TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 3, pp. 819-848, June 2015
DOI: 10.11650/tjm.19.2015.4449
This paper is available online at http://journal.taiwanmathsoc.org.tw

STABILITY AND MORREY SPACES RELATED TO MULTIPLIERS

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Abstract. We apply wavelets to study the generalized local Morrey-Campanato spaces $M_{\phi,p}(\mathbb{R}^n)$ and their preduals. As applications, we characterize the multipliers on $M_{\phi,p}(\mathbb{R}^n)$ and the stability of these spaces under the perturbation of Calderón-Zygmund operators. Our results indicate that there exist some $M_{\phi,p}(\mathbb{R}^n)$ without unconditional basis. This fact shows that $M_{\phi,p}(\mathbb{R}^n)$ have some different characteristics unlike the classical Morrey spaces.

1. Introduction

Morrey spaces $M_{\alpha,p}(\mathbb{R}^n)$ were introduced by Morrey [19] in 1938 when solving PDE problems. In the last decades, Morrey spaces and their generalization have been studied extensively and play an important role in the study of harmonic analysis and PDE. We refer the reader to Lin-Yang [11], Essen-Janson-Peng-Xiao [3], Yang-Yuan [30, 31, 32], Yuan-Sickel-Yang [35, 36], Xiao [29] and Yang-Zhu [34] for further information.

A cube Q centered at x and with radius r is defined as

$$Q = \Big\{ y \in \mathbb{R}^n : |y_i - x_i| < \frac{r}{2}, i = 1, \cdots, n \Big\}.$$

Denote by f_Q the mean value of f on Q:

$$f_Q = |Q|^{-1} \int_Q f(x)dx.$$

The generalized local Morrey-Campanato spaces $M_{\phi,p}(\mathbb{R}^n)$ are defined as follows.

Received February 18, 2014, accepted August 15, 2014.
Communicated by Chin-Cheng Lin.
2010 *Mathematics Subject Classification*: 42B35, 46E30.
Key words and phrases: Generalized local Morrey-Campanato spaces, Multiplier spaces, Wavelets, Calderón-Zygmund operator, Stability.
Project supported by NSFC No. 11171203, No. 11271209, No. 11201280; New Teacher's Fund for Doctor Stations, Ministry of Education No.20114402120003; Guangdong Natural Science Foundation S2011040004131; Foundation for Distinguished Young Talents in Higher Education of Guangdong, China, LYM11063.
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Definition 1.1. For $1 \leq p < \infty$, we say $f \in M_{\phi,p}(\mathbb{R}^n)$ if

$$\sup_{y\in\mathbb{R}^n, r\leq\frac{1}{2}} \left(\frac{1}{r^n\phi^p(r)} \int_{Q(y,r)} |f(x) - f_{Q(y,r)}|^p dx \right)^{\frac{1}{p}} + \sup_{y\in\mathbb{R}^n, \frac{1}{2}\leq r\leq 1} |f_{Q(y,r)}| < \infty. \tag{1.1}$$

Remark 1.2. The spaces $M_{\phi,p}(\mathbb{R}^n)$ are generalizations of many classical function spaces.

(i) When $\phi(r) = r^{-\alpha}, 0 \leq \alpha < \frac{n}{p}$, $M_{\phi,p}(\mathbb{R}^n) = M_{\alpha,p}(\mathbb{R}^n)$, the local Morrey-Campanato spaces.

(ii) For $1 \leq p < \infty$, $M_{0,p}(\mathbb{R}^n) = bmo(\mathbb{R}^n)$, the space of functions with local bounded mean oscillations.

Our aim is to characterize $M_{\phi,p}(\mathbb{R}^n)$ and its predual via the orthogonal regular wavelet basis. Wavelet characterizations of classical Morrey spaces are studied by many authors. See Rosenthal [21], Sawano [22] and Yuan-Sickel-Yang [36] for details. Liang et al. [10] considered the generalized Morrey spaces associated with increasing functions ϕ satisfying

$$\int_r^\infty \frac{1}{\phi(t)} \frac{dt}{t} \lesssim \frac{1}{\phi(r)},\ r > 0.$$

In this paper, we assume that the function ϕ satisfies the following conditions. Let ϕ be a function defined on the interval $(0, 1]$ and satisfy the following conditions.

(1) Suppose that

$$0 < \phi(x) < \infty, x \in (0, 1]. \tag{1.2}$$

(2) There exists a positive constant C_0 such that

$$\sup_{1\leq s\leq 2} \phi(sr) \leq C_0\phi(r),\ 0 < r \leq \frac{1}{2}. \tag{1.3}$$

(3) For $s \in \mathbb{N}$,

$$\sum_{1\leq j\leq s} 2^j\phi(2^{-j}) \leq C2^s\phi(2^{-s}). \tag{1.4}$$

Let $f = \sum_{(\varepsilon,j,k)\in\Lambda_n} f^{\varepsilon}_{j,k}\Phi^{\varepsilon}_{j,k}$ and ϕ satisfy (1.2), (1.3) and (1.4). In Theorem 3.1, we prove that $f \in M_{\phi,p}(\mathbb{R}^n)$ if and only if for any dyadic cube Q with volume $|Q| \leq 1$,

$$|Q|^{-\frac{1}{p}}\phi^{-1}(|Q|^{\frac{1}{n}}) \Big\| \Big(\sum_{(\varepsilon,j,k):Q_{j,k}\subset Q} 2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx - k)\Big)^{\frac{1}{2}} \Big\|_{L^p} \leq C_f < \infty. \tag{1.5}$$

Remark 1.3. Let $k+Q$ denote the cube

$$\Big\{(x_1,\cdots,x_n): k_i \le x_i < 1+k_i, k_i\in\mathbb{Z}, i=1,\cdots,n\Big\}.$$

If $r^n\phi^p(r) \ge C > 0, r\in(0,1]$, then $M_{\phi,p} = \{f : \sup_k \|f\|_{L^p(k+Q)} < \infty\}$. If $\overline{\lim_{r\to 0}} r^{-1}\phi(r) = 0$, then $M_{\phi,p}(\mathbb{R}^n)$ consists of all constants. Hence, in this paper, we always assume that there exist two positive constants C_1 and C_2 such that $C_1 r \le \phi(r) \le C_2 r^{-\frac{n}{p}}$.

In Section 3.2, we study the predual of $M_{\phi,p}(\mathbb{R}^n)$. Fefferman [4] proved that the predual of $BMO(\mathbb{R}^n)$ is the Hardy space $H^1(\mathbb{R}^n)$. Kalita [8] used a group of Borel measures to characterize the predual of $M_{\alpha,2}(\mathbb{R}^n)$. We introduce two classes of Hardy spaces $H^{\phi,p}(\mathbb{R}^n)$ and $H^{\phi,p}_w(\mathbb{R}^n)$. For $1<p<\infty$, we prove that $H^{\phi,p}(\mathbb{R}^n)$ and $H^{\phi,p}_w(\mathbb{R}^n)$ are equivalent. In Theorem 3.8, we obtain the following duality relations.

$$\begin{cases} (H^{\phi,\infty}(\mathbb{R}^n))' = M_{\phi,1}(\mathbb{R}^n); \\ (H^{\phi,p}(\mathbb{R}^n))' = M_{\phi,p'}(\mathbb{R}^n), 1<p<\infty. \end{cases}$$

In Section 4, we devote to the applications of the results obtained above. The first one is the multipliers on $M_{\phi,p}(\mathbb{R}^n)$. The multiplier spaces are defined as follows.

Definition 1.4. For two spaces X and Y, $f\in M(X,Y)$ means $\sup\limits_{\|g\|_X\le 1}\|fg\|_Y < \infty$. Specially, we write $M(X,X)$ as $M(X)$.

Multiplier spaces were introduced in 1950s and studied extensively since then. See Maz'ya-Shaposhnikova [12, 13] for details. Janson [6] and Stegenga [23] studied the multipliers on $M_{\phi,1}(\mathbb{R}^n)$ and the predual $H^{\phi,\infty}(\mathbb{R}^n)$. They obtained

Proposition 1.5. ([6, Theorem 2]). *Let $\phi(r)r^{-1}$ be 'almost decreasing' in the sense that*

$$\phi(\rho)\rho^{-1} \le C\phi(r)r^{-1},\ \rho \ge r. \tag{1.6}$$

Then $M(M_{\phi,1}(\mathbb{R}^n)) = M_{\psi,1}(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$, where $\psi(r) = \phi(r)/(\int_r^1 \phi(t)t^{-1}dt)$.

For any $f\in M_{\phi,1}(\mathbb{R}^n)$, Janson [6] constructed a special function h to characterize $M(M_{\phi,1}(\mathbb{R}^n))$. In this paper, we apply a different method to study the multipliers on $M_{\phi,p}(\mathbb{R}^n), p>1$. Define

$$\psi(r) = \frac{\phi(r)}{\left\{\sum\limits_{j=1}^{1+[|\log_2 r|]} \phi(2^{-j})\right\}}.$$

Let $f \in M_{\psi,1}(\mathbb{R}^n)$ and a be a wavelet atom of $H^{\phi,p}(\mathbb{R}^n)$. We use multi-resolution analysis to decompose the product of $f \cdot a$ as

$$f(x)a(x) = I(x) + II(x) + III(x) + IV(x) + V(x) + VI(x) + VII(x).$$

Thus, the estimates of fa can be converted to the computations of the wavelet coefficients $\{f^{\varepsilon}_{j,k}\}$ and $\{a^{\varepsilon}_{j,k}\}$. By Theorem 3.5, we obtain a characterization of $M(H^{\phi,p}(\mathbb{R}^n))$, $1 < p \leq \infty$. See Section 4.1. The characterization of $M(M_{\phi,p}(\mathbb{R}^n)), 1 \leq p < \infty$, can be deduced from Theorem 4.4 and the duality between $H^{\phi,p}(\mathbb{R}^n)$ and $M_{\phi,p'}(\mathbb{R}^n)$. See Theorem 4.8.

Remark 1.6.

(i) For $M_{\phi,1}(\mathbb{R}^n)$, we only assume that ϕ satisfies (1.2) and (1.3). Hence the function ϕ used here is more general than that in (1.6). Our result is slightly stronger than that of [6].

(ii) One usual tool to characterize $M(X,Y)$ is the capacity on arbitrary compact set. Sometimes, it is difficult to compute the capacity on arbitrary compact set. Recently, wavelets have been used to characterize the multipliers on Sobolev spaces. We refer the reader to Yang-Zhu [34] for details.

The second application is the stability under the action of Calderón- Zygmund operator. The stability of function spaces under the perturbation of operators plays an important role in many problems. See Maz'ya-Verbitsky [14] and Alvarez [1]. In Sections 4.2 and 4.3, under the perturbation of T, we discuss the stability of $M_{\phi,p}(\mathbb{R}^n)$, $M(H^{\phi,p})$ and $M(M_{\phi,p})$, respectively. Unlike the case of $M_{\alpha,p}(\mathbb{R}^n)$, Theorems 4.6 and 4.9 imply that $M_{\phi,1}(\mathbb{R}^n)$ and $M(M_{\phi,p})$ may be unstable.

Remark 1.7. It is well-known that BMO space can be characterized via Carleson measure. See Stein [24]. The Carleson measure characterization of $M_{\alpha,2}(\mathbb{R}^n)$ was obtained by Essen-Janson-Peng-Xiao [3]. For the spaces $M_{\phi,p}(\mathbb{R}^n)$, we could introduce a class of Carleson measures related to the function ϕ. By a similar method, we could characterize $M_{\phi,p}(\mathbb{R}^n)$ via Carleson measure related to the function ϕ. We will discuss this problem in another paper.

