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GRAPHS WITH 4-STEINER CONVEX BALLS

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Abstract. Recently a new graph convexity was introduced, arising from Steiner intervals in graphs that are a natural generalization of geodesic intervals. The Steiner tree of a set W on k vertices in a connected graph G is a tree with the smallest number of edges in G that contains all vertices of W. The Steiner interval $I(W)$ of W consists of all vertices in G that lie on some Steiner tree with respect to W. Moreover, a set S of vertices in a graph G is k-Steiner convex, denoted g_k-convex, if the Steiner interval $I(W)$ of every set W on k vertices is contained in S. In this paper we consider two types of local convexities. In particular, for every $k > 3$, we characterize graphs with g_k-convex closed neighborhoods around all vertices of the graph. Then we follow with a characterization of graphs with g_4-convex closed neighborhoods around all g_4-convex sets of the graph.

1. Introduction

The study of abstract convexity began in the early fifties and is extensively studied in van de Vel's monograph [14], where the interval convexity is used for introducing it. The theory of axiomatic convexity is based on just three natural conditions, imposed on a family of subsets of a given set.

Definition 1. [14] A family $\mathcal{C}$ of subsets of a set X is called a convexity on X if

(C1) the empty set $\emptyset$ and universal set X are in $\mathcal{C}$;

(C2) $\mathcal{C}$ is stable for intersections, that is, if $\mathcal{D} \subseteq \mathcal{C}$ is non-empty, then $\bigcap \mathcal{D}$ is in $\mathcal{C}$;

(C3) $\mathcal{C}$ is stable for nested unions, that is, if $\mathcal{D} \subseteq \mathcal{C}$ is non-empty and totally ordered by inclusion, then $\bigcup \mathcal{D}$ is in $\mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a convex structure (convexity space) and the members of $\mathcal{C}$ are called convex sets.

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From axioms (C1) and (C2) it follows that for any subset A of a convex structure X there exists the smallest convex set $[A] = \bigcap\{C;\ A \subseteq C \in \mathcal{C}\}$ that contains A and it is called the *convex hull* of A.

Several natural convexities were studied also in graphs, and problems such as determining the convex hull of points and sets in graphs have been investigated. The most known graph convexities are defined using the intervals of a certain type, such as the geodesic or the monophonic intervals. The corresponding convexities, introduced by Farber and Jamison [8], are the *geodesic* and the *monophonic convexities*, in which the convex sets are defined simply as the sets S in which all intervals (of a certain type) between elements from S lie in S. So a set S of vertices in a graph G is *g-convex (m-convex)* if it contains the geodesic (monophonic) interval between every pair of vertices in S. For a more extensive study on this topic see [5].

Recently Cáceres and Oellermann [3] introduced a new convexity in graphs that is defined in terms of Steiner intervals in graphs. Recall that for a connected graph G, the *Steiner distance* $d(S)$ of a set S with at least two vertices is the size of the smallest connected subgraph of G containing S. Such connected subgraph is a tree and it is called *Steiner tree* for S. The *Steiner interval* $I(S)$ of S is the set of all vertices that lie on some Steiner tree for S. Steiner intervals have been studied in several papers, e.g. [1, 10, 13]. Now, a set S of vertices in a graph G is *k-Steiner convex* [3], denoted by g_k-convex, if the Steiner interval $I(R)$ of every set R on k vertices is contained in S, that is $I(R) \subseteq S$. Note that a set S is g_2-convex if and only if it is g-convex, thus the Steiner convexity is a natural generalization of the geodesic convexity.

Several properties of convexities have been investigated. One of them involves extreme points, where a point x of a convex set X is an *extreme point* of X if $X - \{x\}$ is convex. The problem in this case is to find characterizations of graphs for which a given convexity presents a convex geometry, i.e., a convexity where each convex set is the convex hull of its extreme points. Such characterizations for the g- and m-convexity are given in [8], for the g_3-convexity in [12] and for the m-convexity in [4].

Farber and Jamison [9] introduced local convexities of graphs and investigated them for geodesic and monophonic convexities. They introduced four types of local convexities, arising from convexities of j-balls of vertices and sets, respectively.

