TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 2, pp. 629-652, April 2015
DOI: 10.11650/tjm.19.2015.4404
This paper is available online at http://journal.taiwanmathsoc.org.tw

ENLARGING THE CONVERGENCE DOMAIN OF SECANT-LIKE METHODS FOR EQUATIONS

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Abstract. We present two new semilocal convergence analyses for secant-like methods in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. These methods include the secant, Newton's method and other popular methods as special cases. The convergence analysis is based on our idea of recurrent functions. Using more precise majorizing sequences than before we obtain weaker convergence criteria. These advantages are obtained because we use more precise estimates for the upper bounds on the norm of the inverse of the linear operators involved than in earlier studies. Numerical examples are given to illustrate the advantages of the new approaches.

1. INTRODUCTION

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and $\mathcal{D}$ be a non-empty, convex and open subset in $\mathcal{X}$. Let $U(x,r)$ and $\overline{U}(x,r)$ stand, respectively, for the open and closed ball in $\mathcal{X}$ with center x and radius $r > 0$. Denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. In the present paper we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet continuously differentiable operator defined on $\mathcal{D}$ with values in $\mathcal{Y}$.

A lot of problems from computational sciences and other disciplines can be brought in the form of equation (1.1) using Mathematical Modelling [8, 10, 14]. The solution of

Received February 6, 2014, accepted September 19, 2014.
Communicated by Wen-Wei Lin.
2010 *Mathematics Subject Classification*: 65H10, 65G99, 47H99, 49M15.
Key words and phrases: Secant-like methods, Newton's method, The secant method, Majorizing sequence, Semilocal convergence, Divided difference operator, Nonlinear equation, Banach space.
The research of the second and third authors has been supported in part by the project MTM2011-28636-C02-01 of the Spanish Ministry of Science and Innovation.

these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [8, 10, 14, 22, 24, 26, 31].

A very important aspect in the study of iterative procedures is the convergence domain. In general the convergence domain is small. This is why it is important to enlarge it without additional hypotheses. Then, this is our goal in this paper.

In the present paper we study the secant-like method defined by

$$\begin{aligned} &x_{-1},\ x_0 \text{ are initial points}\\ &y_n = \lambda\, x_n + (1-\lambda)\, x_{n-1},\ \ \lambda \in [0,1] \qquad (1.2)\\ &x_{n+1} = x_n - B_n^{-1}\, F(x_n), \quad B_n = [y_n, x_n; F] \qquad \text{for each} \ \ n = 0, 1, 2, \cdots . \end{aligned}$$

The family of secant-like methods reduces to the secant method if $\lambda = 0$ and to Newton's method if $\lambda = 1$. It was shown in [26] (see also [7, 8, 21] and the references therein) that the R-order of convergence is at least $(1+\sqrt{5})/2$ if $\lambda \in [0,1)$, the same as that of the secant method. In the real case the closer x_n and y_n are, the higher the speed of convergence. Moreover in [19], it was shown that as λ approaches 1 the speed of convergence is close to that of Newton's method. Moreover, the advantages of using secant-like method instead of Newton's method is that the former method avoids the computation of $F'(x_n)^{-1}$ at each step. The study about convergence matter of iterative procedures is usually centered on two types: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There is a plethora of studies on the weakness and/or extension of the hypothesis made on the underlying operators; see for example [1-33].

The hypotheses used for the semilocal convergence of secant-like method are (see [8, 18, 19, 21]):

($\mathcal{C}_1$) There exists a divided difference of order one denoted by $[x, y; F] \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ satisfying

$$[x, y; F](x - y) = F(x) - F(y) \quad \text{for all} \quad x, y \in \mathcal{D};$$

($\mathcal{C}_2$) There exist x_{-1}, x_0 in $\mathcal{D}$ and $c > 0$ such that

$$\| x_0 - x_{-1} \| \leq c;$$

($\mathcal{C}_3$) There exist $x_{-1}, x_0 \in \mathcal{D}$ and $M > 0$ such that $A_0^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\| A_0^{-1}([x, y; F] - [u, v; F]) \| \leq M\,(\| x - u \| + \| y - v \|) \quad \text{for all} \quad x, y, u, v \in \mathcal{D};$$

($\mathcal{C}_3^\star$) There exist $x_{-1}, x_0 \in \mathcal{D}$ and $L > 0$ such that $A_0^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\| A_0^{-1}([x,y;F] - [v,y;F]) \| \leq L \ \| x - v \| \quad \text{for all} \quad x, y, v \in \mathcal{D};$$

($\mathcal{C}_3^{\star\star}$) There exist $x_{-1}, x_0 \in \mathcal{D}$ and $K > 0$ such that $F(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\| F'(x_0)^{-1} \left([x,y;F] - [v,y;F]\right) \| \leq K \ \| x - v \| \quad \text{for all} \quad x, y, v \in \mathcal{D};$$

($\mathcal{C}_4$) There exists $\eta > 0$ such that

$$\| A_0^{-1} F(x_0) \| \leq \eta;$$

($\mathcal{C}_4^\star$) There exists $\eta > 0$ for each $\lambda \in [0,1]$ such that

$$\| B_0^{-1} F(x_0) \| \leq \eta.$$

We shall refer to **($\mathcal{C}_1$)-($\mathcal{C}_4$)** as the **($\mathcal{C}$)** conditions. From analyzing the semilocal convergence of the simplified secant method, it was shown [18] that the convergence criteria are milder than those of secant-like method given in [20]. Consequently, the decreasing and accessibility regions of (1.2) can be improved. Moreover, the semilocal convergence of (1.2) is guaranteed.

In the present paper we show: an even larger convergence domain can be obtained under the same or weaker sufficient convergence criteria for method (1.2). In view of **($\mathcal{C}_3$)** we have that

($\mathcal{C}_5$) There exists $M_0 > 0$ such that

$$\begin{aligned} &\| A_0^{-1} \left([x,y;F] - [x_{-1},x_0;F]\right) \| \\ &\leq M_0 \left(\| x - x_{-1} \| + \| y - x_0 \|\right) \quad \text{for all} \quad x, y \in \mathcal{D}. \end{aligned}$$

We shall also use the conditions

($\mathcal{C}_6$) There exist $x_0 \in \mathcal{D}$ and $M_1 > 0$ such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\| F'(x_0)^{-1} \left([x,y;F] - F'(x_0)\right) \| \leq M_1 \left(\| x - x_0 \| + \| y - x_0 \|\right) \text{ for all } x, y \in \mathcal{D};$$

($\mathcal{C}_7$) There exist $x_0 \in \mathcal{D}$ and $M_2 > 0$ such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\| F'(x_0)^{-1} \left(F'(x) - F'(x_0)\right) \| \leq M_2 \left(\| x - x_0 \| + \| y - x_0 \|\right) \quad \text{for all } x, y \in \mathcal{D}.$$

Note that $M_0 \le M$, $M_2 \le M_1$, $L \le M$ hold in general and M/M_0, M_1/M_2, M/L can be arbitrarily large [6, 7, 8, 9, 10, 14]. We shall refer to **(**$\mathcal{C}_1$**)**, **(**$\mathcal{C}_2$**)**, **(**$\mathcal{C}_3^{\star\star}$**)**, **(**$\mathcal{C}_4^{\star}$**)**, **(**$\mathcal{C}_6$**)** as the **(**$\mathcal{C}^{\star}$**)** conditions and **(**$\mathcal{C}_1$**)**, **(**$\mathcal{C}_2$**)**, **(**$\mathcal{C}_3^{\star}$**)**, **(**$\mathcal{C}_4^{\star}$**)**, **(**$\mathcal{C}_5$**)** as the **(**$\mathcal{C}^{\star\star}$**)** conditions. Note that **(**$\mathcal{C}_5$**)** is not additional hypothesis to **(**$\mathcal{C}_3$**)**, since in practice the computation of constant M requires that of M_0. Note that if **(**$\mathcal{C}_6$**)** holds, then we can set $M_2 = 2\,M_1$ in **(**$\mathcal{C}_7$**)**.

