseteq V(H)$, let

$$g(S) = \varepsilon(S) + \sum_{u \in S, v \in V(H)-S} \min\{c(u), \mu(uv)\} - k \sum_{w \in S} c(w).$$

Proving (iii) is equivalent to proving $g(S) \geq 0$.

Let $H' = H[A \cup B_0]$. For $s > 0$ or odd t, let $A'_0 = \{a_0, a_1, \ldots, a_{\lceil(\delta k+s)/2\rceil-1}\}$, $A''_0 = A_0 - A'_0$. Let $X'_0 = S \cap A'_0$ and $X''_0 = S \cap A''_0$. Obviously, $X_0 = X'_0 \cup X''_0$. We have

for $u \in V(H')$,

$$\deg_{H'} u = \begin{cases} \lambda k - t & \text{if } s = 0,\ t \text{ is even, and } u \in A_0, \\ \lambda k - 2\lfloor t/2 \rfloor - 2 & \text{if } s > 0 \text{ or } t \text{ is odd, and } u \in A'_0 - \{a_{\lceil(\delta k+s)/2\rceil-1}\}, \\ \lambda k - 2\lfloor t/2 \rfloor - \rho & \text{if } s > 0 \text{ or } t \text{ is odd, and } u = a_{\lceil(\delta k+s)/2\rceil-1}, \\ \lambda k - 2\lfloor t/2 \rfloor & \text{if } s > 0 \text{ or } t \text{ is odd, and } u \in A''_0, \end{cases}$$

where $\rho = 1$ if $\delta k + s$ is odd, and $\rho = 2$ if $\delta k + s$ is even.
Hence

$$\begin{aligned} & \sum_{u \in X_0, v \in Y_0} \mu(uv) + \sum_{u \in X_0, v \in B_0 - Y_0} \mu(uv) \\ = & \sum_{u \in X_0, v \in B_0} \mu(uv) \\ = & \sum_{u \in X_0} deg_{H'} u \\ = & \begin{cases} |X_0|(\lambda k - t) & \text{if } s=0 \text{ and } t \text{ is even}, \\ |X_0|(\lambda k - 2\lfloor t/2 \rfloor) - 2|X'_0| & \text{if } s>0 \text{ or } t \text{ is odd, and } a_{\lceil(\delta k+s)/2\rceil-1} \notin X'_0, \\ |X_0|(\lambda k - 2\lfloor t/2 \rfloor) - 2|X'_0| + 2 - \rho & \text{if } s>0 \text{ or } t \text{ is odd, and } a_{\lceil(\delta k+s)/2\rceil-1} \in X'_0. \end{cases} \end{aligned}$$

Let

$$m = \begin{cases} -t|X_0| & \text{if } s = 0 \text{ and } t \text{ is even}, \\ -2(|X_0|\lfloor t/2 \rfloor + |X'_0|) & \text{if } s > 0 \text{ or } t \text{ is odd, and } a_{\lceil(\delta k+s)/2\rceil-1} \notin X'_0, \\ -2(|X_0|\lfloor t/2 \rfloor + |X'_0| - 1) - \rho & \text{if } s > 0 \text{ or } t \text{ is odd, and } a_{\lceil(\delta k+s)/2\rceil-1} \in X'_0. \end{cases}$$

Then

$$\sum_{u \in X_0, v \in Y_0} \mu(uv) + \sum_{u \in X_0, v \in B_0 - Y_0} \mu(uv) = |X_0|\lambda k + m. \tag{7}$$

By the definition of $g(S)$, and $(4)-(7)$, we have

$$\begin{aligned} g(S) &= \lambda(|X||Y_1|+|X_1||Y_0|)+\sum_{u\in X_0,v\in Y_0}\mu(uv) \\ &\quad +\lambda(|X|(r-|Y_1|)+|X_1|(k-|Y_0|)+|Y_1|(k+r-|X|)) \\ &\quad +\sum_{u\in X_0,v\in B_0-Y_0}\mu(uv)-k\lambda(|X|+|Y_1|) \\ &= \lambda(|X||Y_1|+|X_1||Y_0|+|X|(r-|Y_1|)+|X_1|(k-|Y_0|) \\ &\quad +|Y_1|(k+r-|X|)-k(|X|+|Y_1|)) \\ &\quad +\sum_{u\in X_0,v\in Y_0}\mu(uv)+\sum_{u\in X_0,v\in B_0-Y_0}\mu(uv) \\ &= \lambda(|X|r+|X_1|k+|Y_1|r-|Y_1||X|-k|X|)+|X_0|\lambda k+m \\ &= \lambda(|X|r+|X_1|k+|Y_1|r-|Y_1||X|-k|X|+|X_0|k)+m. \end{aligned}$$