The rest of this paper is organized as follows. In Section 2, we state some preliminary notations and lemmas which will be used in the sequel. In Section 3, we give a wavelet characterization of $M_{\phi,p}(\mathbb{R}^n)$ and obtain the predual of $M_{\phi,p}(\mathbb{R}^n)$. In Section 3.2, we consider the multipliers and stabilities of $M_{\phi,p}(\mathbb{R}^n)$ and $H^{\phi,p}(\mathbb{R}^n)$. Section 4 is devoted to the stability of $M(H^{\phi,p})$, $M_{\phi,p}(\mathbb{R}^n)$ and $M(M_{\phi,p})$, respectively.

2. Wavelets, Functions and Operators

We state some notations related to wavelets. The wavelets used in this paper are tensorial wavelets and real valued. Let $E_n = \{0,1\}^n$ and $\dot{E}_n = \{0,1\}^n \backslash \{0\}$. For $\epsilon = 0$ (respectively, $\epsilon \in \dot{E}_n$), let

$$\Phi^{\epsilon} \in C_0^m([-2^M, 2^M]^n) \tag{2.1}$$

be Daubechies' scale function (respectively, wavelet), cf [15]. We suppose that Daubechies wavelets are sufficient smooth and have sufficient vanishing moments to adapt our needs.

For $j \in \mathbb{Z}$, $k = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$ and $\epsilon \in E_n$, we write

$$\begin{cases} Q = Q_{j,k} = \prod_{s=1}^{n} [2^{-j}k_s, 2^{-j}(k_s+1)], \\ \Phi_Q^{\epsilon}(x) = \Phi_{j,k}^{\epsilon}(x) = 2^{\frac{jn}{2}} \Phi^{\epsilon}(2^j x - k). \end{cases}$$

For any function f and $(\epsilon, j, k) \in E_n \times \mathbb{Z} \times \mathbb{Z}^n$, we define $f_{j,k}^{\epsilon} = \langle f, \Phi_{j,k}^{\epsilon} \rangle$. For $j \geq 0$, let

$$\begin{cases} P_j f(x) = \sum_{k \in \mathbb{Z}^n} f_{j,k}^0 \Phi_{j,k}^0(x), \\ Q_j f(x) = \sum_{\epsilon \in \dot{E}_n, k \in \mathbb{Z}^n} f_{j,k}^{\epsilon} \Phi_{j,k}^{\epsilon}(x). \end{cases}$$

In the sequel, we always write

$$\begin{cases} \Omega = \{Q_{j,k}, j \in \mathbb{N}, k \in \mathbb{Z}^n\}, \\ \Lambda_n = \{(\epsilon, j, k), \epsilon \in E_n, j \geq 0, k \in \mathbb{Z}^n, \text{ and if } j \neq 0, \text{ then } \epsilon \neq 0\}. \end{cases}$$

Let M be the constant in (2.1). Let χ and $\tilde{\chi}$ be the characteristic functions of the cubes $[0,1]^n$ and $[-2^{M+2}, 2^{M+2}]^n$, respectively.

Now we present some wavelet characterization of function spaces. Let $g(x) = \sum_{(\epsilon,j,k) \in \Lambda_n} g_{j,k}^{\epsilon} \Phi_{j,k}^{\epsilon}(x)$. Denote

$$\begin{cases} S_r g_j(x) = \sum_{\epsilon,k\colon (\epsilon,j,k) \in \Lambda_n} 2^{j(r+\frac{n}{2})} |g_{j,k}^{\epsilon}| \tilde{\chi}(2^j x - k), \ j \in \mathbb{N} \text{ and } r \in \mathbb{R}; \\ S g_j(x) = S_0 g_j(x), \ j \in \mathbb{N}. \end{cases}$$

For $1 \leq p \leq \infty$, we denote by p' the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Let $1 \leq p, q < \infty$ and $r \in \mathbb{R}$. It is well-known that $(F_p^{r,q}(\mathbb{R}^n))' = F_{p'}^{-r,q'}(\mathbb{R}^n)$. Triebel-Lizorkin spaces can be characterized via Daubechies wavelets and M, see [26, Section 1.2.3] for details.

Proposition 2.1. *Let* $g(x) = \sum_{(\epsilon,j,k) \in \Lambda_n} g_{j,k}^{\epsilon} \Phi_{j,k}^{\epsilon}(x)$.

(i) *Let* $1 \leq p, q < \infty$ *and* $r \in \mathbb{R}$ *and* M *be the Hardy-Littlewood maximal operator.*

$$g \in F_p^{r,q}(\mathbb{R}^n) \Leftrightarrow \sum_{\epsilon} \Big\| \Big(\sum_{j\in\mathbb{Z}} (MS_r g_j^{\epsilon}(x))^q \Big)^{\frac{1}{q}} \Big\|_{L^p} < \infty$$
$$\Leftrightarrow \Big\| \Big(\sum_{(\epsilon,j,k)\in\Lambda_n} 2^{jq(r+\frac{n}{2})} |g_{j,k}^{\epsilon}|^q \chi(2^j \cdot -k) \Big)^{\frac{1}{q}} \Big\|_{L^p} < \infty.$$

(ii) $g \in F_{\infty}^{0,\infty}(\mathbb{R}^n) = B_{\infty}^{0,\infty}(\mathbb{R}^n)$ *if and only if*

$$|g_{j,k}^{\epsilon}| \leq C 2^{-\frac{n}{2} j}, \forall (\epsilon, j, k) \in \Lambda_n.$$

Calderón-Zygmund operators play an important role in harmonic analysis. For $N > 0$, we say T belongs to non-homogeneous Calderón-Zygmund operator class $CZO(N)$ if

(i) T is continuous from $C^1(\mathbb{R}^n)$ to $(C^1(\mathbb{R}^n))'$ such that

$$Tf(x) = \int K(x,y) f(y) dy;$$

(ii) For any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq N-1$, T and its dual operator T^* satisfy

$$Tx^{\alpha} = T^* x^{\alpha} = 0; \tag{2.2}$$

(iii) The kernel K satisfies

$$\sup_x \int_{|x-y|\geq 1} \Big[|K(x,y)| + |K(y,x)| \Big] dy < \infty. \tag{2.3}$$

For $x \neq y$ and $|\alpha| + |\beta| \leq N$, we have

$$|\partial_x^{\alpha} \partial_y^{\beta} K(x,y)| \leq C |x-y|^{-(n+|\alpha|+|\beta|)}. \tag{2.4}$$

For simplicity, we suppose that N is big enough such that $N > n + 1 + |\log_2 C_0|$ to satisfy the needs of Lemma 2.3 and Theorem 3.9. For any (ϵ, j, k), $(\epsilon', j', k') \in \Lambda_n$ and a distribution $K(\cdot,\cdot)$ in $S'(\mathbb{R}^n \times \mathbb{R}^n)$, let

$$a_{j,k,j',k'}^{\epsilon,\epsilon'} = \langle K(\cdot,\cdot),\ \Phi_{j,k}^{\epsilon} \Phi_{j',k'}^{\epsilon'} \rangle.$$

If $T \in CZO(N)$, then its distribution-kernel $K(\cdot,\cdot)$ and the corresponding coefficients $a_{j,k,j',k'}^{\epsilon,\epsilon'}$ have the following relations.

Proposition 2.2. ([16, Section 8.3, Proposition 1]). *Let* $T \in CZO(N)$. *Then*

$$\sup_{k\in\mathbb{Z}^n}\sum_{k'\in\mathbb{Z}^n}\Big(|a^{0,0}_{0,k,0,k'}|+|a^{0,0}_{0,k',0,k}|\Big)\leq C, \tag{2.5}$$

$$|a^{\epsilon,\epsilon'}_{j,k,j',k'}|\leq C2^{-|j-j'|(\frac{n}{2}+N)}\Big(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|}\Big)^{n+N}, \tag{2.6}$$
$$\forall|\epsilon|+|\epsilon'|\neq 0.$$

At the end of this section, we give some L^p-estimates of Calderón-Zygmund operators which are useful in the sequel. For $s, j_0\in\mathbb{Z}, k_0\in\mathbb{Z}^n$ and $0\leq s\leq j_0$, there exists a $l_s\in\mathbb{Z}^n$ such that the dyadic cube Q_{j_0,k_0} is contained in the dyadic cube Q_{j_0-s,l_s}. Let

$$a(x)=\sum_{\epsilon\neq 0, Q_{j,k}\subset Q_{j_0,k_0}} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)\in L^p.$$

For any $l\in\mathbb{Z}^n$, we define

$$a_{l,j_0,k_0}(x)=\sum_{Q_{j,k}\subset Q_{j_0,l+l_0}}\ \sum_{Q_{j',k'}\subset Q_{j_0,k_0}} a^{\epsilon,\epsilon'}_{j,k,j',k'}a^{\epsilon'}_{j',k'}\Phi^{\epsilon}_{j,k}(x). \tag{2.7}$$

For any $0\leq j<j_0$ and $(\epsilon,j,l+l_{j_0-j})\in\Lambda_n$, we define

$$a^{\epsilon,j,l}_{j_0,k_0}=\sum_{Q_{j',k'}\subset Q_{j_0,k_0}} a^{\epsilon,\epsilon'}_{j,l+l_{j_0-j},j',k'}a^{\epsilon'}_{j',k'}. \tag{2.8}$$

Lemma 2.3. *Let* $1<p<\infty$. *If* $\{a^{\epsilon,\epsilon'}_{j,k,j',k'}\}$ *satisfy (2.6), then we have*

(*i*) $\|a_{l,j_0,k_0}\|_{L^p}\leq C(1+|l|)^{-N}\|a\|_{L^p}$;

(*ii*) $|a^{\epsilon,j,l}_{j_0,k_0}|\leq C(1+|l|)^{-n-N}2^{(\frac{n}{2}+N)j}2^{-(N+n-\frac{n}{p})j_0}\|a\|_{L^p}$.