The paper is organized as follows. In Section 2 we use the **(**$\mathcal{C}^{\star}$**)** and **(**$\mathcal{C}^{\star\star}$**)** conditions instead of the **(**$\mathcal{C}$**)** conditions to provide new semilocal convergence analyses for method (1.2) under weaker sufficient criteria than those given in [18, 19, 21, 25, 26]. This way we obtain a larger convergence domain and a tighter convergence analysis. Two numerical examples, where we illustrate the improvement of the domain of starting points achieved with the new semilocal convergence results, are given in the Section 3.

2. Semilocal Convergence of Secant-like Method

We present the semilocal convergence of secant-like method. First, we need some results on majorizing sequences for secant-like method.

Lemma 2.1. *Let* $c \ge 0$, $\eta > 0$, $M_1 > 0$, $K > 0$ *and* $\lambda \in [0,1]$. *Set* $t_{-1} = 0$, $t_0 = c$ *and* $t_1 = c + \eta$. *Define scalar sequences* $\{q_n\}$, $\{t_n\}$, $\{\alpha_n\}$ *for each* $n = 0, 1, \cdots$ *by*

$$\begin{aligned} q_n &= (1-\lambda)\,(t_n - t_0) + (1+\lambda)\,(t_{n+1} - t_0), \\ t_{n+2} &= t_{n+1} + \frac{K\,(t_{n+1} - t_n + (1-\lambda)\,(t_n - t_{n-1}))}{1 - M_1\,q_n}\,(t_{n+1} - t_n), \end{aligned} \tag{2.1}$$

$$\alpha_n = \frac{K\,(t_{n+1} - t_n + (1-\lambda)\,(t_n - t_{n-1}))}{1 - M_1\,q_n}, \tag{2.2}$$

function $\{f_n\}$ *for each* $n = 1, 2, \cdots$ *by*

$$\begin{aligned} f_n(t) = {} & K\,\eta\,t^n + K\,(1-\lambda)\,\eta\,t^{n-1} + M_1\,\eta\,((1-\lambda)\,(1 + t + \cdots + t^n) + \\ & (1+\lambda)\,(1 + t + \cdots + t^{n+1})) - 1 \end{aligned} \tag{2.3}$$

and polynomial p *by*

$$p(t) = M_1\,(1+\lambda)\,t^3 + (M_1\,(1-\lambda) + K)\,t^2 - K\,\lambda\,t - K\,(1-\lambda). \tag{2.4}$$

Denote by α *the smallest root of polynomial* p *in* $(0,1)$. *Suppose that*

$$0 < \alpha_0 \le \alpha \le 1 - 2\,M_1\,\eta. \tag{2.5}$$

Then, sequence $\{t_n\}$ *is non-decreasing, bounded from above by* $t^{\star\star}$ *defined by*

$$t^{\star\star} = \frac{\eta}{1-\alpha} + c \tag{2.6}$$

and converges to its unique least upper bound $t^\star$ which satisfies

$$c + \eta \le t^\star \le t^{\star\star}. \tag{2.7}$$

Moreover, the following estimates are satisfied for each $n = 0, 1, \cdots$

$$0 \le t_{n+1} - t_n \le \alpha^n \eta \tag{2.8}$$

and

$$t^\star - t_n \le \frac{\alpha^n \eta}{1-\alpha}. \tag{2.9}$$

Proof. We shall first prove that polynomial p has roots in $(0, 1)$. If $\lambda \ne 1$, $p(0) = -(1-\lambda) K < 0$ and $p(1) = 2 M_1 > 0$. If $\lambda = 1$, $p(t) = t\,\overline{p}(t)$, $\overline{p}(0) = -K < 0$ and $\overline{p}(1) = 2 M_1 > 0$. In either case it follows from the intermediate value theorem that there exist roots in $(0, 1)$. Denote by α the minimal root of p in $(0, 1)$. Note that, in particular for Newton's method (i.e. for $\lambda = 1$) and for Secant method (i.e. for $\lambda = 0$), we have, respectively by (2.4) that

$$\alpha = \frac{2K}{K + \sqrt{K^2 + 4 M_1 K}} \tag{2.10}$$

and

$$\alpha = \frac{2K}{K + \sqrt{K^2 + 8 M_1 K}}. \tag{2.11}$$

It follows from (2.1) and (2.2) that estimate (2.8) is satisfied if

$$0 \le \alpha_n \le \alpha. \tag{2.12}$$

Estimate (2.12) is true by (2.5) for $n = 0$. Then, we have by (2.1) that

$$\begin{aligned} t_2 - t_1 \le \alpha\,(t_1 - t_0) &\Longrightarrow t_2 \le t_1 + \alpha\,(t_1 - t_0) \\ &\Longrightarrow t_2 \le \eta + t_0 + \alpha\,\eta = c + (1+\alpha)\,\eta = c + \frac{1-\alpha^2}{1-\alpha\,\eta} < t^{\star\star}. \end{aligned}$$

Suppose that

$$t_{k+1} - t_k \le \alpha^k \eta \quad \text{and} \quad t_{k+1} \le c + \frac{1-\alpha^{k+1}}{1-\alpha}\,\eta. \tag{2.13}$$

Estimate (2.12) shall be true for $k+1$ replacing n if

$$0 \le \alpha_{k+1} \le \alpha \tag{2.14}$$

or

$$f_k(\alpha) \leq 0. \tag{2.15}$$

We need a relationship between two consecutive recurrent functions f_k for each $k = 1, 2, \cdots$. It follows from (2.3) and (2.4) that

$$f_{k+1}(\alpha) = f_k(\alpha) + p(\alpha)\,\alpha^{k-1}\,\eta = f_k(\alpha), \tag{2.16}$$

since $p(\alpha) = 0$. Define function f_∞ on $(0,1)$ by

$$f_\infty(t) = \lim_{n\to\infty} f_n(t). \tag{2.17}$$

Then, we get from (2.3) and (2.17) that

$$\begin{aligned} f_\infty(\alpha) &= \lim_{n\to\infty} f_n(\alpha) \\ &= K\,\eta \lim_{n\to\infty} \alpha^n + K\,(1-\lambda)\,\eta \lim_{n\to\infty} \alpha^{n-1} \\ &\quad + M_1\,\eta \Bigg((1-\lambda) \lim_{n\to\infty} (1+\alpha+\cdots+\alpha^n) \\ &\quad + (1+\lambda) \lim_{n\to\infty} (1+\alpha+\cdots+\alpha^{n+1}) \Bigg) - 1 \\ &= M_1\,\eta \left(\frac{1-\lambda}{1-\alpha} + \frac{1+\lambda}{1-\alpha} \right) - 1 = \frac{2\,M_1\,\eta}{1-\alpha} - 1, \end{aligned} \tag{2.18}$$

since $\alpha \in (0,1)$. In view of (2.15), (2.16) and (2.18) we can show instead of (2.15) that

$$f_\infty(\alpha) \leq 0, \tag{2.19}$$

which is true by (2.5). The induction for (2.8) is complete. It follows that sequence $\{t_n\}$ is non-decreasing, bounded from above by $t^{\star\star}$ given by (2.6) and as such it converges to $t^\star$ which satisfies (2.7). Estimate (2.9) follows from (2.8) by using standard majorization techniques [8, 10, 22]. The proof of Lemma 2.1 is complete. ■