First some notations. In this paper all graphs will be finite with no loops and multiple edges. Let $G = (V, E)$ be a graph and $x, y \in V$. We will write $x \sim y$ if x and y are adjacent in G and $x \nsim y$, otherwise. The *distance* $d_G(x, y)$ between vertices x and y is the length of an *x, y-geodesic*, i.e., a shortest path between x and y in G. The set $N(v) = \{x \in V : x \sim v\}$ is the *open neighborhood* and $N[v] = N(v) \cup \{v\}$ the *closed neighborhood* of v. For a set S we define its *open neighborhood* as $N(S) = \bigcup_{x\in S} N(x)$ and its *closed neighborhood* as $N[S] = N(S) \cup S$. For $k \geq 1$, the *k-ball* $N_k[v]$ of v is the set of all vertices at distance at most k from v in G and the *k-ball* $N_k[S]$ of a set $S \subseteq V$ is the set of all vertices at distance at most k from some vertex of S in G. For a set S of vertices of a graph G we denote with $G[S]$ the subgraph of G induced by the vertices of S.

Now we present the above mentioned four local conditions which can be investigated with respect to any convexity associated with the vertex set of a graph $G = (V, E)$.

1. $N[v]$ is convex for every $v \in V$.
2. $N_j[v]$ is convex for every $v \in V$ and every $j \geq 1$.
3. $N[S]$ is convex for every convex set $S \subseteq V$.
4. $N_j[S]$ is convex for every convex set $S \subseteq V$ and every $j \geq 1$.

It is not hard to see that graphs in which $N[S]$ is convex for every convex set $S \subseteq V$ are exactly the graphs in which $N_j[S]$ is convex for every convex set $S \subseteq V$ and every $j \geq 1$ in terms of any convexity in graphs, see [6]. Thus it suffices to consider only the first three types of local convexities. Henning, Nielsen and Oellermann recently studied the same local convexities for the g_3-convexity [7]. They characterized the graphs in which $N[v]$ is g_3-convex for every vertex v of a graph and the graphs in which $N[S]$ is g_3-convex for every g_3-convex set S of a graph. Necessary conditions for the family of g_3-convex sets to satisfy the local convexity property 2 are given in [7]. It was subsequently shown in [2] that these conditions are also sufficient. A generalization of the m-convexity based on minimal trees between three or more vertices was introduced in [4] and the corresponding local convexity properties have been investigated in [11]. For $k > 3$, local k-Steiner convexities have not been studied yet, so this problem will be the main topic of this paper.

The paper is organized as follows. In Section 2 we prove several lemmas that are used in the rest of the paper. Then, in Section 3, we characterize graphs in which $N[v]$ is g_n-convex for every vertex v of G. We present such characterization for every $n > 1$. Then we follow with a section in which we characterize graphs with g_4-convex closed neighborhoods around g_4-convex sets. Finally we give some open problems about local Steiner convexities.

2. Preliminary Results

In this section we present some preliminary observations and prove several lemmas, which will be used during the paper.

A *subdivision* of a graph G is a graph obtained from G by inserting vertices of degree 2 into the edges of the graph.

Let G be a graph and T an x_1, x_n-path of G. To simplify notation, we denote such path by $T[x_1, x_n]$. The notation $T[x_1, x_i, x_j, x_n]$ specifies that the path T goes from x_1 to x_n and passes through x_i and x_j, where x_i is before x_j on this path.

First we describe the structure of any Steiner tree of a set R on four vertices. It is readily seen that a Steiner tree for a set of four vertices is isomorphic to one of the trees described in the following lemma.