Proof. We first prove (i). Notice that for $1<p<\infty$, $L^p(\mathbb{R}^n)=\dot{F}^{0,2}_p(\mathbb{R}^n)$. By Theorem 2.1, we have

$$\|a_{l,j_0,k_0}\|_{L^p}=\Big\|\Big(\sum_{Q_{j,k}\subset Q_{j_0,l+k_0}}2^{jn}\Big|\sum_{Q_{j',k'}\subset Q_{j_0,k_0}}a^{\epsilon,\epsilon'}_{j,k,j',k'}a^{\epsilon'}_{j',k'}\Big|^2\chi(2^jx-k)\Big)^{\frac{1}{2}}\Big\|_{L^p}.$$

We divide the estimate of $|\sum_{Q_{j',k'}\subset Q_{j_0,k_0}}a^{\epsilon,\epsilon'}_{j,k,j',k'}a^{\epsilon'}_{j',k'}|$ into two cases:

Case 1. $x \in Q_{j,k}$ and $j_0 \leq j' \leq j$. We have

$$
\begin{aligned}
&|\sum_{Q_{j',k'} \subset Q_{j_0,k_0}} a^{\epsilon,\epsilon'}_{j,k,j',k'} a^{\epsilon'}_{j',k'}| \\
&\leq 2^{-|j-j'|(\frac{n}{2}+N)-\frac{n}{2}j'} Sa_{j'}(x) \sum_{k'} (1+2^{j'-j_0}\max(|l|-2^n,0)+|2^{j'}x-k'|)^{-n-N} \\
&\leq 2^{-|j-j'|(\frac{n}{2}+N)-\frac{n}{2}j'} (1+2^{j'-j_0}\max(|l|-2^n,0))^{-N} Sa_{j'}(x).
\end{aligned}
$$

Case 2. $x \in Q_{j,k}$ and $j_0 \leq j < j'$. We denote $k' = 2^{j'-j}m + \tau$, where $\tau \in \{0, \cdots, 2^{j'-j}-1\}^n$. We have

$$
\begin{aligned}
&\Big|\sum_{Q_{j',k'} \subset Q_{j_0,k_0}} a^{\epsilon,\epsilon'}_{j,k,j',k'} a^{\epsilon'}_{j',k'}\Big| \\
&\leq 2^{-|j-j'|(\frac{n}{2}+N)+\frac{n}{2}j'-nj} \sum_m \frac{Ma_{j'}(2^{-j}m)}{(1+2^{j-j_0}\max(|l|-2^n,0)+|2^jx-m|)^{n+N}} \\
&\leq 2^{-|j-j'|(\frac{n}{2}+N)-\frac{n}{2}j} \frac{MMSa_{j'}(x)}{(1+2^{j-j_0}\max(|l|-2^n,0)+|2^jx-m|)^{n+N}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{Q_{j,k} \subset Q_{j_0,l+l_0}} 2^{jn} \Big|\sum_{Q_{j',k'} \subset Q_{j_0,k_0}} a^{\epsilon,\epsilon'}_{j,k,j',k'} a^{\epsilon'}_{j',k'}\Big|^2 \chi(2^jx-k) \\
&\leq \frac{1}{(1+|l|)^{2N}} \Big\{ \sum_{j \geq j_0} \Big[\sum_{j_0 \leq j' \leq j} 2^{-N|j-j'|} Sa_{j'}(x) \Big]^2 \\
&\quad + \sum_{j \geq j_0} \Big[\sum_{j_0 \leq j \leq j'} 2^{-N|j-j'|} MMSa_{j'}(x) \Big]^2 \Big\} \\
&\leq \frac{1}{(1+|l|)^{2N}} \Big\{ \sum_{j' \geq j_0} [Sa_{j'}(x)]^2 + \sum_{j' \geq j_0} [MMSa_{j'}(x)]^2 \Big\}.
\end{aligned}
$$

Applying Theorem 2.1, we get $\|a_{l,j_0,k_0}\|_{L^p} \leq C(1+|l|)^{-N}\|a\|_{L^p}$.

Then we prove (ii). Fix $j' \geq j_0 > j$. We have

$$
\begin{aligned}
&\Big|\sum_{Q_{j',k'} \subset Q_{j_0,k_0}} a^{\epsilon,\epsilon'}_{j,l+l_{j_0-j},j',k'} a^{\epsilon'}_{j',k'}\Big| \\
&\leq \int \sum_{Q_{j',k'} \subset Q_{j_0,k_0}} 2^{\frac{n}{2}j'} |a^{\epsilon,\epsilon'}_{j,l+l_{j_0-j},j',k'}| 2^{\frac{n}{2}j'} |a^{\epsilon'}_{j',k'}| \chi(2^{j'}x-k') dx \\
&\leq \int \sum_{Q_{j',k'} \subset Q_{j_0,k_0}} \frac{2^{\frac{n}{2}j'-|j-j'|(\frac{n}{2}+N)} 2^{\frac{n}{2}j'}}{(1+|l+l_{j_0-j}-2^{j-j'}k'|)^{n+N}} |a^{\epsilon'}_{j',k'}| \chi(2^{j'}x-k') dx \\
&\leq C(1+|l|)^{-n-N} 2^{(\frac{n}{2}+N)j} \int 2^{-Nj'} Sa_{j'}(x) dx.
\end{aligned}
$$

We can obtain

$$\begin{aligned}
&\Big|\sum_{Q_{j',k'}\subset Q_{j_0,k_0}} a^{\epsilon,\epsilon'}_{j,l+l_{j_0-j},j',k'} a^{\epsilon'}_{j',k'}\Big| \\
&\le C(1+|l|)^{-n-N}2^{(\frac{n}{2}+N)j}\int \sum_{j'\ge j_0} 2^{-Nj'} Sa_{j'}(x)dx \\
&\le C(1+|l|)^{-n-N}2^{(\frac{n}{2}+N)j}2^{-Nj_0}\int \Big(\sum_{j'\ge j_0}|Sa_{j'}(x)|^2\Big)^{\frac{1}{2}}dx \\
&\le C(1+|l|)^{-n-N}2^{(\frac{n}{2}+N)j}2^{-(N+n-\frac{n}{p})j_0}\Big\|\Big(\sum_{j'\ge j_0}|Sa_{j'}(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}. \qquad \blacksquare
\end{aligned}$$

3. Generalized Morrey Spaces $M_{\phi,p}(\mathbb{R}^n)$

3.1. Wavelet characterization

In this section, we use wavelets to characterize $M_{\phi,p}(\mathbb{R}^n)$. Let χ_S be the characteristic function of a set S. We can obtain the following wavelet characterization of $M_{\phi,p}(\mathbb{R}^n)$.

Theorem 3.1. *Let $1 < p < \infty$ and ϕ satisfy (1.2), (1.3) and (1.4). Then the following two statements are equivalent:*

$$f = \sum_{(\epsilon,j,k)\in\Lambda_n} f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k} \in M_{\phi,p}(\mathbb{R}^n). \tag{3.1}$$

There exists a constant C_f such that for all dyadic cube Q with volume $|Q| \le 1$,

$$\frac{1}{|Q|^{1/p}\phi(|Q|^{\frac{1}{n}})}\Big\|\Big(\sum_{(\varepsilon,j,k):Q_{j,k}\subset Q} 2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C_f < \infty. \tag{3.2}$$

Proof. At first we prove that if $f \in M_{\phi,p}$, f satisfies (3.2) for any dyadic cube Q. We divide the proof into two cases.

Case 1. $2^{(M+3)n}|Q| \le 1$. Because the support of Daubechies wavelets is bounded, there exists a cube Q_M such that

$$\begin{cases} |Q_M| \le 2^{(M+2)n}|Q|, \\ \text{supp } \Phi^{\epsilon}_{j,k} \subset Q_M, \forall(\epsilon,j,k)\in\Lambda_n, Q_{j,k}\subset Q. \end{cases}$$

For any $(\epsilon, j, k) \in \Lambda_n$ and $Q_{j,k} \subset Q$, we have

$$f^{\epsilon}_{j,k} = \Big\langle\Big(\sum_{(\epsilon',j',k')\in\Lambda_n} f^{\epsilon'}_{j',k'}\Phi^{\epsilon'}_{j',k'} - f_{Q_M}\Big)\chi_{Q_M},\ \Phi^{\epsilon}_{j,k}\Big\rangle.$$

We can deduce from Proposition 2.1 that

$$\begin{aligned}&\Big\|\Big(\sum_{(\varepsilon,j,k):Q_{j,k}\subset Q}2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac12}\Big\|_{L^p}\\ \le&\Big\|\Big(\sum_{(\epsilon,j,k)\in\Lambda_n}f^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x)-f_{Q_M}\Big)\chi_{Q_M}(x)\Big\|_{L^p}\\ \le&\|f\|_{M_{\phi,p}}|Q_M|^{\frac1p}\phi(|Q_M|^{\frac1n}).\end{aligned}$$

By (1.3), there exists a constant C depending only on M and C_0, such that

$$\Big\|\Big(\sum_{Q_{j,k}\subset Q}2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac12}\Big\|_{L^p}\le C\|f\|_{M_{\phi,p}}r^{\frac np}\phi(r).$$

Case 2. $|Q|\le 1$ and $2^{(M+3)n}|Q|>1$. Let $j_0,j_Q\in\mathbb{N}$ such that $\frac12<2^{(M+3-j_0)n}|Q|\le 1$, $2^{n(j_0-j_Q)}=|Q|$ and let $E_{n,j_0}=\{m:m=0,\cdots,2^{nj_0}-1\}$. There exist 2^{nj_0} dyadic cubes $Q_{j_0,m}$, $m\in E_{n,j_0}$, such that $|Q_{j_0,m}|=2^{-nj_Q}$ and $Q=\bigcup\limits_{m\in E_{n,j_0}}Q_{j_0,m}$. Thus we have

$$\begin{aligned}\Big\|\Big(\sum_{Q_{j,k}\subset Q}2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac12}\Big\|_{L^p}\le&\Big\|\Big(\sum_{Q_{j,k}\subset Q,j\le j_Q}2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac12}\Big\|_{L^p}\\ &+\sum_{m\in\mathbb{Z}^n,Q_{j_0,m}\subset Q}\Big\|\Big(\sum_{Q_{j,k}\subset Q_{j_0,m}}2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac12}\Big\|_{L^p}.\end{aligned}$$

It is easy to see that

$$\sum_{Q_{j_0,m}\subset Q}\Big\|\Big(\sum_{Q_{j,k}\subset Q_{j_0,m}}2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac12}\Big\|_{L^p}\le C\|f\|_{M_{\phi,p}}\phi(\tfrac12)$$

and

$$\Big\|\Big(\sum_{Q_{j,k}\subset Q,j\le j_Q}2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac12}\Big\|_{L^p}\le C\|f\|_{M_{\phi,p}}\Big[1+\sum_{1\le j\le j_Q}\phi(2^{-j})\Big].$$

By (1.3) again, there exists C, depending on M and C_0, such that

$$\Big\|\Big(\sum_{Q_{j,k}\subset Q}2^{jn}|f^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac12}\Big\|_{L^p}\le C\|f\|_{M_{\phi,p}}\phi(|Q|^{\frac1n}).$$

Conversely, we assume that f satisfies (3.2). It is easy to see that

$$\sup_{y\in\mathbb{R}^n,\frac12\le r\le 1}|f_{Q(y,r)}|<\infty.$$

For any cube $Q(y,r)$ with $r \leq \frac{1}{2}$, take $j_{Q(y,r)} \in \mathbb{N}$ such that $2^{-nj_{Q(y,r)}} \leq |Q(y,r)| < 2^{n(1-j_{Q(y,r)})}$. We decompose f into the following two parts:

$$\begin{cases} f_1(x) = \displaystyle\sum_{(\epsilon,j,k)\in\Lambda_n, j\geq j_{Q(y,r)}} f^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x), \\ f_2(x) = \displaystyle\sum_{(\epsilon,j,k)\in\Lambda_n, j< j_{Q(y,r)}} f^\epsilon_{j,k}\Phi^\epsilon_{j,k}(x). \end{cases}$$

Denote by $f_{1,Q(y,r)}$ and $f_{2,Q(y,r)}$ the means of f_1 and f_2 on the cube $Q(y,r)$, respectively. It is easy to see that

$$|f_{1,Q(y,r)}| \leq |Q(y,r)|^{-\frac{1}{p}}\|f_1\|_{L^p(Q(y,r))}.$$

Hence there exists a constant C depending on C_0, M and C_f such that

$$\phi(r)^{-1}\Big(r^{-n}\int_{Q(y,r)}|f_1(x)-f_{1,Q(y,r)}|^p dx\Big)^{\frac{1}{p}} \leq C.$$