Lemma 2.2. *Let $c \geq 0$, $\eta > 0$, $M_1 > 0$, $K > 0$ and $\lambda \in [0,1]$. Set $r_{-1} = 0$, $r_0 = c$ and $r_1 = c + \eta$. Define scalar sequences $\{r_n\}$ for each $n = 1, \cdots$ by*

$$\begin{aligned} r_2 &= r_1 + \beta_1\,(r_1 - r_0) \\ r_{n+2} &= r_{n+1} + \beta_n\,(r_{n+1} - r_n), \end{aligned} \tag{2.20}$$

where

$$\beta_1 = \frac{M_1\,(r_1 - r_0 + (1-\lambda)\,(r_0 - r_{-1}))}{1 - M_1\,q_1},$$

$$\beta_n = \frac{K\,(r_{n+1} - r_n + (1-\lambda)\,(r_n - r_{n-1}))}{1 - M_1\,q_n} \quad for\ each \quad n = 2, 3, \cdots$$

and function $\{g_n\}$ *on* $[0, 1)$ *for each* $n = 1, 2, \cdots$ *by*

$$\begin{aligned} & g_n(t) \\ & = K\,(t+(1-\lambda))\,t^{n-1}\,(r_2-r_1) \\ & +M_1\,t\left((1-\lambda)\,\frac{1-t^{n+1}}{1-t}+(1+\lambda)\,\frac{1-t^{n+2}}{1-t}\right)(r_2-r_1)+(2\,M_1\,\eta-1)\,t. \end{aligned} \tag{2.21}$$

Suppose that

$$0 \le \beta_1 \le \alpha \le 1 - \frac{2\,M_1\,(r_2 - r_1)}{1 - 2\,M_1\,\eta}, \tag{2.22}$$

where α *is defined in Lemma 2.1. Then, sequence* $\{r_n\}$ *is non-decreasing, bounded from above by* $r^{\star\star}$ *defined by*

$$r^{\star\star} = c + \eta + \frac{r_2 - r_1}{1 - \alpha} \tag{2.23}$$

and converges to its unique least upper bound $r^\star$ *which satisfies*

$$c + \eta \le r^\star \le r^{\star\star}. \tag{2.24}$$

Moreover, the following estimates are satisfied for each $n = 1, \cdots$

$$0 \le r_{n+2} - r_{n+1} \le \alpha^n\,(r_2 - r_1). \tag{2.25}$$

Proof. We shall use mathematical induction to show that

$$0 \le \beta_n \le \alpha. \tag{2.26}$$

Estimate (2.26) is true for $n = 0$ by (2.22). Then, we have by (2.20) that

$$\begin{aligned} 0 \le r_3 - r_2 \le \alpha\,(r_2 - r_1) & \Longrightarrow r_3 \le r_2 + \alpha\,(r_2 - r_1) \\ & \Longrightarrow r_3 \le r_2 + (1+\alpha)\,(r_2 - r_1) - (r_2 - r_1) \\ & \Longrightarrow r_3 \le r_1 + \frac{1-\alpha^2}{1-\alpha}\,(r_2 - r_1) \le r^{\star\star}. \end{aligned}$$

Suppose (2.26) holds for each $n \le k$, then, using (2.20), we obtain that

$$0 \le r_{k+2} - r_{k+1} \le \alpha^k\,(r_2 - r_1) \quad \text{and} \quad r_{k+2} \le r_1 + \frac{1-\alpha^{k+1}}{1-\alpha}\,(r_2 - r_1). \tag{2.27}$$

Estimate (2.26) is certainly satisfied, if

$$g_k(\alpha) \le 0, \tag{2.28}$$

where g_k is defined by (2.21). Using (2.21), we obtain the following relationship between two consecutive recurrent functions g_k for each $k = 1, 2, \cdots$

$$g_{k+1}(\alpha) = g_k(\alpha) + p(\alpha)\, \alpha^{k-1}\, (r_2 - r_1) = g_k(\alpha), \tag{2.29}$$

since $p(\alpha) = 0$. Define function g_∞ on $[0, 1)$ by

$$g_\infty(t) = \lim_{k\to\infty} g_k(t). \tag{2.30}$$

Then, we get from (2.21) and (2.30) that

$$g_\infty(\alpha) = \alpha \left(\frac{2\, M_1\, (r_2 - r_1)}{1 - \alpha} + 2\, M_1\, \eta - 1 \right). \tag{2.31}$$

In view of (2.28)-(2.31) to show (2.28), it suffices to have $g_\infty(\alpha) \le 0$, which true by the right hand hypothesis in (2.22). The induction for (2.26) (i.e. for (2.25)) is complete. The rest of the proof is omitted (as identical to the proof of Lemma 2.1). The proof of Lemma 2.2 is complete. ∎

Remark 2.3. Let us see how sufficient convergence criterion on (2.5) for sequence $\{t_n\}$ simplifies in the interesting case of Newton's method. That is when $c = 0$ and $\lambda = 1$. Then, (2.5) can be written for $L_0 = 2\, M_1$ and $L = 2\, K$ as

$$h_0 = \frac{1}{8}\, (L + 4\, L_0 + \sqrt{L^2 + 8\, L_0\, L})\, \eta \le \frac{1}{2}. \tag{2.32}$$

The convergence criterion in [18] reduces to the famous for it simplicity and clarity Kantorovich hypothesis

$$h = L\, \eta \le \frac{1}{2}. \tag{2.33}$$

Note however that $L_0 \le L$ holds in general and L/L_0 can be arbitrarily large [6, 7, 8, 9, 10, 14]. We also have that

$$h \le \frac{1}{2} \Longrightarrow h_0 \le \frac{1}{2} \tag{2.34}$$

but not necessarily vice versa unless if $L_0 = L$ and

$$\frac{h_0}{h} \longrightarrow \frac{1}{4} \quad \text{as} \quad \frac{L}{L_0} \longrightarrow \infty. \tag{2.35}$$

Similarly, it can easily be seen that the sufficient convergence criterion (2.22) for sequence $\{r_n\}$ is given by

$$h_1 = \frac{1}{8}\, (4\, L_0 + \sqrt{L_0\, L + 8\, L_0^2} + \sqrt{L_0\, L})\, \eta \le \frac{1}{2}. \tag{2.36}$$

We also have that

$$h_0 \le \frac{1}{2} \Longrightarrow h_1 \le \frac{1}{2} \tag{2.37}$$

and

$$\frac{h_1}{h} \longrightarrow 0,\ \frac{h_1}{h_0} \longrightarrow 0 \quad \text{as} \quad \frac{L_0}{L} \longrightarrow 0. \tag{2.38}$$

Note that sequence $\{r_n\}$ is tighter than $\{t_n\}$ and converges under weaker conditions. Indeed, a simple inductive argument shows that for each $n = 2, 3, \cdots$, if $M_1 < K$, then

$$r_n < t_n,\ r_{n+1} - r_n < t_{n+1} - t_n \quad \text{and} \quad r^\star \le t^\star. \tag{2.39}$$

We have the following usefull and obvious extensions of Lemma 2.1 and Lemma 2.2, respectively.