Lemma 2. *Let G be a connected graph and $R = \{a, b, c, d\} \subseteq V(G)$. Then a Steiner tree T of R is one of the following graphs:*

- *a subdivision of $K_{1,4}$;*
- *a tree with four leaves, two vertices of degree 3 and some vertices (if any) of degree 2;*
- *a subdivision of $K_{1,3}$;*
- *a path.*

Let G be a graph, $v \in V(G)$ and let S be an arbitrary g_4-convex set such that $N[S]$ is not g_4-convex. From the definition of g_4-convexity it follows that there exists a set $R = \{a, b, c, d\}$ of four vertices in $N[S]$ such that $I(R) \nsubseteq N[S]$. Thus $I(\{a, b, c, d\}) \nsubseteq \{a, b, c, d\}$, which implies that the subgraph of G induced by the vertices of R is not connected, and therefore $d(R) \geq 4$.

The next two lemmas give some properties of g_4-convex sets whose closed neighborhoods are not g_4-convex. Similar results were established for g_3-convex sets [7].

Lemma 3. *Let G be a connected graph that contains a g_4-convex set S such that $N[S]$ is not g_4-convex. Let $R = \{a, b, c, d\} \subseteq N[S]$ be a set with the smallest $d(R)$ such that $I(R) \nsubseteq N[S]$. Then for any Steiner tree T of R with $V(T) \nsubseteq N[S]$ and each $y \in R$ the neighbors of y (if any) in $V(T) \setminus R$ are not in $N[S]$.*

Proof. Suppose some vertex in R, say a, has a neighbor $a' \notin R$ in T and assume that $a' \in N[S]$. Since $V(T) \nsubseteq N[S]$, $G[R]$ is disconnected and T contains at least one vertex $x \notin N[S]$. If a is a leaf of T, then $T - a$ is a Steiner tree for $\{a', b, c, d\} \subseteq N[S]$ of a smaller size than T, containing a vertex not in $N[S]$, contrary to our choice of R. Hence a is not a leaf of T. Therefore, using Lemma 2, T is either a path or a subdivision of $K_{1,3}$ with leaves b, c and d.

First let T be a path. Without loss of generality let $T = T[b, a, c, d]$. Note that the neighbors of b and d in T are either from R or $V \setminus N[S]$, otherwise we get a contradiction with the choice of R (in the same way as we did in the case where a is a leaf of T, since now b and d are leaves of T). First suppose that the neighbor of b in T is from $V \setminus N[S]$. Then the b, c-path of T is a Steiner tree for $\{b, a, a', c\} \subseteq N[S]$ of smaller size than T, containing a vertex not in $N[S]$, contrary to our choice of R. Hence a is the neighbor of b in T. In this case the a, d-path of T is a Steiner tree for $\{a, a', c, d\} \subseteq N[S]$ with smaller size than T containing a vertex not in $N[S]$, contrary to our choice of R.

Finally let T be a subdivision of $K_{1,3}$ with leaves b, c and d. Let z be the vertex of degree 3 in T and let x_b, x_c and x_d be the neighbors of b, c and d on T, respectively. Again it is clear that x_b, x_c, x_d are from R or $V \setminus N[S]$, otherwise we get a contradiction with the choice of R. Furthermore, if one of these three vertices is from R, say $x_b = a$, then the tree obtained from T by deleting b is a Steiner tree for $\{a, a', c, d\} \subseteq N[S]$

with smaller size than T, containing a vertex not in $N[S]$, which is a contradiction. Hence $x_b, x_c, x_d \in V \setminus N[S]$. Without loss of generality let a lies on the b, c-path of T (if a lies on the b, d-path of T then we change the role of c and d). If a' lies on the z, d-path of T then the subtree of T obtained from T by deleting the a', d-path of T except a' is a Steiner tree for $\{a, a', b, c\} \subseteq N[S]$ of a smaller size than T, containing a vertex not in $N[S]$, contrary to our choice of R. Let now a' lies on the b, c-path of T. If a or a' is the vertex of degree 3 in T then the b, c-path of T is a Steiner tree for $\{a, a', b, c\} \subseteq N[S]$ of a smaller size than T, containing a vertex not in $N[S]$, contrary to our choice of R. Hence both a and a' lie either on the b, z- or on the c, z-path of T. Without loss of generality let a, a' lie on the b, z-path of T and without loss of generality let a lies on the b, a'-path of T (otherwise we change the role of a and a'). Hence the subtree of T obtained from T by deleting all verteices, except a, on the b, a-path of T is a Steiner tree for $\{a, a', c, d\} \subseteq N[S]$ of a smaller size than T, containing a vertex not in $N[S]$, which is a final contradiction. ■

A graph G that contains no graph F as an induced subgraph is called an *F-free* graph. Furthermore, if G contains no graph from a family $\mathcal{F}$ as induced subgraph, then we say that G is *$\mathcal{F}$-free*.