For any j with $0 \leq j < j_{Q(y,r)}$, there exists $x^\epsilon_{j,k} \in Q(y,r)$ such that

$$\int_{Q(y,r)}\Phi^\epsilon(2^jx-k)dx = \int_{Q(y,r)}\Phi^\epsilon(2^jx^\epsilon_{j,k}-k)dx.$$

Let

$$Q_{\epsilon,y,r} = \Big\{(\epsilon,j,k): \ (\epsilon,j,k)\in\Lambda_n, Q(y,r)\bigcap \text{supp } \Phi^\epsilon_{j,k} \neq \emptyset\Big\}.$$

We have

$$\begin{aligned} &\int_{Q(y,r)}|f_2(x)-f_{2,Q(y,r)}|^p dx \\ &\leq \int_{Q(y,r)}\Big|\sum_{(\epsilon,j,k)\in Q_{\epsilon,y,r}, j<j_{Q(y,r)}} f^\epsilon_{j,k}\Big[\Phi^\epsilon_{j,k}(x)-\Phi^\epsilon_{j,k}(x^\epsilon_{j,k})\Big]\Big|^p dx \\ &\leq C\int_{Q(y,r)}\Big(\sum_{(\epsilon,j,k)\in Q_{\epsilon,y,r}, j<j_{Q(y,r)}} |f^\epsilon_{j,k}|2^{(\frac{n}{2}+1)j}|Q(y,r)|^{\frac{1}{n}}\Big)^p dx. \end{aligned}$$

By (3.2), we have

$$\begin{aligned} \int_{Q(y,r)}|f_2(x)-f_{2,Q(y,r)}|^p dx &\leq C\int_{Q(y,r)}\Big(\sum_{0\leq j<j_{Q(y,r)}}\phi(2^{-j})2^j|Q(y,r)|^{\frac{1}{n}}\Big)^p dx \\ &\leq C\int_{Q(y,r)}\phi(r)^p dx \leq C\phi(r)^p r^n, \end{aligned}$$

where in the last inequality we have used (1.3) and (1.4). This completes the proof of Theorem 3.1. ∎

3.2. Duality between $M_{\phi,p}(\mathbb{R}^n)$ and $H^{\phi,p'}(\mathbb{R}^n)$

Fefferman-Stein [5] proved that $BMO(\mathbb{R}^n)$ is the dual of Hardy space $H^1(\mathbb{R}^n)$. Kalita [8] characterized the predual of $M_{\alpha,2}(\mathbb{R}^n)$ by the method of functional analysis. The predual of Q spaces is obtained by Dafni-Xiao [2] via Hausdorff capacities. In this section, we adopt Fefferman-Stein's ideas and use wavelet atoms to study the predual of $M_{\phi,p}(\mathbb{R}^n)$.

At first, we introduce two classes of Hardy spaces associated with ϕ.

Definition 3.2. Let $1 < p < \infty$.

(i) A distribution $g = \sum_{\epsilon,j,k} g^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}$ is called a $(\phi,p)-$wavelet atom on dyadic cube $Q(y,r)$, if the support of $\sum_{\epsilon,j,k} 2^{jn}|g^{\epsilon}_{j,k}|^2\chi(2^jx-k)$ is contained in $Q(y,r)$ and

$$\Big\|\Big(\sum_{\epsilon,j,k} 2^{jn}|g^{\epsilon}_{j,k}|^2\chi(2^jx-k)\Big)^{\frac{1}{2}}\Big\|_{L^p} \leq r^{\frac{n}{p}-n}\phi^{-1}(r).$$

(ii) We call a distribution $f \in H^{\phi,p}_w(\mathbb{R}^n)$ if there exist a sequence $\{\lambda_u\} \in l^1$ and a sequence of $(\phi,p)-$wavelet atoms $\{g_u\}$ such that $f(x) = \sum_u \lambda_u g_u(x)$. The norm of $H^{\phi,p}_w(\mathbb{R}^n)$ is defined as

$$\|f\|_{H^{\phi,p}_w} = \inf\sum_u |\lambda_u|,$$

where the infimum is take over all the possible decompositions.

Another class of atoms are introduced without using wavelets.

Definition 3.3. Fix $1 < p \leq \infty$.

(i) A distribution g is said to be a $(\phi,p)-$atom on cube $Q(y,r)$, if $\operatorname{supp} g \subset Q(y,r)$, $\|g\|_{L^p} \leq r^{\frac{n}{p}-n}\phi^{-1}(r)$ and for $r \leq \frac{1}{2}$, $\int g(x)dx = 0$ is true in the sense of distributions.

(ii) We call a distribution $f \in H^{\phi,p}(\mathbb{R}^n)$ if there exist a sequence $\{\lambda_u\} \in l^1$ and a sequence of $(\phi,p)-$atoms $\{g_u(x)\}$ such that $f(x) = \sum_u \lambda_u g_u(x)$. The norm of $H^{\phi,p}(\mathbb{R}^n)$ is defined as

$$\|f\|_{H^{\phi,p}} = \inf\sum_u |\lambda_u|,$$

where the infimum is take over all the possible decompositions.

For $\phi(r) = 1$, we denote $H^{\phi,p}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$. Then we have

Lemma 3.4. (i) *If there exists $c > 0$ such that $\phi(r) > c$, then*

$$H^{\phi,\infty}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n).$$

(ii) *If $\varliminf_{r\to 0} \phi(r) = 0$, then $H^1(\mathbb{R}^n) \subset H^{\phi,\infty}(\mathbb{R}^n)$.*

Now we prove that $H_w^{\phi,p}(\mathbb{R}^n)$ and $H^{\phi,p}(\mathbb{R}^n)$ are equivalent.

Theorem 3.5. *Let $1 < p < \infty$ and ϕ satisfy (1.2), (1.3) and (1.4). Then $H^{\phi,p}(\mathbb{R}^n) = H_w^{\phi,p}(\mathbb{R}^n)$.*

Proof. Let a be a wavelet atom on dyadic cube $Q(y, r)$. The support of a is contained in $Q(y, 2^{M+2}r)$, $\|a\|_{L^p} \le r^{\frac{n}{p}-n}\phi^{-1}(r)$ and $\int a(x)dx = 0$ for $r \le \frac{1}{2}$. We can get $\|a\|_{H^{\phi,p}} \le C$. This gives $H_w^{\phi,p}(\mathbb{R}^n) \subseteq H^{\phi,p}(\mathbb{R}^n)$.

Conversely, for a $(\phi, p)-$atom a, we can write

$$a(x) = \sum_{\epsilon\neq 0, j\ge j_Q, k\in\mathbb{Z}^n} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x) + \sum_{(\epsilon,j,k)\in\Lambda_n, 0\le j<j_Q} a^{\epsilon}_{j,k}\Phi^{\epsilon}_{j,k}(x) =: I + II.$$

There exists a constant C such that $\|I\|_{H_w^{\phi,p}} \le C$. For II, we have

$$|a^{\epsilon}_{j,k}| \le C2^{\frac{nj}{2}}2^{j-j_Q}\phi^{-1}(2^{-j_Q}).$$

By (1.4), we can obtain

$$\|II\|_{H_w^{\phi,p}} \le C \sum_{0\le j<j_Q} 2^{j-j_Q}\phi(2^{-j})\phi^{-1}(2^{-j_Q}) \le C.$$

This implies that $H^{\phi,p}(\mathbb{R}^n) \subseteq H_w^{\phi,p}(\mathbb{R}^n)$. ■

For any $k \in \mathbb{Z}^n$, we call $u \in k + Q$ if $k_i \le u_i < 1 + k_i$. We have the following relations among $H^{\phi,p}(\mathbb{R}^n)$.

Theorem 3.6. *Suppose that ϕ satisfies (1.2) and (1.3).*

(i) *If ϕ is unbounded, then $H^{\phi,q}(\mathbb{R}^n) \subsetneq H^{\phi,p}(\mathbb{R}^n)$, $1 < p < q \le \infty$.*

(ii) *If $\sum_{j\ge 1} \phi^{-1}(2^{-j}) < \infty$, then $H^{\phi,\infty}(\mathbb{R}^n) = l^1(L^\infty(k + Q))$.*

(iii) *$r^{\frac{n}{p}}\phi(r) \ge C > 0$, then $H^{\phi,p}(\mathbb{R}^n) = l^1(L^p(k + Q))$.*

Proof. (ii) and (iii) are obvious. We only prove (i). By Hölder's inequality, we know that $H^{\phi,q}(\mathbb{R}^n) \subseteq H^{\phi,p}(\mathbb{R}^n)$. Now we prove that $H^{\phi,q}(\mathbb{R}^n) \neq H^{\phi,p}(\mathbb{R}^n)$. For any integer $j \ge 1$, let $Q_j = [0, 2^{-j})^n$ and

$$a_j(x) = 2^{nj}\phi^{-1}(2^{-j})2^{\frac{nj}{p}}\Phi^{(1,0,\cdots,0)}(2^{2j}x).$$

Then $\|a_j\|_{H^{\phi,p}} = 1$. But we know that

$$\|a_j\|_{H^{\phi,q}} = 2^{\frac{nj}{p}} - 2^{\frac{nj}{q}} \to \infty.$$

This implies that $H^{\phi,q}(\mathbb{R}^n) \subsetneq H^{\phi,p}(\mathbb{R}^n)$, $1 < p < q \leq \infty$. ∎

When ϕ is bounded, we have

Theorem 3.7. *If ϕ is bounded and satisfies (1.2) and (1.3), then for $1 < p \neq q \leq \infty$, we have $H^{\phi,p}(\mathbb{R}^n) = H^{\phi,q}(\mathbb{R}^n)$.*

Proof. We know that a $(\phi,\infty)-$atom is a $(\phi,p)-$atom for $1 < p < \infty$. Further, if $\phi(r) = 1$, we decompose each $(1,p)-$atom into a group of $(1,\infty)-$atoms, see Stein [24, Section 3.2]. Hence by the definition of $(\phi,p)-$atom, if ϕ is bounded, we can decompose each $(\phi,p)-$atom into a group of $(\phi,\infty)-$atoms. ∎

Next we prove that the dual of $H^{\phi,p}(\mathbb{R}^n)$ is $M_{\phi,p}(\mathbb{R}^n)$.

Theorem 3.8. *(i) If ϕ satisfies (1.2) and (1.3), then the dual $H^{\phi,\infty}(\mathbb{R}^n)$ is $M_{\phi,1}(\mathbb{R}^n)$.*

(ii) If $1 < p < \infty$ and ϕ satisfies (1.2), (1.3) and (1.4), then the dual of $H^{\phi,p}(\mathbb{R}^n)$ is $M_{\phi,p'}(\mathbb{R}^n)$.

Proof. It is easy to see that $M_{\phi,p'}(\mathbb{R}^n) \subseteq (H^{\phi,p}(\mathbb{R}^n))'$. Hence it is enough to prove the reverse inclusion $(H^{\phi,p}(\mathbb{R}^n))' \subseteq M_{\phi,p'}(\mathbb{R}^n)$.