Lemma 2.4. *Let $N = 0, 1, 2, \cdots$ be fixed. Suppose that*

$$t_1 \le t_2 \le \cdots \le t_N \le t_{N+1}, \tag{2.40}$$

$$\frac{1}{M_1} > (1-\lambda)\,(t_N - t_0) + (1+\lambda)\,(t_{N+1} - t_0) \tag{2.41}$$

and

$$0 \le \alpha_N \le \alpha \le 1 - 2\,M_1\,(t_{N+1} - t_N). \tag{2.42}$$

Then, sequence $\{t_n\}$ generated by (2.1) is nondecreasing, bounded from above by $t^{\star\star}$ and converges to $t^\star$ which satisfies $t^\star \in [t_{N+1}, t^{\star\star}]$. Moreover, the following estimates are satisfied for each $n = 0, 1, \cdots$

$$0 \le t_{N+n+1} - t_{N+n} \le \alpha^n\,(t_{N+1} - t_N) \tag{2.43}$$

and

$$t^\star - t_{N+n} \le \frac{\alpha^n}{1-\alpha}\,(t_{N+1} - t_N). \tag{2.44}$$

Lemma 2.5. *Let $N = 1, 2, \cdots$ be fixed. Suppose that*

$$r_1 \le r_2 \le \cdots \le r_N \le r_{N+1}, \tag{2.45}$$

$$\frac{1}{M_1} > (1-\lambda)\,(r_N - r_0) + (1+\lambda)\,(r_{N+1} - r_0) \tag{2.46}$$

and

$$0 \leq \beta_N \leq \alpha \leq 1 - \frac{2\, M_1\, (r_{N+1} - r_N)}{1 - 2\, M_1\, (r_N - r_{N-1})}. \tag{2.47}$$

Then, sequence $\{r_n\}$ *generated by (2.20) is nondecreasing, bounded from above by* $r^{\star\star}$ *and converges to* $r^\star$ *which satisfies* $r^\star \in [r_{N+1}, r^{\star\star}]$. *Moreover, the following estimates are satisfied for each* $n = 0, 1, \cdots$

$$0 \leq r_{N+n+1} - r_{N+n} \leq \alpha^n\, (r_{N+1} - r_N) \tag{2.48}$$

and

$$r^\star - r_{N+n} \leq \frac{\alpha^n}{1-\alpha}\, (r_{N+1} - r_N). \tag{2.49}$$

Next, we present the following semilocal convergence result for secant-like method under the **(**$\mathcal{C}^\star$**)** conditions.

Theorem 2.6. *Suppose that the* **(**$\mathcal{C}^\star$**)**, *Lemma 2.1 (or Lemma 2.4) conditions and*

$$\overline{U}(x_0, t^\star) \subseteq \mathcal{D} \tag{2.50}$$

hold. Then, sequence $\{x_n\}$ *generated by the secant-like method is well defined, remains in* $\overline{U}(x_0, t^\star)$ *for each* $n = -1, 0, 1, \cdots$ *and converges to a solution* $x^\star \in \overline{U}(x_0, t^\star - c)$ *of equation* $F(x) = 0$. *Moreover, the following estimates are satisfied for each* $n = 0, 1, \cdots$

$$\| x_{n+1} - x_n \| \leq t_{n+1} - t_n \tag{2.51}$$

and

$$\| x_n - x^\star \| \leq t^\star - t_n. \tag{2.52}$$

Furthemore, if there exists $r \geq t^\star$ *such that*

$$\overline{U}(x_0, r) \subseteq \mathcal{D} \tag{2.53}$$

and

$$r + t^\star < \frac{1}{M_1} \quad \text{or} \quad r + t^\star < \frac{2}{M_2}, \tag{2.54}$$

then, the solution $x^\star$ *is unique in* $\overline{U}(x_0, r)$.

Proof. We use mathematical induction to prove that

$$\| x_{k+1} - x_k \| \leq t_{k+1} - t_k \tag{2.55}$$

and

$$\overline{U}(x_{k+1}, t^\star - t_{k+1}) \subseteq \overline{U}(x_k, t^\star - t_k) \tag{2.56}$$

for each $k = -1, 0, 1, \cdots$. Let $z \in \overline{U}(x_0, t^\star - t_0)$. Then, we obtain that

$$\| z - x_{-1} \| \leq \| z - x_0 \| + \| x_0 - x_{-1} \| \leq t^\star - t_0 + c = t^\star = t^\star - t_{-1},$$

which implies $z \in \overline{U}(x_{-1}, t^\star - t_{-1})$. Let also $w \in \overline{U}(x_0, t^\star - t_1)$. We get that

$$\| w - x_0 \| \leq \| w - x_1 \| + \| x_1 - x_0 \| \leq t^\star - t_1 + t_1 - t_0 = t^\star = t^\star - t_0.$$

That is $w \in \overline{U}(x_0, t^\star - t_0)$. Note that

$$\| x_{-1} - x_0 \| \leq c = t_0 - t_{-1} \quad \text{and} \quad \| x_1 - x_0 \| = \| B_0^{-1} F(x_0) \| \leq \eta = t_1 - t_0 < t^\star,$$

which implies $x_1 \in \overline{U}(x_0, t^\star) \subseteq \mathcal{D}$. Hence, estimates (2.51) and (2.52) hold for $k = -1$ and $k = 0$. Suppose (2.51) and (2.52) hold for all $n \leq k$. Then, we obtain that

$$\| x_{k+1} - x_0 \| \leq \sum_{i=1}^{k+1} \| x_i - x_{i-1} \| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 \leq t^\star$$

and

$$\| y_k - x_0 \| \leq \lambda \, \| x_k - x_0 \| + (1-\lambda) \, \| x_{k-1} - x_0 \| \leq \lambda \, t^\star + (1-\lambda) \, t^\star = t^\star.$$

Hence, $x_{k+1}, y_k \in \overline{U}(x_0, t^\star)$. Let $E_k := [x_{k+1}, x_k; F]$ for each $k = 0, 1, \cdots$. Using (1.2), Lemma 2.1 and the induction hypotheses, we get that

$$\begin{aligned} &\| F'(x_0)^{-1} (B_{k+1} - F'(x_0)) \| \leq M_1 (\| y_{k+1} - x_0 \| + \| x_{k+1} - x_0 \|) \\ &\leq M_1 ((1-\lambda) \, \| x_k - x_0 \| + \lambda \, \| x_{k+1} - x_0 \| + \| x_{k+1} - x_0 \|) \\ &\leq M_1 ((1-\lambda)(t_k - t_0) + (1+\lambda)(t_{k+1} - t_0)) < 1, \end{aligned} \tag{2.57}$$

since, $y_{k+1} - x_0 = \lambda \, (x_{k+1} - x_0) + (1-\lambda)(x_k - x_0)$ and

$$\begin{aligned} &\| y_{k+1} - x_0 \| = \| \lambda \, (x_{k+1} - x_0) + (1-\lambda)(x_k - x_0) \| \\ &\leq \lambda \, \| x_{k+1} - x_0 \| + (1-\lambda) \, \| x_k - x_0 \| . \end{aligned}$$

It follows from (2.57) and the Banach lemma on invertible operators that B_{k+1}^{-1} exists and

$$\| B_{k+1}^{-1}\, F'(x_0) \| \leq \frac{1}{1-\Theta_k} \leq \frac{1}{1-M_1\, q_{k+1}}, \tag{2.58}$$

where $\Theta_k = M_1\,((1-\lambda)\,\| x_k - x_0 \| + (1+\lambda)\,\| x_{k+1} - x_0 \|)$. In view of (1.2), we obtain the identity

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - B_k\,(x_{k+1} - x_k) = (E_k - B_k)\,(x_{k+1} - x_k). \tag{2.59}$$