Lemma 4. *Let G be a connected P_5-free graph which contains a g_4-convex set S such that $N[S]$ is not g_4-convex. Let $R = \{a, b, c, d\} \subseteq N[S]$ be a set with the smallest $d(R)$ such that $I(R) \nsubseteq N[S]$. Then for any Steiner tree T of R with $V(T) \nsubseteq N[S]$, we have*

$$V(T) \cap N[S] = R.$$

Proof. Assume, to the contrary, that there exists $a' \in V(T) \setminus R$ such that $a' \in N[S]$. From Lemma 3 it follows that a neighbor of $y \in R$ in T is either from R or from $V \setminus N[S]$. Indeed if y is a leaf of T and the neighbor x_y of y in T is from R then the tree obtained from T by the removal of y is a Steiner tree for $(R \setminus \{y\}) \cup \{a'\} \subseteq N[S]$ of a smaller size than T, containing a vertex not in $N[S]$, contrary to our choice of R. Therefore $x_y \in V \setminus N[S]$ for every leaf y of T. Now we distinguish three cases with respect to Lemma 2.

First let T be a path $T[a, b, c, d]$. Without loss of generality let a' lies on the a, c-path of T. Then the a, c-path of T is a Steiner tree for $\{a, b, c, a'\}$ of smaller size than T containing a vertex not in $N[S]$, a contradiction.

Now let T be a subdivision of $K_{1,3}$ with leaves a, b, c and with z being the vertex of degree 3, or let T be a subdivision of $K_{1,4}$ with leaves a, b, c, d and with z being the vertex of degree 4. Without loss of generality let a' lies on the a, z-path of T. Then the tree obtained from T by the removal of the a, a'-path in T, with the exception of a' (or the a, d-path in T with the exception of d if d lies on the a, a'-path of T) is a Steiner tree for $\{a', b, c, d\} \subseteq N[S]$ of smaller size than T containing a vertex not in $N[S]$, a contradiction.

Finally let T be a tree with four leaves, two vertices z_1, z_2 of degree 3 and maybe some other vertices of degree 2. Furthermore let z_1 lies on the a, z_2- and b, z_2-path of T. Since G is P_5-free and T is a Steiner tree, a' cannot lies on the z_1, z_2-path of T. Without loss of generality let a' lies on the a, z_1-path of T. Then the tree obtained from T by the removal of a, a'-path of T, with the exception of a' is a Steiner tree for $\{a', b, c, d\} \subseteq N[S]$ of smaller size than T containing a vertex not in $N[S]$, a contradiction. ■

Lemma 4 says that if there is a Steiner tree T of a set $R \subseteq N[S]$, with $|R| = 4$ and if $V(T) \nsubseteq N[S]$ then all vertices of T except those form R are outside $N[S]$. That is, vertices from R are the only vertices of T that lie in $N[S]$.

3. Graphs for which Closed Neighborhoods of Vertices Are g_n-Convex

In this section we use forbidden induced subgraphs to characterize graph $G = (V, E)$ in which $N[v]$ is g_n-convex ($n \geq 2$) for every $v \in V$.

Theorem 5. *Let $G = (V, E)$ be a graph and $n \geq 2$. Then the following assertions are equivalent:*

1. *$N[v]$ is g_n-convex for every $v \in V$.*
2. *G contains no induced subgraph G', where G' is a graph with*
   (a) $|V(G')| = n + 2$;
   (b) G' contains two nonadjacent vertices v_1, v_2;
   (c) $N_{G'}(v_2) \subseteq N_{G'}(v_1) = V(G') \setminus \{v_1, v_2\}$;
   (d) $G'[N_{G'}(v_1)]$ is disconnected;
   (e) $G'[V(G') - \{v_1\}]$ is a connected subgraph of G'.