If $p' = 1$ and $f \notin M_{\phi,1}(\mathbb{R}^n)$, then for any positive integer u, there exists a cube Q_u such that

$$\int_{Q_u} |f(x) - f_{Q_u}|^p dx \geq u|Q_u|\phi^p(|Q_u|^{\frac{1}{n}}).$$

Denote

$$\begin{cases} f_u(x) = (f(x) - f_{Q_u})\chi_{Q_u}(x), \\ E_{u,+} = \Big\{x \in Q_u : f - f_{Q_u} > 0\Big\}, \\ E_{u,-} = \Big\{x \in Q_u : f - f_{Q_u} < 0\Big\}. \end{cases}$$

Now we construct a $(\phi,\infty)-$atom g_u on cube Q_u. Since $\int_{Q_u}(f - f_{Q_u})dx = 0$, we have $|E_{u,+}| > |E_{u,-}|$ or $|E_{u,-}| \geq |E_{u,+}|$. Without loss of generality, we assume that $|E_{u,+}| > |E_{u,-}|$. We decompose $E_{u,+}$ into two sets by choosing a measurable subset $F_u \subset E_{u,+}$ such that $|F_u| = |E_{u,-}|$. Then we define a $(\phi,\infty)-$atom by

$$g_u = |Q_u|^{-1}\phi^{-1}(|Q_u|^{\frac{1}{n}})(\chi_{F_u} - \chi_{E_{u,-}}).$$

We have $\int f_u g_u dx \geq \frac{u}{2}$. Let $u \to \infty$. We know that $f \notin (H^{\phi,\infty}(\mathbb{R}^n))'$, that is, $(H^{\phi,\infty}(\mathbb{R}^n))' \subset M_{\phi,1}(\mathbb{R}^n)$.

Now we consider the case $1 < p < \infty$. If $f \notin M_{\phi,p'}(\mathbb{R}^n)$, by Theorem 3.1, for any $u > 0$, there exists a cube Q_u such that $|Q_u| \leq 1$ and

$$\int_{Q_u} \left(\sum_{Q_{j,k} \subset Q_u} 2^{jn} |f^{\epsilon}_{j,k}|^2 \chi(2^j x - k) \right)^{\frac{p'}{2}} dx \geq u^{p'} |Q_u| \phi^{p'}(|Q_u|^{\frac{1}{n}}).$$

Let $f_u(x) = \sum\limits_{Q_{j,k} \subset Q_u} f^{\epsilon}_{j,k} \Phi^{\epsilon}_{j,k}(x)$. We can choose a $(\phi, p)-$ wavelet atom g_u on Q_u such that $\int f_u(x) g_u(x) dx \geq \frac{1}{8} u$. Because $\{\Phi^{\epsilon}_{j,k}\}$ are a regular Daubechies' wavelet basis,

$$\int f(x) g_u(x) dx = \int f_u(x) g_u(x) dx \geq \frac{1}{8} u.$$

This implies that $f \notin (H^{\phi,p}(\mathbb{R}^n))'$. ■

3.3. Calderón-Zygmund operators on $H^{\phi,p}(\mathbb{R}^n)$

Alvarez [1] proved the Calderón-Zygmund operators are bounded on the predual of $M_{\alpha,p}(\mathbb{R}^n)$. In this section, we consider the boundedness of Calderón-Zygmund operators on $H^{\phi,p}(\mathbb{R}^n)$.

Theorem 3.9. *Suppose that ϕ satisfies (1.2), (1.3) and (1.4). If $1 < p < \infty$ or ϕ is bounded and $p = \infty$, Calderón-Zygmund operators are bounded on $H^{\phi,p}(\mathbb{R}^n)$.*

Proof. By Theorem 3.7, we only consider the case $1 < p < \infty$. We know that if $f \in H^{\phi,p}(\mathbb{R}^n)$, then

$$f(x) = \sum_m \lambda_m a_{j_m,k_m}(x) + \sum_{k \in \mathbb{Z}^n} \rho_k \Phi^0(x - k),$$

where $\{\lambda_m\}, \{\rho_k\} \in l^1$ and $a_m(x) = \sum\limits_{\epsilon \neq 0, Q_{j,k} \subset Q_{j_m,k_m}} a^{\epsilon}_{j,k} \Phi^{\epsilon}_{j,k}(x)$ are $(\phi, p)-$wavelet atoms on dyadic cube Q_{j_m,k_m}. It follows that

$$T\Phi^0(x - k) = \sum_{k' \in \mathbb{Z}^n} a^{0,0}_{0,k',0,k} \Phi^0(x - k') + \sum_{l \in \mathbb{Z}^n} \sum_{\epsilon' \neq 0, Q_{j',k'} \subset Q_{0,l+k}} a^{\epsilon',0}_{j',k',0,k} \Phi^{\epsilon'}_{j',k'}(x).$$

Similar to the proof of Lemma 2.2, we have

$$\Big\| \sum_{\epsilon' \neq 0, Q_{j',k'} \subset Q_{0,l+k}} a^{\epsilon',0}_{j',k',0,k} \Phi^{\epsilon'}_{j',k'} \Big\|_{H^{\phi,p}} \leq C(1 + |l|)^{-N}.$$

Hence $\|T\Phi^0(x - k)\|_{H^{\phi,p}} \leq C$.

We write

$$Ta_m(x) = \sum_{l\in\mathbb{Z}^n} a_{l,j_m,k_m}(x) + \sum_{l\in\mathbb{Z}^n}\sum_{0\le j<j_m} a^{\epsilon,j,l}_{j_m,k_m}\Phi^{\epsilon'}_{j',k'}(x),$$

where $\{a_{l,j_m,k_m}(x)\}$ and $\{a^{\epsilon,j,l}_{j_m,k_m}\}$ are defined in (2.7) and (2.7), respectively. By Lemma 2.3,

$$\begin{aligned}\|Ta_m\|_{H^{\phi,p}} &\le C\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\\ &\quad + C\sum_{l\in\mathbb{Z}^n}\sum_{0\le j<j_m}(1+|l|)^{-n-N}2^{N(j-j_m)}\phi(2^{-j})\phi^{-1}(2^{-j_m}).\end{aligned}$$

Applying (1.3), we get

$$\|Ta_m\|_{H^{\phi,p}} \le C\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N} + C\sum_{l\in\mathbb{Z}^n}\sum_{0\le j<j_m}(1+|l|)^{-n-N}2^{N(j-j_m)}C_0^{j_m-j} \le C,$$

which gives $Tf\in H^{\phi,p}(\mathbb{R}^n)$. This completes the proof of Theorem 3.9. ∎

4. Stability of Morrey Spaces

In this section, we discuss stabilities of $M_{\phi,p}(\mathbb{R}^n)$, $H^{\phi,p}(\mathbb{R}^n)$ and $M(M_{\phi,p}(\mathbb{R}^n))$ under the perturbation of Calderón-Zygmund operators.

4.1. Multiplier on $H^{\phi,p}(\mathbb{R}^n)$

In this section, we consider multipliers on $H^{\phi,p}(\mathbb{R}^n)$. Let f and g be two functions. Before considering the multiplier spaces $M(H^{\phi,p}(\mathbb{R}^n))$, we first give three useful lemmas.

Lemma 4.1. (i) *If $1<p\le\infty$, then there exists a positive constant C depending on ϕ and M such that*

$$C^{-1}\sum_{j=1}^{j_0}\phi(2^{-j}) \le \|2^{nj_0}\Phi^0(2^{j_0}x)\|_{H^{\phi,p}} \le C\sum_{j=1}^{j_0}\phi(2^{-j}).$$

(ii) *There exists a positive constant C depending on ϕ such that*

$$C\sum_{j=1}^{j_0}\phi(2^{-j}) \le \|2^{nj_0}\chi(2^{j_0}x)\|_{H^{\phi,\infty}} \le C^{-1}\sum_{j=1}^{j_0}\phi(2^{-j}).$$

Proof. We only prove (i). The proof of (ii) is similar. Since

$$2^{nj_0}\Phi^0(2^{j_0}x) = \Big\{2^{nj_0}\Phi^0(2^{j_0}x) - 2^{n(j_0-1)}\Phi^0(2^{(j_0-1)}x)\Big\} \\ \cdots + \Big\{2\Phi^0(2x) - \Phi^0(x)\Big\} + \Phi^0(x),$$

we have $\|2^{nj_0}\Phi^0(2^{j_0}x)\|_{H^{\phi,p}} \leq C\sum_{j=1}^{j_0}\phi(2^{-j})$.

To get the another inequality, we construct a special non-negative function f. Taking the smallest positive number $r_0 \sim 2^{-j_0}$ such that $\mathrm{supp}\Phi^0(2^{j_0}x) \subset B(0,r_0)$, we know that $r_0 \sim 2^{-j_0}$. Let g be a linear non-increasing Lipschitz function on $[0,\infty)$ such that

(1) if $x \in [0, r_0]$, then $g(x) = \sum_{j=1}^{j_0}\phi(2^{-j})$;

(2) if $x \in [\frac{1}{2},\infty)$, then $g(x) = 0$;

(3) if $x \in [r_0, \frac{1}{4}]$, then $g(x) \sim \sum_{j=1}^{[-\log_2 x]}\phi(2^{-j})$.

Let $f(x) = g(|x|)$. Then

$$\sup_{y\in\mathbb{R}^n, \frac{1}{2}\leq r\leq 1}|f_{Q(y,r)}| \leq C\sum_{s=1}^{j_0}2^{-ns}\sum_{j=1}^{s}\phi(2^{-j}).$$

Since $g(|x|)$ is a linear non-increasing Lipschitz function, by a similar procedure, we have

$$\sup_{y\in\mathbb{R}^n, r\leq\frac{1}{2}}\phi(r)^{-1}(r^{-n}\int_{Q(y,r)}|f(x)-f_{Q(y,r)}|^{p'}dx)^{\frac{1}{p'}} \leq C.$$

This gives $f \in M_{\phi,p'}(\mathbb{R}^n)$. Further,

$$\int f(x)2^{nj_0}\Phi^0(2^{j_0}x)dx \geq C\sum_{j=1}^{j_0}\phi(2^{-j}). \qquad \blacksquare$$

Lemma 4.2. *For any* $j \geq 0, P_jf \in L^\infty(\mathbb{R}^n)$ *if and only if* $f \in L^\infty(\mathbb{R}^n)$.

Proof. For any $j \geq 0$, it is obvious that $f \in L^\infty(\mathbb{R}^n)$ implies $P_jf \in L^\infty(\mathbb{R}^n)$. Further, if $P_jf \in L^\infty(\mathbb{R}^n), j \geq 0$, then $f \in B^{0,\infty}_\infty(\mathbb{R}^n)$. Let $\tilde{f}^0_{j,k} = \langle f, 2^{\frac{nj}{2}}\chi(2^jx-k)\rangle$ and $\tilde{P}_jf = \sum_k \tilde{f}^0_{j,k}2^{\frac{nj}{2}}\chi(2^jx-k)$. Since $f \in B^{0,\infty}_\infty(\mathbb{R}^n)$, we know that

$$|\langle f, \Phi^0_{j,k} - 2^{\frac{nj}{2}}\chi(2^jx-k)\rangle| \leq C2^{-\frac{nj}{2}}.$$

Hence for any $j \geq 0$, $P_j f \in L^\infty(\mathbb{R}^n)$ implies $\tilde{P}_j f \in L^\infty(\mathbb{R}^n)$. Now we prove that if $j \geq 0$, $\tilde{P}_j f \in L^\infty(\mathbb{R}^n), j \geq 0$, then $f \in L^\infty(\mathbb{R}^n)$.