Then, using the induction hypotheses, the **($\mathcal{C}^\star$)** condition and (2.59), we get in turn that

$$\begin{aligned} & \| F'(x_0)^{-1}\, F(x_{k+1}) \| \\ &= \| F'(x_0)^{-1}\,(E_k - B_k)\,(x_{k+1} - x_k) \| \\ &\leq K\ \| x_{k+1} - y_k \|\, \| x_{k+1} - x_k \| \\ &\leq K\,(\| x_{k+1} - x_k \| + (1-\lambda)\ \| x_k - x_{k-1} \|)\ \| x_{k+1} - x_k \| \\ &\leq K\,(t_{k+1} - t_k + (1-\lambda)\,(t_k - t_{k-1}))\,(t_{k+1} - t_k), \end{aligned} \tag{2.60}$$

since, $x_{k+1} - y_k = x_{k+1} - x_k + (1-\lambda)\,(x_k - x_{k-1})$ and

$$\| x_{k+1} - y_k \| \leq \| x_{k+1} - x_k \| + (1-\lambda)\ \| x_k - x_{k-1} \| \leq t_{k+1} - t_k + (1-\lambda)\,(t_k - t_{k-1}).$$

It now follows from (1.2), (2.1), (2.58)-(2.60) that

$$\begin{aligned} \| x_{k+2} - x_{k+1} \| &\leq \| B_{k+1}^{-1}\ F'(x_0) \|\, \| F'(x_0)^{-1}\, F(x_{k+1}) \| \\ &\leq \frac{K\,(t_{k+1} - t_k + (1-\lambda)\,(t_{k+1} - x_k))\,(t_{k+1} - t_k)}{1 - M_1\, q_{k+1}} = t_{k+2} - t_{k+1}, \end{aligned}$$

which completes the induction for (2.55). Furthemore, let $v \in \overline{U}(x_{k+2}, t^\star - t_{k+2})$. Then, we have that

$$\begin{aligned} \| v - x_{k+1} \| &\leq \| v - x_{k+2} \| + \| x_{k+2} - x_{k+1} \| \\ &\leq t^\star - t_{k+2} + t_{k+2} - t_{k+1} = t^\star - t_{k+1}, \end{aligned}$$

which implies $v \in \overline{U}(x_{k+1}, t^\star - t_{k+1})$. The induction for (2.55) and (2.56) is complete. Lemma 2.1 implies that $\{t_k\}$ is a complete sequence. It follows from (2.55) and (2.56) that $\{x_k\}$ is a complete sequence in a Banach space $\mathcal{X}$ and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \longrightarrow \infty$ in (2.60), we get that $F(x^\star) = 0$. Moreover, estimate (2.52) follows from (2.51) by using standard majorization techniques [8, 10, 22]. To show the uniqueness part, let $y^\star \in \overline{U}(x_0, r)$ be such $F(y^\star) = 0$, where r satisfies (2.53) and (2.54). We have that

$$\begin{aligned} \| F'(x_0)^{-1}\,([y^\star, x^\star; F] - F'(x_0)) \| &\leq M_1\,(\| y^\star - x_0 \| + \| x^\star - x_0 \|) \\ &\leq M_1\,(t^\star + r) < 1. \end{aligned} \tag{2.61}$$

It follows by (2.61) and the Banach lemma on invertible operators that linear operator $[y^\star, x^\star; F]^{-1}$ exists. Then, using the identity $0 = F(y^\star) - F(x^\star) = [y^\star, x^\star; F]\,(y^\star - x^\star)$, we deduce that $x^\star = y^\star$. The proof of Theorem 2.6 is complete. ■

In order for us to present the semilocal result for secant-like method under the **(**$\mathcal{C}^{\star\star}$**)** conditions, we first need a result on a majorizing sequence. The proof in given in Lemma 2.1.

Remark 2.7. Clearly, (2.22) (or (2.47)), $\{r_n\}$ can replace (2.5) (or (2.42)), $\{t_n\}$, respectively in Theorem 2.6.

Lemma 2.8. *Let* $c \geq 0$, $\eta > 0$, $L > 0$, $M_0 > 0$ *with* $M_0\, c < 1$ *and* $\lambda \in [0, 1]$. *Set*

$$s_{-1} = 0,\; s_0 = c,\; s_1 = c + \eta,\; \tilde{K} = \frac{L}{1 - M_0\, c} \quad \text{and} \quad \tilde{M}_1 = \frac{M_0}{1 - M_0\, c}.$$

Define scalar sequences $\{\tilde{q}_n\}$, $\{s_n\}$, $\{\tilde{\alpha}_n\}$ *for each* $n = 0, 1, \cdots$ *by*

$$\begin{aligned} \tilde{q}_n &= (1-\lambda)\,(s_n - s_0) + (1+\lambda)\,(s_{n+1} - s_0), \\ s_{n+2} &= s_{n+1} + \frac{\tilde{K}\,(s_{n+1} - s_n + (1-\lambda)\,(s_n - s_{n-1}))}{1 - \tilde{M}_1\, \tilde{q}_n}\,(s_{n+1} - s_n), \\ \tilde{\alpha}_n &= \frac{\tilde{K}\,(s_{n+1} - s_n + (1-\lambda)\,(s_n - s_{n-1}))}{1 - \tilde{M}_1\, \tilde{q}_n}, \end{aligned}$$

function $\{\tilde{f}_n\}$ *for each* $n = 1, 2, \cdots$ *by*

$$\begin{aligned} \tilde{f}_n(t) \;=\; & \tilde{K}\,\eta\, t^n + \tilde{K}\,(1-\lambda)\,\eta\, t^{n-1} + \tilde{M}_1\,\eta\,((1-\lambda)\,(1 + t + \cdots + t^n) + \\ & (1+\lambda)\,(1 + t + \cdots + t^{n+1})) - 1 \end{aligned}$$

and polynomial $\tilde{p}$ *by*

$$\tilde{p}(t) = \tilde{M}_1\,(1+\lambda)\,t^3 + (\tilde{M}_1\,(1-\lambda) + \tilde{K})\,t^2 - \tilde{K}\,\lambda\, t - \tilde{K}\,(1-\lambda).$$

Denote by $\tilde{\alpha}$ *the smallest root of polynomial* $\tilde{p}$ *in* $(0, 1)$. *Suppose that*

$$0 \leq \tilde{\alpha}_0 \leq \tilde{\alpha} \leq 1 - 2\,\tilde{M}_1\,\eta. \tag{2.62}$$

Then, sequence $\{s_n\}$ *is non-decreasing, bounded from above by* $s^{\star\star}$ *defined by*

$$s^{\star\star} = \frac{\eta}{1 - \tilde{\alpha}} + c$$

and converges to its unique least upper bound $s^\star$ *which satisfies* $c + \eta \leq s^\star \leq s^{\star\star}$. *Moreover, the following estimates are satisfied for each* $n = 0, 1, \cdots$

$$0 \leq s_{n+1} - s_n \leq \tilde{\alpha}^n\,\eta \quad \text{and} \quad s^\star - s_n \leq \frac{\tilde{\alpha}^n\,\eta}{1 - \tilde{\alpha}}.$$

Next, we present the semilocal convergence result for secant-like method under the **($\mathcal{C}^{\star\star}$)** conditions.

Theorem 2.9. *Suppose that the* **($\mathcal{C}^{\star\star}$)** *conditions, (2.62) (or Lemma 2.2 conditions with $\tilde{\alpha}_n$, $\tilde{\alpha}$, $\tilde{M}_1$ replacing, respectively, α_n, α, M_1) and $\overline{U}(x_0, s^\star) \subseteq \mathcal{D}$ hold. Then, sequence $\{x_n\}$ generated by the secant-like method is well defined, remains in $\overline{U}(x_0, s^\star)$ for each $n = -1, 0, 1, \cdots$ and converges to a solution $x^\star \in \overline{U}(x_0, s^\star)$ of equation $F(x) = 0$. Moreover, the following estimates are satisfied for each $n = 0, 1, \cdots$*

$$\| x_{n+1} - x_n \| \leq s_{n+1} - s_n \quad \text{and} \quad \| x_n - x^\star \| \leq s^\star - s_n.$$

Furthemore, if there exists $r \geq s^\star$ such that $\overline{U}(x_0, r) \subseteq \mathcal{D}$ and $r + s^\star + c < 1/M_0$, then, the solution $x^\star$ is unique in $\overline{U}(x_0, r)$.