Since $|X|=|X_0|+|X_1|$, we have

$$g(S)=\lambda(|X|r+|Y_1|(r-|X|))+m. \tag{8}$$

We first show that $m\geq -\lambda r^2$. Note that $|X_0|\leq k$. When $s=0$ and t is even, $m\geq -kt=-\lambda r^2$. When $s>0$ or t is odd,

$$|X'_0|\leq \begin{cases} |A'_0|-1 & \text{if } a_{\lceil(\delta k+s)/2\rceil-1}\notin X'_0, \\ |A'_0| & \text{if } a_{\lceil(\delta k+s)/2\rceil-1}\in X'_0. \end{cases}$$

Thus

$$\begin{aligned} m &\geq \begin{cases} -2(|X_0|\lfloor t/2\rfloor+|A'_0|)+2 & \text{if } a_{\lceil(\delta k+s)/2\rceil-1}\notin X'_0, \\ -2(|X_0|\lfloor t/2\rfloor+|A'_0|)+2-\rho & \text{if } a_{\lceil(\delta k+s)/2\rceil-1}\in X'_0. \end{cases} \\ &\geq -2(|X_0|\lfloor t/2\rfloor+|A'_0|)+2-\rho. \end{aligned}$$

In addition, $\lfloor t/2\rfloor=(t-\delta)/2$, and $|A'_0|=\lceil(\delta k+s)/2\rceil=(\delta k+s+2-\rho)/2$. Thus $m\geq -2(k(t-\delta)/2+(\delta k+s+2-\rho)/2-1)-\rho=-(kt+s)=-\lambda r^2$.

Therefore, if $|X|\geq r$, we have from (8), that

$$g(S)\geq \lambda(r|X|-|Y_1|(|X|-r))-\lambda r^2=\lambda(|X|-r)(r-|Y_1|)\geq 0.$$

So it remains to consider the case $|X|<r$.

We first show that $m\geq -\lambda r|X_0|$.

Suppose that k is even and $(\lambda,r)=(1,k-1)$. Then $\lambda r^2=(k-1)^2=(k-2)k+1=(r-1)k+1$. This implies $s=1$ and $t=r-1$ where t is even. Thus $\lfloor t/2\rfloor=(\lambda r-1)/2$ and $A'_0=\{a_0\}$. Thus if $a_{\lceil(\delta k+s)/2\rceil-1}\notin X'_0$, we have $|X'_0|=0$, which implies $m=$

$-2(|X_0|\lfloor\frac{t}{2}\rfloor+|X_0'|)=-2|X_0|\frac{\lambda r-1}{2}\geq -2|X_0|\frac{\lambda r}{2}=-\lambda r|X_0|$; if $a_{\lceil(\delta k+s)/2\rceil-1}\in X_0'$, we have $|X_0'|=1$, which implies $m=-2(|X_0|\lfloor\frac{t}{2}\rfloor+|X_0'|-1)-\rho=-2|X_0|\frac{\lambda r-1}{2}-1=-\lambda r|X_0|+|X_0|-1\geq -\lambda r|X_0|$. Hence $m\geq -\lambda r|X_0|$.

Suppose that either k is odd or $(\lambda,r)\neq(1,k-1)$â. Recall that $t=\lfloor\lambda r^2/k\rfloor$ in the beginning of the proof. Lemma 3.5 implies $\lfloor t/2\rfloor\leq \lambda r/2-1$. Hence

$$\begin{aligned} m &\geq -2\left(|X_0|\lfloor\frac{t}{2}\rfloor+|X_0'|\right)\\ &\geq -2(|X_0|(\lambda r/2-1)+|X_0'|)\\ &= -\lambda r|X_0|+2(|X_0|-|X_0'|)\\ &\geq -\lambda r|X_0|. \end{aligned}$$

Therefore, for $|X|<r$, we have from (8), that

$$g(S)\geq \lambda(r|X|+|Y_1|(r-|X|))-\lambda r|X_0|=\lambda(r|X_1|+|Y_1|(r-|X|))\geq 0.$$

This settles (iii).