For $\|g(x)\|_{L^1} \leq 1$ and $s, s' \in \mathbb{N}$, there exists $g_{s,s'}(x) = \sum\limits_Q g_Q \chi_Q(x)$, where $\{Q\}$ are dyadic cubes whose interiors are mutually disjoint, $|Q| \geq 2^{-s'n}$ and $\|g(x) - g_{s,s'}(x)\|_{L^1} \leq 2^{-s}$. Hence

$$\Big|\int f(x) g_{s,s'}(x)dx\Big| = \Big|\int \tilde{P}_{s'} f(x) g_{s,s'}(x)dx\Big| \leq C.$$

We can see that $|\int f(x)g(x)dx| \leq C$ and $f \in L^\infty$. ■

Assume that ϕ satisfies (1.2) and (1.3). We define

$$\psi(r) = \frac{\phi(r)}{\Big\{\sum\limits_{j=1}^{1+[|\log_2 r|]} \phi(2^{-j})\Big\}}. \tag{4.1}$$

The following result can be obtained immediately.

Lemma 4.3. *Let ψ be the function defined in (4.1). $M_{\psi,1}(\mathbb{R}^n) \subset bmo(\mathbb{R}^n)$.*

Proof. If $f \in M_{\psi,1}(\mathbb{R}^n)$, then

$$\sup_{\frac{1}{2}\leq r\leq 1} |f_{Q_r}| + \sup_{r\leq\frac{1}{2}} \psi(r)^{-1}\Big(r^{-n}\int_{Q_r} |f(x) - f_{Q_r}|^p dx\Big)^{\frac{1}{p}} < \infty.$$

According to the definition of ψ, we know that ψ is bounded. Hence

$$\sup_{\frac{1}{2}\leq r\leq 1} |f_{Q_r}| + \sup_{r\leq\frac{1}{2}} \Big\{r^{-n}\int_{Q_r} |f(x) - f_{Q_r}|^p dx\Big\}^{\frac{1}{p}} < \infty.$$

It follows that $f \in bmo(\mathbb{R}^n)$. ■

Now we give the characterization of the multiplier spaces $M(H^{\phi,p})$.

Theorem 4.4. *Suppose that ϕ satisfies (1.2) and (1.3). If ϕ also satisfies (1.4) for $1 < p < \infty$, we have*

(i) *If $\sum\limits_{j=1}^{\infty} \psi(2^{-j}) \leq C_\psi$, then $M(H^{\phi,p}(\mathbb{R}^n)) = M_{\psi,1}(\mathbb{R}^n)$;*

(ii) *If $\psi \geq C_\psi > 0$, then $M(H^{\phi,p}(\mathbb{R}^n)) = L^\infty(\mathbb{R}^n)$;*

(iii) *If $\varliminf\limits_{r\to 0} \psi(r) = 0$ and $\sum\limits_{j=1}^{\infty} \psi(2^{-j}) = \infty$, then*

$$M(H^{\phi,p}(\mathbb{R}^n)) = M_{\psi,1}(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n).$$

Proof. We divide the proof of this theorem into eight steps. In the first six steps, the constant C depends on $\|f\|_{M_{\psi,1}}$, $\|f\|_{L^\infty}$, C_ψ, the constants in (1.2), (1.3), (1.4) and M. In the proof of Steps 7 and 8, the constant C depends on M, $\|f\|_{M(H^{\phi,p})}$, the constant in (1.4) (if $1 < p < \infty$), the constants in (1.2) and (1.3).

Step 1. We first prove that if $\sum\limits_{j=1}^{\infty} \psi(2^{-j}) \leq C_\psi$, $1 < p < \infty$ and $f \in M_{\psi,1}(\mathbb{R}^n)$, then $f \in M(H^{\phi,p}(\mathbb{R}^n))$. By Theorem 3.8 and Lemma 4.1, $\|P_j f\|_{L^\infty} \leq C \sum\limits_{s=1}^{\infty} \psi(2^{-s})$. Hence, if $f \in M_{\psi,1}(\mathbb{R}^n)$ and $\sum\limits_{j=1}^{\infty} \psi(2^{-j}) \leq C_\psi$, then $P_j f \in L^\infty(\mathbb{R}^n), j \geq 0$. We have $f \in L^\infty(\mathbb{R}^n)$. For each (ϕ, p)-wavelet atom a on dyadic cube Q_{j_0,k_0}, we have

$$\|fa\|_{L^p} \leq C 2^{(n-\frac{n}{p})j_0} \phi^{-1}(2^{-j_0}).$$

Let

$$\begin{cases} A(x) = \sum\limits_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} f^\epsilon_{j,k} a^\epsilon_{j,k} 2^{jn} \chi(2^j x - k), \\ B(x) = \Big\{ \sum\limits_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} f^\epsilon_{j,k} a^\epsilon_{j,k} \Big\} 2^{j_0 n} \Phi^0(2^{j_0} x - k_0). \end{cases}$$

We know that

$$\begin{aligned} &\Big| \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} f^\epsilon_{j,k} a^\epsilon_{j,k} \Big| \leq \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} \int |f^\epsilon_{j,k} a^\epsilon_{j,k} 2^{jn} \chi(2^j x - k) dx| \\ &\leq \int \Big\{ \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} 2^{jn} |f^\epsilon_{j,k}|^2 \chi(2^j x - k) \Big\}^{\frac{1}{2}} \Big\{ \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} 2^{jn} |a^\epsilon_{j,k}|^2 \chi(2^j x - k) \Big\}^{\frac{1}{2}} dx \\ &\leq \Big\| \Big\{ \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} 2^{jn} |f^\epsilon_{j,k}|^2 \chi(2^j x - k) \Big\}^{\frac{1}{2}} \Big\|_{L^{p'}} \Big\| \Big\{ \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} 2^{jn} |a^\epsilon_{j,k}|^2 \chi(2^j x - k) \Big\}^{\frac{1}{2}} \Big\|_{L^p} \end{aligned}$$

and

$$\begin{aligned} &\Big| \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} f^\epsilon_{j,k} a^\epsilon_{j,k} 2^{jn} \chi(2^j x - k) \Big| \\ &\leq \Big\{ \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} 2^{jn} |f^\epsilon_{j,k}|^2 \chi(2^j x - k) \Big\}^{\frac{1}{2}} \Big\{ \sum_{\epsilon, Q_{j,k} \subset Q_{j_0,k_0}} 2^{jn} |a^\epsilon_{j,k}|^2 \chi(2^j x - k) \Big\}^{\frac{1}{2}}. \end{aligned}$$

Hence, if $f \in M_{\psi,1}(\mathbb{R}^n)$, by Lemma 4.3, $f \in BMO(\mathbb{R}^n)$. So

$$\|A\|_{L^p} + \|B\|_{L^p} \leq C 2^{(n-\frac{n}{p})j_0} \phi^{-1}(2^{-j_0}).$$

Now we prove that $fa \in H^{\phi,p}(\mathbb{R}^n)$. By multi-resolution analysis, for $j_0 \in \mathbb{N}$, the product $f \cdot g$ can be decomposed into the following parts.

$$\begin{aligned} f(x)g(x) = \; & P_{j_0}f(x)P_{j_0}g(x) + \sum_{j\geq j_0} P_jf(x)Q_jg(x) \\ & + \sum_{j\geq j_0} Q_jf(x)P_jg(x) + \sum_{j\geq j_0} Q_jf(x)Q_jg(x). \end{aligned} \tag{4.2}$$

According to (4.2), we can write

$$\begin{aligned} f(x)a(x) = \; & P_{j_0}f(x)P_{j_0}a(x) + \sum_{j\geq j_0} P_jfQ_ja + \sum_{j\geq j_0} Q_jfP_ja \\ & + \sum_{j\geq j_0,(\epsilon,k)\neq(\epsilon',l)} f^\epsilon_{j,k}a^{\epsilon'}_{j,l}\Phi^\epsilon_{j,k}(x)\Phi^{\epsilon'}_{j,l}(x) \\ & + \sum_{\epsilon,Q_{j,k}\subset Q_{j_0,k_0}} f^\epsilon_{j,k}a^\epsilon_{j,k}[(\Phi^\epsilon_{j,k}(x))^2 - 2^{jn}\chi(2^jx-k)] \\ & + (A(x)-B(x)) + B(x) \\ \equiv \; & \sum_{i=1}^{7} M_i(x). \end{aligned} \tag{4.3}$$

It is easy to see that the supports of $M_i(x), i = 1, 2, \ldots, 7$, are contained in the multiple cube $\widetilde{Q_{j_0,k_0}}$ of dyadic cube Q_{j_0,k_0}. If $j_0 > 0$, then $P_{j_0}a(x) = 0$ and $M_1(x) = 0$. If $j_0 = 0$, then $\|P_0f(x)P_0a(x)\|_{L^p} \leq C$ and $M_1(x) \in H^{\phi,p}(\mathbb{R}^n)$. By Lemma 4.1, we have

$$\|2^{j_0n}\Phi^0(2^{j_0}x-k_0)\|_{H^{\phi,p}} \leq C\sum_{j=1}^{j_0}\phi(2^{-j}).$$

Hence, $M_7(x) \in H^{\phi,p}(\mathbb{R}^n)$.

For any $\epsilon, \epsilon' \neq 0, k, k' \in \mathbb{Z}^n$, by wavelet property,

$$\begin{cases} \int \Phi^0_{j,k}(x)\Phi^\epsilon_{j,k'}(x)dx = 0, \\ \int \Phi^\epsilon_{j,k}(x)\Phi^{\epsilon'}_{j,k'}(x)dx = \delta_{\epsilon,\epsilon'}\delta_{k.k'}. \end{cases}$$

So we have

$$\int M_2(x)dx = \int M_3(x)dx = \int M_4(x)dx = 0.$$

Further

$$\int (\Phi^\epsilon_{j,k}(x))^2dx = \int 2^{jn}\chi(2^jx-k)dx = \int 2^{jn}\Phi^0(2^jx-k)dx = 1,$$

which implies $\int M_5(x)dx = \int M_6(x)dx = 0$.

(1) According to the property of Daubechies wavelets, we have

$$\|M_2\|_{L^p}^p \leq C \int \Big(\sum_j |P_j f(x) Q_j a(x)|^2 \Big)^{\frac{p}{2}} dx.$$

For any $j \geq 0, P_j f \in L^\infty(\mathbb{R}^n)$, hence $\|M_2(x)\|_{L^p} \leq C2^{(n-\frac{n}{p})j_0}\phi^{-1}(2^{-j_0})$.

(2) Similarly, we have

$$\begin{cases} \|M_4(x)\|_{L^p} \leq C2^{(n-\frac{n}{p})j_0}\phi^{-1}(2^{-j_0}), \\ \|M_5(x)\|_{L^p} \leq C2^{(n-\frac{n}{p})j_0}\phi^{-1}(2^{-j_0}). \end{cases}$$

(3) $\|M_6(x)\|_{L^p} \leq \|A(x)\|_{L^p} + \|B(x)\|_{L^p} \leq C2^{(n-\frac{n}{p})j_0}\phi^{-1}(2^{-j_0})$.

(4) $M_3(x) = f(x)a(x) - M_1(x) - M_2(x) - M_6(x) - M_5(x) - A(x)$. By the above estimates, we get $\|M_3(x)\|_{L^p} \leq C2^{(n-\frac{n}{p})j_0}\phi^{-1}(2^{-j_0})$.

That is to say, all terms $M_i(x), i = 2, 3, \ldots, 6$, are bounded in $H^{\phi,p}(\mathbb{R}^n)$.