Proof. Assume first that G satisfies Condition 1 of Theorem 5. Suppose that G contains an induced subgraph G' described in Condition 2 of Theorem 5 and let $R = N_{G'}(v_1) \subseteq N_{G'}[v_1] \subseteq N_G[v_1]$. Thus R contains n vertices. From Statements (2c) and (2d) it follows that $d(R) = n$. Using Statement (2e), we can find a tree T' with $V(T') = V(G') \setminus \{v_1\}$, which is a Steiner tree for R and is not contained in $N_{G'}[v_1]$ or $N_G[v_1]$, which is a contradiction.

For the converse suppose that G satisfies Condition 2 of Theorem 5. Let $v \in V$ and assume that $N[v]$ is not g_n-convex. Then there exists a subset $R = \{x_1, \ldots, x_n\} \subseteq N[v]$ and a Steiner tree T for R such that T contains a vertex x not in $N[v]$. It is clear that $d(R) = n$, $R \subseteq N(v)$, $G[R]$ is disconnected and thus $V(T) = R \cup \{x\}$. Let G' be a subgraph of G induced by $R \cup \{v, x\}$ and let $v_1 = v, v_2 = x$. Then G' satisfies all the statements from Condition 2, which is a contradiction. ■

The above theorem gives rise to the following problem.

Problem 1. Is there a characterization of graphs which satisfies any other type of local n-Steiner convexities, for $n > 3$?

In the following section we restrict ourselves to the case: $n = 4$.

4. Graphs for which Closed Neighborhoods of g_4-Convex Sets Are g_4-Convex

In this section we characterize graphs with g_4-convex closed neighborhoods around g_4-convex sets using forbidden induced subgraphs.

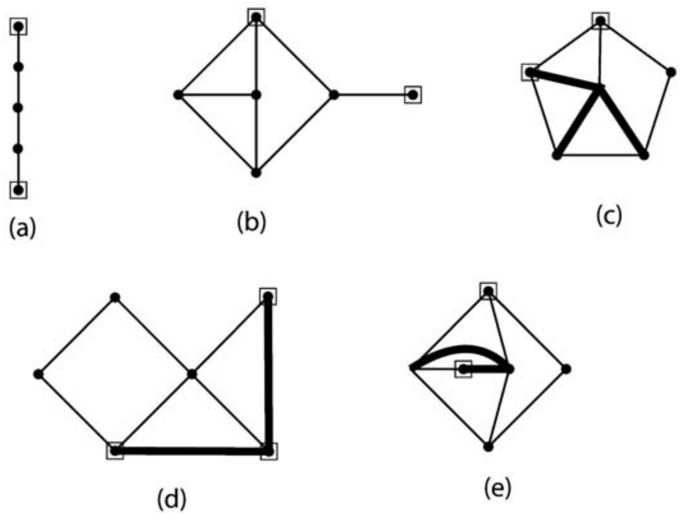

Fig. 1. Family $\mathcal{F}$ of forbidden induced subgraphs, where bold edges are optional (in $\mathcal{F}$ we have all combinations of graphs with or without such edges).

Let $\mathcal{F}$ be a family of graphs from Figure 1 and let

$\mathcal{F}' = \mathcal{F} \cup \{G';\ G'$ is a graph satisfying conditions (2a)-(2e) of Theorem 5 for $n = 4\}$.

To prove the main theorem we need the following result.

Lemma 6. *Let G be a connected $\mathcal{F}'$-free graph which contains a g_4-convex set S such that $N[S]$ is not g_4-convex. Let $R = \{a, b, c, d\} \subseteq N[S]$ be a set with the smallest $d(R)$ such that $I(R) \nsubseteq N[S]$. Then*

$$R \cap S = \emptyset.$$

Proof. Let T be a Steiner tree for R with $V(T) \nsubseteq N[S]$. Note that $G[R]$ is disconnected with at most three edges and $d(R) \geq 4$. Since G is a P_5-free it follows from Lemma 4 that $V(T) \cap N[S] = R$. If R is an independent set of vertices in T, then it follows from Lemma 3 that a neighbor of $y \in R$ is from $V \setminus N[S]$, and so y belongs to $N[S] \setminus S$. Now we will distinguish three cases with respect to the number of edges in $G[R]$.