Proof. The proof is analogous to Theorem 2.6. Simply notice that in view of **($\mathcal{C}_5$)**, we obtain instead of (2.57) that

$$\begin{aligned} &\| A_0^{-1} (B_{k+1} - A_0) \| \leq M_0 (\| y_{k+1} - x_{-1} \| + \| x_{k+1} - x_0 \|) \\ &\leq M_0 ((1-\lambda) \| x_k - x_0 \| + \lambda \| x_{k+1} - x_0 \| + \| x_0 - x_{-1} \| + \| x_{k+1} - x_0 \|) \\ &\leq M_0 ((1-\lambda)(s_k - s_0) + (1+\lambda)(s_{k+1} - s_0) + c) < 1, \end{aligned}$$

leading to B_{k+1}^{-1} exists and

$$\| B_{k+1}^{-1} A_0 \| \leq \frac{1}{1 - \Xi_k},$$

where $\Xi_k = M_0 ((1-\lambda)(s_k - s_0) + (1+\lambda)(s_{k+1} - s_0) + c)$. Moreover, using **($\mathcal{C}_3^\star$)** instead of **($\mathcal{C}_3^{\star\star}$)**, we get that

$$\| A_0^{-1} F(x_{k+1}) \| \leq L (s_{k+1} - s_k + (1-\lambda)(s_k - s_{k-1}))(s_{k+1} - s_k).$$

Hence, we have that

$$\begin{aligned} \| x_{k+2} - x_{k+1} \| &\leq \| B_{k+1}^{-1} A_0 \| \| A_0^{-1} F(x_{k+1}) \| \\ &\leq \frac{L (s_{k+1} - s_k + (1-\lambda)(s_k - s_{k-1}))(s_{k+1} - s_k)}{1 - M_0 ((1+\lambda)(s_{k+1} - s_0) + (1-\lambda)(s_k - s_0) + c)} \\ &\leq \frac{\tilde{K} (s_{k+1} - s_k + (1-\lambda)(s_k - s_{k-1}))(s_{k+1} - s_k)}{1 - \tilde{M}_1 ((1+\lambda)(s_{k+1} - s_0) + (1-\lambda)(s_k - s_0))} = s_{k+2} - s_{k+1}. \end{aligned}$$

The uniqueness part is given in Theorem 2.6 with r, $s^\star$ replacing R_2 and R_0, respectively. The proof of Theorem 2.9 is complete. ∎

Remark 2.10. (a) Condition (2.50) can be replaced by

$$\overline{U}(x_0, t^{\star\star}) \subseteq \mathcal{D}, \tag{2.63}$$

where $t^{\star\star}$ is given in the closed form by (2.55).

(b) The majorizing sequence $\{u_n\}$ essentially used in [18] is defined by
(2.64)
$$u_{-1} = 0,\ u_0 = c,\ u_1 = c + \eta$$
$$u_{n+2} = u_{n+1} + \frac{M\,(u_{n+1} - u_n + (1-\lambda)\,(u_n - u_{n-1}))}{1 - M\, q_n^\star}\,(u_{n+1} - u_n),$$

where
$$q_n^\star = (1-\lambda)\,(u_n - u_0) + (1+\lambda)\,(u_{n+1} - u_0).$$

Then, if $K < M$ or $M_1 < M$, a simple inductive argument shows that for each $n = 2, 3, \cdots$

$$t_n < u_n, \quad t_{n+1} - t_n < u_{n+1} - u_n \quad \text{and} \quad t^\star \leq u^\star = \lim_{n\to\infty} u_n. \tag{2.65}$$

Clearly $\{t_n\}$ converges under the **($\mathcal{C}$)** conditions and conditions of Lemma 2.1. Moreover, as we already showed in Remark 2.3, the sufficient convergence criteria of Theorem 2.6 can be weaker than those of Theorem 2.9. Similarly if $L \leq M$, $\{s_n\}$ is a tighter sequence than $\{u_n\}$. In general, we shall test the convergence criteria and use the tightest sequence to estimate the error bounds.

(c) Clearly the conclusions of Theorem 2.9 hold if $\{s_n\}$, (2.62) are replaced by $\{\tilde{r}_n\}$, (2.22), where $\{\tilde{r}_n\}$ is defined as $\{r_n\}$ with M_0 replacing M_1 in the definition of β_1 (only at the numerator) and the tilda letters replacing the non-tilda letters in (2.22).

3. Numerical Examples

Now, we check numerically with two examples that the new semilocal convergence results obtained in Theorems 2.6 and 2.9 improve the domain of starting points obtained by the following classical result given in [20].

Theorem 3.1. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear operator defined on a non-empty open convex domain Ω. Let $x_{-1}, x_0 \in \Omega$ and $\lambda \in [0,1]$. Suppose that there exists $[u, v; F] \in \mathcal{L}(X, Y)$, for all $u, v \in \Omega$ ($u \neq v$), and the following four conditions*

- $\|x_0 - x_{-1}\| = c \neq 0$ *with* $x_{-1}, x_0 \in \Omega$,

- *Fixed* $\lambda \in [0,1]$, *the operator* $B_0 = [y_0, x_0; F]$ *is invertible and such that* $\|B_0^{-1}\| \le \beta$,
- $\|B_0^{-1} F(x_0)\| \le \eta$,
- $\|[x,y;F] - [u,v;F]\| \le Q(\|x-u\| + \|y-v\|); Q \ge 0; x,y,u,v \in \Omega; x \ne y; u \ne v$,

are satisfied. If $B(x_0, \rho) \subseteq \Omega$, *where* $\rho = \dfrac{1-a}{1-2a}\eta$,

$$a = \frac{\eta}{c+\eta} < \frac{3-\sqrt{5}}{2} \qquad \textit{and} \qquad b = \frac{Q\beta c^2}{c+\eta} < \frac{a(1-a)^2}{1+\lambda(2a-1)}, \tag{3.1}$$

then the secant-like methods defined by (1.2)converge to a solution x^* *of equation* $F(x) = 0$ *with* R*-order of convergence at least* $\frac{1+\sqrt{5}}{2}$. *Moreover,* $x_n, x^* \in \overline{B(x_0, \rho)}$, *the solution* x^* *is unique in* $B(x_0, \tau) \cap \Omega$, *where* $\tau = \frac{1}{Q\beta} - \rho - (1-\lambda)\alpha$.

3.1. Example 1

We illustrate the above-mentioned with an application, where a system of nonlinear equations is involved. We see that Theorem 3.1 cannot guarantee the semilocal convergence of secant-like methods (1.2) , but Theorem 2.6 can do it.

It is well known that energy is dissipated in the action of any real dynamical system, usually through some form of friction. However, in certain situations this dissipation is so slow that it can be neglected over relatively short periods of time. In such cases we assume the law of conservation of energy, namely, that the sum of the kinetic energy and the potential energy is constant. A system of this kind is said to be conservative.