Step 2. We prove that if $\psi \geq C_\psi > 0$ and $f \in L^\infty(\mathbb{R}^n)$, then $f \in M(H^{\phi,p}(\mathbb{R}^n))$ for $1 < p < \infty$. For each $(\phi, p)-$atom a on cube $Q = Q(y,r)$, the support of $f(x)a(x)$ belongs to $Q(y,r)$. If $f \in L^\infty$, then $\|f(x)a(x)\|_{L^p} \leq Cr^{\frac{n}{p}-n}\phi^{-1}(r)$ and

$$\begin{aligned} |(fa)_Q| &= |Q(y,r)|^{-1} | \int_{Q(y,r)} f(x)a(x)dx| \\ &\leq |Q(y,r)|^{-1} \int_{Q(y,r)} |a(x)|dx \\ &\leq |Q(y,r)|^{-\frac{1}{p}} \|a(x)\|_{L^p} \leq Cr^{-n}\phi^{-1}(r). \end{aligned}$$

Hence

$$\begin{aligned} \|f(x)a(x) - (fa)_Q\|_{L^p(Q(y,r))} &\leq \|f(x)a(x) - (fa)_Q\|_{L^p(Q(y,r))} \\ &\quad + \|(fa)_Q \chi_{Q(y,r)}(x)\|_{L^p(Q(y,r))} \\ &\leq Cr^{\frac{n}{p}-n}\phi^{-1}(r), \end{aligned}$$

that is, $\|f(x)a(x) - (fa)_Q\chi_Q(x)\|_{H^{\phi,p}} \leq C$. Furthermore, if $\psi \geq C_\psi > 0$, then $\|(fa)_Q\chi_Q(x)\|_{H^{\phi,p}} \leq C$. Thus, $\|fa\|_{H^{\phi,p}} \leq C$.

Step 3. We prove that if $f \in M_{\psi,1}(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$, then $f \in M(H^{\phi,p}(\mathbb{R}^n))$, $1 < p < \infty$. For each $(\phi, p)-$atom a on cube $Q = Q(y,r)$, the support of fa belongs to $Q(y,r)$. If $f\in L^{\infty}(\mathbb{R}^n)$, then $\|fa\|_{L^p}\leq Cr^{\frac{n}{p}-n}\phi^{-1}(r)$ and

$$\begin{aligned}(fa)_Q &= |Q(y,r)|^{-1}\int_{Q(y,r)} f(x)a(x)dx\\ &\leq |Q(y,r)|^{-1}\int_{Q(y,r)}|a(x)|dx\\ &\leq |Q(y,r)|^{-\frac{1}{p}}\|a(x)\|_{L^p}\leq Cr^{-n}\phi^{-1}(r).\end{aligned}$$

For each cube $Q=Q(y,r)$, there exists a biggest dyadic cube $Q_{j,k}$ such that $\text{supp}\Phi^0(2^jx-k)\subset Q(y,r)$. Hence

$$\begin{aligned}&\|f(x)a(x)-(fa)_Q 2^{nj}\Phi^0(2^jx-k)\|_{L^p(Q(y,r))}\\ \leq&\ \|f(x)a(x)-(fa)_Q\|_{L^p(Q(y,r))}+\|(fa)_Q\chi_{Q(y,r)}(x)\|_{L^p(Q(y,r))}\\ \leq&\ Cr^{\frac{n}{p}-n}\phi^{-1}(r).\end{aligned}$$

Furthermore, if $f\in M_{\psi,1}$, then $\|(fa)_Q 2^{nj}\Phi^0(2^jx-k)\|_{H^{\phi,p}}\leq C$.

Step 4. We prove that, if $\sum\limits_{j=1}^{\infty}\psi(2^{-j})\leq C_\psi$ and $f\in M_{\psi,1}(\mathbb{R}^n)$, then $f\in M(H^{\phi,\infty}(\mathbb{R}^n))$. By Theorem 3.8 and Lemma 4.1, $\|P_jf\|_{L^\infty}\leq C\sum\limits_{s=1}^{\infty}\psi(2^{-s})$. Hence, if $f\in M_{\psi,1}$, and $\sum\limits_{j=1}^{\infty}\psi(2^{-j})\leq C_\psi$, then $P_jf\in L^\infty,\forall j\geq 0$. And $f\in L^\infty$ by Lemma 4.2, . For each $(\phi,\infty)-$ atom $a(x)$ on cube $Q=Q(y,r)$, $\|fa\|_{L^\infty}\leq r^{-n}\phi^{-1}(r)$ and

$$\begin{aligned}|(fa)_Q| &= |Q(y,r)|^{-1}|\int_{Q(y,r)}(f(x)-f_Q)a(x)dx|\\ &\leq Cr^n\psi(r)r^{-n}\phi^{-1}(r)\\ &\leq C(\sum_{j=1}^{[|\log_2 r|]}\phi(2^{-j}))^{-1}.\end{aligned}$$

By Lemma 4.1, we have

$$\|(fa)_Q\chi_Q(x)\|_{H^{\phi,\infty}}\leq C\Big(\sum_{j=1}^{[|\log_2 r|]}\phi(2^{-j})\Big)^{-1}\sum_{j=1}^{[|\log_2 r|]}\phi(2^{-j})\leq C.$$

Step 5. We prove that, if $\psi\geq C_\psi>0$ and $f\in L^\infty(\mathbb{R}^n)$, then $f\in M(H^{\phi,\infty}(\mathbb{R}^n))$. For each $(\phi,\infty)-$ atom $a(x)$ on cube $Q=Q(y,r)$, the support of $f(x)a(x)$ belongs to $Q(y,r)$. If $f \in L^\infty(\mathbb{R}^n)$, then $\|f(x)a(x)\|_{L^\infty} \le Cr^{-n}\phi^{-1}(r)$ and

$$(fa)_Q = |Q(y,r)|^{-1}\int_{Q(y,r)} f(x)a(x)dx \le |Q(y,r)|^{-1}\int_{Q(y,r)} |a(x)|dx$$
$$\le \|a(x)\|_{L^\infty} \le Cr^{-n}\phi^{-1}(r).$$

Hence

$$\begin{aligned}&\|f(x)a(x) - (fa)_Q\|_{L^\infty(Q(y,r))}\\ \le\ &\|f(x)a(x) - (fa)_Q\|_{L^\infty(Q(y,r))} + \|(fa)_Q\chi_{Q(y,r)}(x)\|_{L^\infty(Q(y,r))}\\ \le\ &Cr^{-n}\phi^{-1}(r).\end{aligned}$$

Step 6. We prove that, if $f \in M_{\psi,1}(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$, then $f \in M(H^{\phi,\infty}(\mathbb{R}^n))$. For each $(\phi,\infty)-$atom a on cube $Q = Q(y,r)$, the support of fa belongs to $Q(y,r)$. If $f \in L^\infty(\mathbb{R}^n)$, then $\|fa\|_{L^\infty} \le Cr^{-n}\phi^{-1}(r)$ and

$$(fa)_Q = |Q(y,r)|^{-1}\int_{Q(y,r)} f(x)a(x)dx \le |Q(y,r)|^{-1}\int_{Q(y,r)} |a(x)|dx$$
$$\le \|a\|_{L^\infty} \le Cr^{-n}\phi^{-1}(r).$$

For each cube $Q = Q(y,r)$, there exists a biggest dyadic cube $Q_{j,k}$ such that supp $\Phi^0(2^jx-k) \subset Q(y,r)$. Hence

$$\begin{aligned}&\|f(x)a(x) - (fa)_Q 2^{nj}\Phi^0(2^jx-k)\|_{L^\infty(Q(y,r))}\\ \le\ &\|f(x)a(x) - (fa)_Q\|_{L^\infty(Q(y,r))} + \|(fa)_Q\chi_{Q(y,r)}(x)\|_{L^\infty(Q(y,r))}\\ \le\ &Cr^{-n}\phi^{-1}(r).\end{aligned}$$

Furthermore, if $f \in M_{\psi,1}(\mathbb{R}^n)$, then $\|(fa)_Q 2^{nj}\Phi^0(2^jx-k)\|_{H^{\phi,\infty}} \le C$.

Step 7. We prove that, if $1 < p \le \infty$ and $f \in M(H^{\phi,p}(\mathbb{R}^n))$, then $f \in L^\infty(\mathbb{R}^n)$. In fact, we choose $a(x) = \Phi^\epsilon_{j,k}(x)$ and $b(x) = \tau_{j,k}(x) - \tau_{j,k+2^{M+4}}(x)$, where $k + 2^{M+4} = (k_1 + 2^{M+4}, \cdots, k_n + 2^{M+4})$, $\tau(x) \in C_0^2(B(0,2^{M+3}))$ and $\tau(x) = 1$ if $x \in \bigcup_{\epsilon\in\{0,1\}^n} \mathrm{supp}\Phi^\epsilon$. Then $\|a\|_{H^{\phi,p}} \le \phi(2^{-j})$, $\|a\|_{M^{\phi,p'}} \le \phi^{-1}(2^{-j})$ and $\|b\|_{M^{\phi,p'}} \le \phi^{-1}(2^{-j})$. Furthermore, $|\int faadx| \le C$ and

$$\int f2^{jn}\Phi^0(2^jx-k)dx = \int faadx - \int f(aa - 2^{jn}\Phi^0(2^jx-k))dx.$$

So we have $P_jf \in L^\infty(\mathbb{R}^n),\ j \ge 0$. It follows that $f \in L^\infty(\mathbb{R}^n)$.

Step 8. We prove that if $f \in M(H^{\phi,p}(\mathbb{R}^n)), 1 < p \leq \infty$, then $f \in M_{\psi,1}(\mathbb{R}^n)$. For each cube $Q = Q(y,r)$, denote

$$a(x) = \frac{\overline{f(x) - f_Q}}{|f(x) - f_Q|}\chi_Q(x) r^{-n}\phi^{-1}(r).$$

Then $\|a\|_{H^{\phi,p}} \leq 1$. Let $j_Q = 1 + [-\log_2 r]$ and let b be a function such that $b(x) = \sum_{j=1}^{j_Q} \phi(2^{-j})$ for $x \in Q(y,r)$, $b(x) = 0$ for $x \notin Q(y, 2^{j_Q} r)$ and

$$b(x) = \sum_{j=1}^{j_Q - s} \phi(2^{-j}) + (s - \log_2 t)\phi(2^{s-j_Q})$$

for $0 \leq s \leq j_Q - 1, 2^s \leq t \leq 2^{s+1}$ and $x \in \partial Q(y, tr)$. Then according to (1.3), $\|b\|_{M_{\phi,p'}} \leq C$ and $b(x) \geq \sum_{1 \leq j \leq j_Q} \phi(2^{-j})$ on cube $Q(y,r)$. So $|\int f(x)a(x)b(x)dx| \leq C$. According to Step 7, $f_Q \in l^\infty$ and we can get $|\int f_Q a(x)b(x)dx| \leq C$. Then

$$\Big|\int (f(x) - f_Q)a(x)b(x)dx\Big| \leq \Big|\int f(x)a(x)b(x)dx\Big| + \Big|\int f_Q a(x)b(x)dx\Big| \leq C.$$

It follows that

$$\int_Q |f(x) - f_Q| dx \sum_{1 \leq j \leq j_Q} \phi(2^{-j}) r^{-n}\phi^{-1}(r) \leq C.$$

Hence

$$\int_Q |f(x) - f_Q| dx \leq C r^n \Big(\sum_{1 \leq j \leq j_Q} \phi(2^{-j})\Big)^{-1}\phi(r).$$

We get $f \in M_{\psi,1}(\mathbb{R}^n)$. ∎

4.2. Stability of Morrey spaces

In this section, we establish a stability condition for $M_{\phi,p}(\mathbb{R}^n)$. Given a function space A and an operator space B. If the facts that $f \in A$ and $T \in B$ always imply $Tf \in A$, one calls the space A is stable under the perturbation of operators in B. We will see that there exist some $M_{\phi,p}(\mathbb{R}^n)$ which are unstable under the perturbation of Calderón-Zygmund operators. Compared with the classical Morrey spaces, $M_{\phi,p}(\mathbb{R}^n)$ have a distinctive characteristic.