In the above we show that the function $c:V(H)\to\mathbb{N}$ satisfies (i), (ii), (iii) in Proposition 3.1. Thus H has an S_k-decomposition, say $\mathscr{D}$.

Let $\mathscr{P}=\mathscr{D}\cup\mathscr{P}_0$. Clearly, $\mathscr{P}$ is a (P_{k+1},S_k)-packing of $\lambda K_{n,n}$ with leave P_{s+1} and $|\mathscr{P}|=\lfloor\lambda n^2/k\rfloor$. Let

$$\mathscr{C}=\begin{cases}\mathscr{P} & \text{if } s=0,\\ \mathscr{P}\cup\{Q\} & \text{if } s\geq 1,\end{cases}$$

where Q is a $(k+1)$-path containing the leave of $\mathscr{P}$. It is easy to see that $\mathscr{C}$ is a (P_{k+1},S_k)-covering and $|\mathscr{C}|=\lceil\lambda n^2/k\rceil$. ■

Combining Lemma 3.3, Lemma 3.4 and Lemma 3.6, we obtain the following lemma.

Lemma 3.7. *If λ, k, and n be positive integers with $3\leq k\leq n<2k$, then $\lambda K_{n,n}$ has a (P_{k+1},S_k)-packing $\mathscr{P}$ with $|\mathscr{P}|=\lfloor\lambda n^2/k\rfloor$ and a (P_{k+1},S_k)-covering $\mathscr{C}$ with $|\mathscr{C}|=\lceil\lambda n^2/k\rceil$.*

Now, we are ready for the main result of this section.

Theorem 3.8. *If λ, k, and n are positive integers with $3\leq k\leq n$, then $\lambda K_{n,n}$ has a (P_{k+1},S_k)-packing $\mathscr{P}$ with $|\mathscr{P}|=\lfloor\lambda n^2/k\rfloor$ and a (P_{k+1},S_k)-covering $\mathscr{C}$ with $|\mathscr{C}|=\lceil\lambda n^2/k\rceil$.*

Proof. Due to Lemma 3.7, we only need consider $n\geq 2k$.

Let $n=qk+r$ where q and r are integers with $q\geq 2$, $0\leq r<k$. We have $\lambda K_{n,n}=\lambda K_{k+r,k+r}\cup\lambda K_{k+r,(q-1)k}\cup\lambda K_{(q-1)k,n}$. Note that by Lemma 3.7 $\lambda K_{k+r,k+r}$

has a (P_{k+1}, S_k)-packing $\mathscr{P}$ with $|\mathscr{P}| = \lfloor \lambda(k+r)^2/k \rfloor$ and a (P_{k+1}, S_k)-covering $\mathscr{C}$ with $|\mathscr{C}| = \lceil \lambda(k+r)^2/k \rceil$. Trivially, $\lambda K_{k+r,(q-1)k}$ and $\lambda K_{(q-1)k,n}$ have S_k-decompositions, say $\mathscr{D}$ and $\mathscr{D}'$, respectively, where $|\mathscr{D}| = \lambda(k+r)(q-1)$ and $|\mathscr{D}'| = \lambda(q-1)n$. Then $\mathscr{P} \cup \mathscr{D} \cup \mathscr{D}'$ is a (P_{k+1}, S_k)-packing of $\lambda K_{n,n}$, obviously with cardinality $\lfloor \lambda n^2/k \rfloor$ and $\mathscr{C} \cup \mathscr{D} \cup \mathscr{D}'$ is a (P_{k+1}, S_k)-covering of $\lambda K_{n,n}$, obviously with cardinality $\lceil \lambda n^2/k \rceil$. This completes the proof. ■

Clearly, if $\lambda K_{n,n}$ admits a (P_{k+1}, S_k)-decomposition, then $k \leq n$ and λn^2 is divisible by k. Thus the following corollary follows from Theorem 3.8.

Corollary 3.9. *For positive integers λ, k and n with $k \geq 3$, the balanced complete bipartite multigraph $\lambda K_{n,n}$ is (P_{k+1}, S_k)-decomposable if and only if $k \leq n$ and λn^2 is divisible by k.*

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