If φ and ψ are arbitrary functions with the property that $\varphi(0) = 0$ and $\psi(0) = 0$, the general equation

$$\mu \frac{d^2 x(t)}{dt^2} + \psi\left(\frac{dx(t)}{dt}\right) + \varphi(x(t)) = 0, \tag{3.2}$$

can be interpreted as the equation of motion of a mass μ under the action of a restoring force $-\varphi(x)$ and a damping force $-\psi(dx/dt)$. In general these forces are nonlinear, and equation (3.2) can be regarded as the basic equation of nonlinear mechanics. In this paper we shall consider the special case of a nonlinear conservative system described by the equation

$$\mu \frac{d^2 x(t)}{dt^2} + \varphi(x(t)) = 0,$$

in which the damping force is zero and there is consequently no dissipation of energy. Extensive discussions of (3.2), with applications to a variety of physical problems, can be found in classical references [4] and [30].

Now, we consider the special case of a nonlinear conservative system described by the equation

$$\frac{d^2x(t)}{dt^2} + \phi(x(t)) = 0 \tag{3.3}$$

with the boundary conditions

$$x(0) = x(1) = 0. \tag{3.4}$$

After that, we use a process of discretization to transform problem (3.3)-(3.4) into a finite-dimensional problem and look for an approximated solution of it when a particular function ϕ is considered. So, we transform problem (3.3)-(3.4) into a system of nonlinear equations by approximating the second derivative by a standard numerical formula.

Firstly, we introduce the points $t_j = jh$, $j = 0, 1, \ldots, m+1$, where $h = \frac{1}{m+1}$ and m is an appropriate integer. A scheme is then designed for the determination of numbers x_j, it is hoped, approximate the values $x(t_j)$ of the true solution at the points t_j. A standard approximation for the second derivative at these points is

$$x_j'' \approx \frac{x_{j-1} - 2x_j + x_{j+1}}{h^2}, \quad j = 1, 2, \ldots, m.$$

A natural way to obtain such a scheme is to demand that the x_j satisfy at each interior mesh point t_j the difference equation

$$x_{j-1} - 2x_j + x_{j+1} + h^2\phi(x_j) = 0. \tag{3.5}$$

Since x_0 and x_{m+1} are determined by the boundary conditions, the unknowns are $x_1, x_2, \ldots, x_m$.

A further discussion is simplified by the use of matrix and vector notation. Introducing the vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad v_{\mathbf{x}} = \begin{pmatrix} \phi(x_1) \\ \phi(x_2) \\ \vdots \\ \phi(x_m) \end{pmatrix}$$

and the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix},$$

the system of equations, arising from demanding that (3.5) holds for $j = 1, 2, \ldots, m$, can be written compactly in the form

$$F(\mathbf{x}) \equiv A\mathbf{x} + h^2 v_{\mathbf{x}} = 0, \tag{3.6}$$

where F is a function from $\mathbb{R}^m$ into $\mathbb{R}^m$.

From now on, the focus of our attention is to solve a particular system of form (3.6). We choose $m = 8$ and the infinity norm.

The steady temperature distribution is known in a homogeneous rod of length 1 in which, as a consequence of a chemical reaction or some such heat-producing process, heat is generated at a rate $\phi(x(t))$ per unit time per unit length, $\phi(x(t))$ being a given function of the excess temperature x of the rod over the temperature of the surroundings. If the ends of the rod, $t = 0$ and $t = 1$, are kept at given temperatures, we are to solve the boundary value problem given by (3.3)-(3.4), measured along the axis of the rod. For an example we choose an exponential law $\phi(x(t)) = \exp(x(t))$ for the heat generation.

Taking into account that the solution of (3.3)-(3.4) with $\phi(x(t)) = \exp(x(t))$ is of the form

$$x(s) = \int_0^1 G(s,t) \exp(x(t))\, dt,$$

where $G(s,t)$ is the Green function in $[0,1] \times [0,1]$, we can locate the solution $x^*(s)$ in some domain. So, we have

$$\|x^*(s)\| - \frac{1}{8} \exp(\|x^*(s)\|) \leq 0,$$

so that $\|x^*(s)\| \in [0, \varrho_1] \cup [\varrho_2, +\infty]$, where $\varrho_1 = 0.1444$ and $\varrho_2 = 3.2616$ are the two positive real roots of the scalar equation $8t - \exp(t) = 0$.

Observing the semilocal convergence results presented in this work, we can only guarantee the semilocal convergence to a solution $x^*(s)$ such that $\|x^*(s)\| \in [0, \varrho_1]$. For this, we can consider the domain

$$\Omega = \{x(s) \in C^2[0,1]\,;\ \|x(s)\| < \log(7/4),\ s \in [0,1]\},$$

since $\varrho_1 < \log\left(\frac{7}{4}\right) < \varrho_2$.

In view of what the domain Ω is for equation (3.3), we then consider (3.6) with $F : \widetilde{\Omega} \subset \mathbb{R}^8 \to \mathbb{R}^8$ and

$$\widetilde{\Omega} = \{\mathbf{x} \in \mathbb{R}^8;\ \|\mathbf{x}\| < \log(7/4)\}.$$

According to the above-mentioned, $v_{\mathbf{x}} = (\exp(x_1), \exp(x_2), \ldots, \exp(x_8))^t$ if $\phi(x(t)) = \exp(x(t))$. Consequently, the first derivative of the function F defined in (3.6) is given by

$$F'(\mathbf{x}) = A + h^2 \mathrm{diag}(v_{\mathbf{x}}).$$

Moreover,

$$F'(\mathbf{x}) - F'(\mathbf{y}) = h^2 \mathrm{diag}(\mathbf{z}),$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_8)^t$ and $\mathbf{z} = (\exp(x_1) - \exp(y_1), \exp(x_2) - \exp(y_2), \ldots, \exp(x_8) - \exp(y_8))$. In addition,

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \leq h^2 \max_{1 \leq i \leq 8} |\exp(\ell_i)| \, \|\mathbf{x} - \mathbf{y}\|,$$

where $\ell = (\ell_1, \ell_2, \ldots, \ell_8)^t \in \widetilde{\Omega}$ and $h = \frac{1}{9}$, so that

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \leq \frac{7}{4} h^2 \|\mathbf{x} - \mathbf{y}\|. \tag{3.7}$$

Considering (see [26])

$$[\mathbf{x}, \mathbf{y}; F] = \int_0^1 F'(\tau \mathbf{x} + (1-\tau)\mathbf{y}) \, d\tau,$$

taking into account

$$\int_0^1 \|\tau(\mathbf{x} - \mathbf{u}) + (1-\tau)(\mathbf{y} - \mathbf{v})\| d\tau \leq \frac{1}{2} \left(\|\mathbf{x} - \mathbf{u}\| + \|\mathbf{y} - \mathbf{v}\| \right),$$

and (3.7), we have

$$\begin{aligned} \|[\mathbf{x}, \mathbf{y}; F] - [\mathbf{u}, \mathbf{v}; F]\| &\leq \int_0^1 \|F'(\tau \mathbf{x} + (1-\tau)\mathbf{y}) - F'(\tau \mathbf{u} + (1-\tau)\mathbf{v})\| \, d\tau \\ &\leq \frac{7}{4} h^2 \int_0^1 (\tau \|\mathbf{x} - \mathbf{u}\| + (1-\tau)\|\mathbf{y} - \mathbf{v}\|) \, d\tau \\ &= \frac{7}{8} h^2 \left(\|\mathbf{x} - \mathbf{u}\| + \|\mathbf{y} - \mathbf{v}\| \right). \end{aligned}$$

From the last, we have $L = \frac{7}{648}$ and $M_1 = \frac{7}{648} \|[F'(x_0)]^{-1}\|$.