For $M_{\phi,p}(\mathbb{R}^n)$, the assumption whether ϕ is bounded makes a great difference. In fact, by Theorems 3.6, 3.7 and 3.8, we have

Theorem 4.5. *Let* $1 \leq p \neq q < \infty$.

(i) *If ϕ is bounded, then $M_{\phi,p}(\mathbb{R}^n) = M_{\phi,q}(\mathbb{R}^n)$.*

(ii) *If ϕ is unbounded, then $M_{\phi,p}(\mathbb{R}^n) \neq M_{\phi,q}(\mathbb{R}^n)$.*

We denote by $B(0,r)$ the ball centered at the origin and with radius r. Let $B(0,\tilde{N})$ be the smallest ball containing supp Φ^0. We choose $\tau \in C_0^\infty(B(0,4\tilde{N}))$ such that $\tau(x) = 1$ if $x \in B(0,2\tilde{N})$. Let Φ be a positive function in $C_0^{n+2}(B(0,2^{n+2}))$ with $\Phi(x) = 1$ in $B(0,2^{n+1})$. Take

$$K(x,y) = \Phi(|x-y|)(x_1 - y_1)|x-y|^{-n-1}.$$

Then $Tf(x) = \int K(x,y)f(y)dy$ is a Calderón-Zygmund operator.

The following theorem tells us that $M_{\phi,1}(\mathbb{R}^n)$ is unstable under the perturbation of Calderón-Zygmund operators if $\phi(r)$ is unbounded.

Theorem 4.6. *If ϕ is unbounded, then there exist a Calderón-Zygmund operator T and a function $f \in M_{\phi,1}(\mathbb{R}^n)$ such that $Tf \notin M_{\phi,1}(\mathbb{R}^n)$.*

Proof. Based on the convergence of $\sum_{j\geq 1} \phi^{-1}(2^{-j})$, we divide the proof into two cases.

Case 1. $\sum_{j\geq 1} \phi^{-1}(2^{-j}) < \infty$. Then $M_{\phi,1}(\mathbb{R}^n) = l^\infty(L^1(k+Q))$. This space is unstable under the perturbation of Calderón-Zygmund operators. In fact, for any $j \geq 2$, let $f_j(x) = 2^{nj}\Phi^0(2^j x)$. We can get $\|f_j\|_{l^\infty(L^1(k+Q))} \leq C$ and

$$\begin{aligned} Tf_j(x) &= \tau(2^{-j}x)Tf_j(x) + (1-\tau(2^{-j}x))\int (K(x,y) - K(x,0))f_j(x) \\ &+ (1-\tau(2^{-j}x))K(x,0). \end{aligned}$$

It is easy to see that the first two terms are bounded in $l^\infty(L^1(k+Q))$ for all $j \geq 2$. Denote by g the third term. We have $\|g\|_{l^\infty(L^1(k+Q))} \geq Cj$.

Case 2. $\sum_{j\geq 1} \phi^{-1}(2^{-j}) = \infty$ and $\phi(2^{-j})$ is unbounded. For any $j \geq 2$, let $f_j(x) = \phi(2^{-j})\Phi^0(2^j x)$. We have $\|f_j\|_{M_{\phi,1}} \leq C$ and

$$\begin{aligned} Tf_j(x) &= \tau(2^{-j}x)Tf_j(x) + (1-\tau(2^{-j}x))\int (K(x,y) - K(x,0))f_j(x) \\ &+ (1-\tau(2^{-j}x))K(x,0)\phi(2^{-j})2^{-jn}. \end{aligned}$$

It is easy to see that the first two terms are bounded on $M_{\phi,1}(\mathbb{R}^n)$ for all $j \geq 2$. Denote the third term by $h(x)$. We have

$$h_{B(0,r)} = r^{-n}\int_{B(0,r)} (1-\tau(2^{-j}x))K(x,0)\phi(2^{-j})2^{-jn}dx = 0, \forall r > 0$$

and

$$I_r = \int_{B(0,r)} |h - h_{B(0,r)}| dx = \int_{B(0,r)} |(1-\tau(2^{-j}x))||K(x,0)|\phi(2^{-j})2^{-jn} dx.$$

If $r \sim 2^{-j}$, then $I_{-j} \sim j\phi(2^{-j})2^{-jn}$. It follows that $\|h\|_{M_{\phi,1}} \geq Cj$. ■

At last, we establish the stability condition.

Theorem 4.7. *Suppose that ϕ satisfies (1.2), (1.3) and (1.4). Let T be a Calderón-Zygmund operator.*

(i) *If ϕ is bounded, then T is bounded on $M_{\phi,\infty}(\mathbb{R}^n)$.*

(ii) *If $1 < p < \infty$, then T is bounded on $M_{\phi,p}(\mathbb{R}^n)$.*

Proof. By the duality between $H^{\phi,p'}(\mathbb{R}^n)$ and $M_{\phi,p}(\mathbb{R}^n)$, it is enough to prove the continuity of Calderón-Zygmund operators on $H^{\phi,p'}(\mathbb{R}^n)$. The desired conclusion follows from Theorem 3.9. ■

4.3. Multipler spaces on Morrey spaces and stability

Janson [6] and Stegenga [23] studied the multipliers on $M_{\phi,1}(\mathbb{R}^n)$. Xiao [29] considered the multipliers on complex Q-spaces. However, the skills of [6] and [23] can not be extended to $p \neq 1$. Also it is very difficult to apply the method of Xiao [29] to deal with $M(M_{\phi,p}(\mathbb{R}^n))$. In this section, we will use wavelets to characterize the multipliers on $H^{\phi,p}(\mathbb{R}^n)$ and $M_{\phi,p}(\mathbb{R}^n)$. Further, we consider the stability of these spaces.

By the theorems in Sections 3.1 and 4.1, we have

Theorem 4.8. *Let ψ be defined by (4.1). Suppose that ϕ satisfies (1.2) and (1.3); and additionally, if $1 < p < \infty$, ϕ also satisfies (1.4). For $1 \leq p < \infty$, we have*

(i) *If $\sum_{j=1}^{\infty} \psi(2^{-j}) \leq C$, then $M(M_{\phi,p}(\mathbb{R}^n)) = M_{\psi,1}(\mathbb{R}^n)$.*

(ii) *If $\psi \geq C > 0$, then $M(M_{\phi,p}(\mathbb{R}^n)) = L^{\infty}(\mathbb{R}^n)$.*

(iii) *If $\underline{\lim}_{r\to 0} \psi(r) = 0$ and $\sum_{j=1}^{\infty} \psi(2^{-j}) = \infty$, then*

$$M(M_{\phi,p}(\mathbb{R}^n)) = M_{\psi,1}(\mathbb{R}^n) \bigcap L^{\infty}(\mathbb{R}^n).$$

Proof. By duality of $H^{\phi,p'}(\mathbb{R}^n)$ and $M_{\phi,p}(\mathbb{R}^n)$ and Theorem 4.4, we get the desired conclusion. ■

We consider the stability of $M(M_{\phi,p}(\mathbb{R}^n))$ under the perturbation of Calderón-Zygmund operators.

Theorem 4.9. *Suppose that $\phi(x)$ satisfies (1.2), (1.3) and (1.4). ψ is defined by (4.1). For $1 \leq p < \infty$, we have*

(i) *If $\sum\limits_{j=1}^{\infty} \psi(2^{-j}) \leq C$, then $M(M_{\phi,p}(\mathbb{R}^n))$ is stable under the perturbation of Calderón-Zygmund operators.*

(ii) *Otherwise, $M(M_{\phi,p}(\mathbb{R}^n))$ is unstable under the perturbation of Calderón-Zygmund operators.*

Proof. By Theorem 5.1 and Theorem 5.4 (i), if $\sum\limits_{j=1}^{\infty} \psi(2^{-j}) \leq C$, then

$$M(M_{\phi,p}(\mathbb{R}^n)) = M_{\psi,q}(\mathbb{R}^n),\ 1 \leq q < \infty.$$

Using wavelet characterization of $M_{\psi,q}(\mathbb{R}^n)$, we know these multiplier spaces are stable under the perturbation of Calderón-Zygmund operators.

By Theorems 5.4 (ii), if $\psi(r) \geq C > 0$, then the related multiplier space is $L^\infty(\mathbb{R}^n)$, so it is unstable under the perturbation of Calderón-Zygmund operators.

We will construct some special multiplier to show that the rest multiplier spaces under these conditions are not stable. Denote $(1) = (1, 0, \cdots, 0)$. We know that $\Phi^{(1)}(0)$ is not zero. There exists a ball $B(0, r_0)$ such that $|\Phi^{(1)}(x)| \geq C > 0, \forall x \in B(0, r_0)$. On the other hand, because the Daubechies wavelets are real valued, we can suppose that $\Phi^{(1)}(x) \geq C > 0$ for all $x \in B(0, r_0)$. By (1.3), if $\tilde{N}$ is large enough, then $\{\psi(2^{-j\tilde{N}})\}$ is a decreasing sequence, $\psi(2^{-\tilde{N}}r) \leq \psi(r), \forall 0 < r < \frac{1}{2}$ and $\sum\limits_{j=1}^{\infty} \psi(2^{-j\tilde{N}}) = \infty$. Let $\tilde{N}$ be the smallest positive integer which satisfies the above requirements and $\mathrm{supp}\Phi^{(1)}(2^{\tilde{N}}x) \subset B(0, r_0)$.

Let $\Phi(x) \in C_0^\infty(B(0,2)), \Phi(x) \geq 0$ and $\Phi(x) = 1$ if $|x| \leq 1$. Let

$$\begin{cases} f(x) = \sum\limits_{j\in\mathbb{N}} (-1)^j \psi(2^{-j\tilde{N}}) \Phi^{(1)}(2^{j\tilde{N}}x), \\ g(x) = \sum\limits_{j\in\mathbb{N}} \psi(2^{-j\tilde{N}}) \Phi(2^{j\tilde{N}}x). \end{cases}$$

Then $f(x) \in M_{\psi,1}(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$, $f(x) \in M(M_{\phi,p}(\mathbb{R}^n))$ and $g(x) \notin L^\infty(\mathbb{R}^n)$. Let T be the operator defined by $Th(x) = \int K(x,y)h(y)dy$, where

$$K(x,y) = \sum_{j\in\mathbb{N}} (-1)^j 2^{nj\tilde{N}} \Phi(2^{j\tilde{N}}x) \Phi^{(1)}(2^{j\tilde{N}}y).$$

Then T is a Calderón-Zygmund operator and $g(x) = Tf(x)$. If $\sum\limits_{j=1}^{\infty} \psi(2^{-j}) = \infty$, then by Theorem 4.8, $M(M_{\phi,p}(\mathbb{R}^n)) \subset L^\infty(\mathbb{R}^n)$. Hence $g(x) \notin M(M_{\phi,p}(\mathbb{R}^n))$. ∎

Acknowledgments

The authors thank sincerely the referee for reading carefully and suggesting some improvements.

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