First let $G[R]$ contain three edges. Since $G[R]$ is disconnected one vertex from R, say d, is isolated and hence from $N[S] \setminus S$, and the other three vertices from R form a triangle. Let d' be a neighbor of d in S. For the purpose of contradiction suppose that at least one vertex from $\{a, b, c\}$ is from S. Note that a vertex from S can not be adjacent to a vertex from $V \setminus N[S]$. Since a, b, c induce a triangle, there exists a Steiner tree T' for R which is a path and with $V(T') \nsubseteq N[S]$. Without loss of generality let $T' = T'[a, b, c, d]$ which implies that a vertex from S is a or b. We may also assume that $a \in S$, otherwise we investigate the path $T'' = T''[d, c, a, b]$ instead of T'. Since G is P_5-free we get, using Lemma 4, that $d_{T'}(c, d) = 2$. Let x be the neighbor of d and c on T'. Note that since $x \notin N[S]$ and $a \in S$, a and x are not adjacent. Since G is P_5-free, G contains at least one edge from $\{d'c, d'a\}$ and at least one edge from $\{bd', bx, cd'\}$. If $d' \nsim c$, then G contains a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)). On the other hand, if $d' \sim c$, then G contains a graph from $\mathcal{F}$ (shown in Figures 1(c), (d) or (e)) or a graph from $\mathcal{F}' \setminus \mathcal{F}$ as induced subgraph, depending which of the edges from $\{ad', bd', xb\}$ appear in G, a contradiction.

Now let $G[R]$ contains two edges. We distinguish two possibilities. First let $G[R]$ contain one isolated vertex, say d, and two adjacent edges ab and bc. Since isolated vertices are from $N[S] \setminus S$, there exists a neighbor d' of d in S. For the purpose of contradiction suppose that at least one vertex from $\{a, b, c\}$ is from S. Using Lemma 2, we get that there exists Steiner tree T' for R, with $V(T') \nsubseteq N[S]$, which is either a path $T' = T'[a, b, c, d]$ or a path $T' = T'[c, b, a, d]$ or a subdivision of $K_{1,3}$ with leaves a, c, d in which the vertex of degree 3 is b. Since a and c are symmetrical we can assume without loss of generality that T' is either a path $T' = T'[a, b, c, d]$ or a subdivision of $K_{1,3}$ with leaves a, c, d and with b being the vertex of degree 3.

First let $T' = T'[a, b, c, d]$. Thus $c, d \notin S$ and at least one vertex from $\{a, b\}$ is from S. In any case we get that $d_{T'}(c, d) = 2$, since G is P_5-free. Let $x \notin N[S]$ be the neighbor of d and c on T'. Since G is P_5-free and at least one vertex from R is from S, x is adjacent to exactly one vertex from $\{a, b\}$. First suppose that $x \sim a$ and $x \nsim b$. Since G is P_5-free there is at least one edge from $\{d'b, d'c\}$. If $d' \nsim c$, then G contains a forbidden induced subgraph from $\mathcal{F}$ (shown in Figure 1(c)). Thus we may assume that $d' \sim c$. Again using the fact that G is P_5-free we get that G contains at least one edge from $\{ad', bd'\}$. Depending on which of those two edges appear in G, we get one of forbidden induced subgraphs from $\mathcal{F}$ (shown in Figures 1(c) or (e)) or from $\mathcal{F}' \setminus \mathcal{F}$. Therefore we may assume that $x \nsim a$ and $x \sim b$. Since G is P_5-free there is at least one edge from $\{ad', bd'\}$. If $a \sim d'$, then using the fact that G is P_5-free, we get that $c \sim d'$ which implies that G contains one of forbidden induced subgraphs from $\mathcal{F}$ (shown in Figure 1(c)) or from $\mathcal{F}' \setminus \mathcal{F}$ depending whether b is adjacent to d' or not. Thus we may assume that $a \nsim d'$ and therefore $b \sim d'$. In this case G contains a forbidden induced subgraph from $\mathcal{F}$ (shown in Figures 1(d) or (e)) depending whether c is adjacent to d' or not, a contradiction.