If we choose $\lambda = \frac{1}{2}$ and the starting points $\mathbf{x_{-1}} = (\frac{1}{10}, \frac{1}{10}, \ldots, \frac{1}{10})^t$ and $\mathbf{x_0} = (0, 0, \ldots, 0)^t$, we obtain $c = \frac{1}{10}$, $\beta = 11.202658\ldots$ and $\eta = 0.138304\ldots$, so that (3.1) of Theorem 3.1 is not satisfied, since

$$a = \frac{\eta}{c + \eta} = 0.580368\ldots > \frac{3 - \sqrt{5}}{2} = 0.381966\ldots$$

Thus, according to Theorem 3.1, we cannot guarantee the convergence of secant-like method (1.2) with $\lambda = \frac{1}{2}$ for approximating a solution of (3.6) with $\phi(s) = \exp(s)$.

However, we can do it by Theorem 2.6, since all the inequalities which appear in (2.5) are satisfied:

$$0 < \alpha_0 = 0.023303\ldots \leq \alpha = 0.577350\ldots \leq 1 - 2M_1\eta = 0.966625\ldots,$$

where $\|[F'(x_0)]^{-1}\| = 11.169433\ldots$, $M_1 = 0.120657\ldots$ and

$$p(t) = (0.180986\ldots)t^3 + (0.180986\ldots)t^2 - (0.060328\ldots)t - (0.060328\ldots).$$

Then, we can use secant-like method (1.2) with $\lambda = \frac{1}{2}$ to approximate a solution of (3.6) with $\phi(u) = \exp(u)$, the approximation given by the vector $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_8^*)^t$ shown in Table 1 and reached after four iterations with a tolerance 10^{-16}. In Table 2 we show the errors $\|\mathbf{x_n} - \mathbf{x}^*\|$ using the stopping criterion $\|\mathbf{x_n} - \mathbf{x_{n-1}}\| < 10^{-16}$. Notice that the vector shown in Table 1 is a good approximation of the solution of (3.6) with $\phi(u) = \exp(u)$, since $\|F(\mathbf{x}^*)\| \leq C \times 10^{-16}$. See the sequence $\{\|F(\mathbf{x_n})\|\}$ in Table 2.

Table 1. Approximation of the solution $\mathbf{x}^*$ of (3.6) with $\phi(u) = \exp(u)$

n	x_i^*	n	x_i^*	n	x_i^*	n	x_i^*
1	$0.05481058\ldots$	3	$0.12475178\ldots$	5	$0.13893761\ldots$	7	$0.09657993\ldots$
2	$0.09657993\ldots$	4	$0.13893761\ldots$	6	$0.12475178\ldots$	8	$0.05481058\ldots$

Table 2. Absolute errors obtained by secant-like method (1.2) with $\lambda = \frac{1}{2}$ and $\{\|F(\mathbf{x_n})\|\}$

n	$\|\mathbf{x_n} - \mathbf{x}^*\|$	$\|F(\mathbf{x_n})\|$
-1	$1.3893\ldots \times 10^{-1}$	$8.6355\ldots \times 10^{-2}$
0	$4.5189\ldots \times 10^{-2}$	$1.2345\ldots \times 10^{-2}$
1	$1.43051\ldots \times 10^{-4}$	$2.3416\ldots \times 10^{-5}$
2	$1.14121\ldots \times 10^{-7}$	$1.9681\ldots \times 10^{-8}$
3	$4.30239\ldots \times 10^{-13}$	$5.7941\ldots \times 10^{-14}$

3.2. Example 2

Consider the following nonlinear boundary value problem

$$\begin{cases} x''(s) = -x(s)^3 - \frac{1}{4}x(s)^2 \\ x(0) = 0, \quad x(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$x(s) = s + \int_0^1 G(s,t)\left(x^3(t) + \frac{1}{4}x^2(t)\right)dt \tag{3.8}$$

where G is the Green function in $[0,1] \times [0,1]$. Observe that equation (3.8) is equivalent to equation (1.1) if we consider a suitable domain $\mathcal{D}$ in $C^2[0,1]$ and

$$[F(x)](s) = x(s) - s - \int_0^1 G(s,t)\left(x^3(t) + \frac{1}{4}x^2(t)\right)\, dt.$$

For this, taking into account the infinity norm, we see that a solution $x^*(s)$ of (3.8) satisfies

$$\|x^*(s)\| - 1 - \frac{1}{8}\left(\|x^*(s)\|^3 + \frac{1}{4}\|x^*(s)\|^2\right) \leq 0.$$

So, we consider the domain

$$\mathcal{D} = \{x(s) \in C^2[0,1]\,;\ \|x(s)\| < 2,\ s \in [0,1]\}.$$

The first Fréchet derivative of the operator F is

$$[F'(x)y]\,(s) = y(s) - 3\int_0^1 G(s,t)x^2(t)y(t)dt - \frac{1}{2}\int_0^1 G(s,t)x(t)y(t)dt.$$

If we choose $x_0(s) = s$, then $\|F(x_0)\| \leq \frac{5}{32}$. Define the divided difference by

$$[x,y;F] = \int_0^1 F'(\tau x + (1-\tau)y)d\tau$$

and, consequently,

$$\begin{aligned}
\|[x,y;F]-[v,y;F]\| &\leq \int_0^1 \|F'\left(\tau x + (1-\tau)y\right) - F'\left(\tau v + (1-\tau)y\right)\|\, d\tau \\
&\leq \frac{1}{8}\int_0^1 \left(3\tau^2\|x^2-v^2\| + 2\tau(1-\tau)\|y\|\|x-v\| + \frac{\tau}{2}\|x-v\|\right) d\tau \\
&\leq \frac{1}{8}\left(\|x^2-v^2\| + \left(\|y\| + \frac{1}{4}\right)\right)\|x-v\| \\
&\leq \frac{1}{8}\left(\|x+v\| + \|y\| + \frac{1}{4}\right)\|x-v\| \\
&\leq \frac{25}{32}\|x-v\|.
\end{aligned}$$

Next, if $x_{-1}(s) = \frac{9}{10}s$, we obtain

$$\|I - A_0\| \leq \int_0^1 \|F'\left(\tau x_0 + (1-\tau)x_{-1}\right)\|\, d\tau \leq 0.409375\ldots,$$

so that, by the Banach Lemma on invertible operators, it follows $\|A_0^{-1}\| \leq 1.69312\ldots$ In addition, $L \geq \dfrac{25}{32}\|A_0^{-1}\| = 1.32275\ldots$ and take then $L = 1.32275\ldots$

On the one hand, if we now choose $\lambda = 0.8$, we obtain, in an analogous way, the following:

$$M_0 = 0.89947\ldots, \quad \|B_0^{-1}\| = 1.75262\ldots, \quad \eta = 0.27384\ldots$$

Observe that we cannot guarantee the convergence of the secant method from Theorem 3.1, since the first condition of (3.1) is not satisfied:

$$a = \frac{\eta}{c+\eta} = 0.732511\ldots > \frac{3-\sqrt{5}}{2} = 0.381966\ldots$$

On the other hand, observe that

$$\tilde{M}_1 = 0.09883\ldots, \quad \tilde{K} = 1.45349\ldots, \quad \alpha_0 = 0.43407\ldots, \quad \alpha = 0.90732\ldots$$

and $1 - 2\tilde{M}_1\eta = 0.945868\ldots$, so that condition (2.62), $0 < \alpha_0 \leq \alpha \leq 1 - 2\tilde{M}_1\eta$, is satisfied and, as a consequence, we can guarantee the convergence of the secant method by Theorem 2.9.

4. Conclusion

We presented a new semilocal convergence analysis of the secant-like method for approximating a locally unique solution of an equation in a Banach space. Using a combination of Lipschitz and center-Lipschitz conditions, instead of only Lipschitz conditions invested in [18], we provided a finer analysis with larger convergence domain and weaker sufficient convergence conditions than in [15, 18, 19, 21, 25, 26]. Numerical examples validate our theoretical results.

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