Now let T' be a subdivision of $K_{1,3}$ with leaves a, c, d where the vertex of degree 3 is b. From Lemma 4 it follows that all vertices of the b, d-path in T' are from $V(G) \setminus N[S]$. Thus also $b \notin S$ and hence at least one vertex a or c is from S and without loss of generality let $a \in S$. Since G is P_5-free and $a \in S$, $d_{T'}(b, d) = 2$. Let $x \notin N[S]$ be the neighbor of d and b on T'. Again using the fact that G is P_5-free, it follows that G contains at least one edge from $\{bd', ad'\}$ and at least one edge from $\{bd', cd', xc\}$. If $x \sim c$, then there exists a Steiner tree $T'' = T''[a, b, c, d]$ on the same vertex set as T', thus we get a contradiction as in the previous case. Hence we may assume that $x \nsim c$. Therefore G contains at least one edge from $\{bd', cd'\}$. First suppose that $b \nsim d'$. Then G contains a forbidden induced subgraph from $\mathcal{F}$ (shown in Figure 1(c)), a contradiction. Thus we may assume that $b \sim d'$ and hence G contains a graph from $\mathcal{F}$ (shown in Figure 1(d)) or from $\mathcal{F}' \setminus \mathcal{F}$ as induced subgraph, a contradiction.

To conclude the case when $G[R]$ contains two edges, without loss of generality let $a \sim b, c \sim d$. For the purpose of contradiction suppose that at least one vertex from $\{a, b, c, d\}$ is from S. Using Lemma 2, we get that there exists a Steiner tree T' for R, with $V(T') \nsubseteq N[S]$, which is a path and without loss of generality we may assume that $T' = T'[a, b, c, d]$, where all vertices of b, c-path in T' are from $V(G) \setminus N[S]$. Thus a vertex from S is either a or d. Without loss of generality let $a \in S$. Since G is P_5-free and $a \in S$, $d_{T'}(b, c)$=2. Let $x \notin N[S]$ be the neighbor of b and c on T'. Since G is P_5-free $d \sim x$ and thus $d \notin S$. Let $d' \in S$ be a neighbor of d in S. Since G is P_5-free, G contains at least one edge from $\{d'b, d'a\}$. If $c \sim d'$, then G contains one of forbidden induced subgraphs from $\mathcal{F}$ (shown in Figures 1(b) or (c)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending which edges from $\{bd', ad'\}$ appear in G, a contradiction. Hence $c \nsim d'$. If just one edge from $\{d'b, d'a\}$ appear in G, then G contains one of forbidden induced subgraphs from $\mathcal{F}$ (shown in Figures 1(a) or (c)), a contradiction. Therefore $d' \sim b, d' \sim a$. Since $c \in N(S)$, there exists a neighbor c' of c in S. Now we interchange the roles of c and d from previous case and get a contradiction (the vertices a, b, c, d, x, c' induce a forbidden subgraph), unless c' is adjacent to both a and b and $d \nsim c'$. Now the vertices b, d', a, c', x, c induce a forbidden induced subgraph from $\mathcal{F}' \setminus \mathcal{F}$, a contradiction.

Finally let $G[R]$ contain one edge and, without loss of generality, let a and b be adjacent. Thus c and d are isolated and hence from $N[S] \setminus S$. For the purpose of contradiction suppose without loss of generality that $a \in S$. Since $N[S] \cap V(T) = R$ and $a \in S$, the only neighbor of a in T is b, which is not a leaf of T and thus it is from $N[S] \setminus S$, because of Lemma 3. Since ab is the only edge in $G[R]$ and G is

P_5-free it follows, from Lemma 4, that T is not a path. Since b is not a leaf of T, T is a subdivision of $K_{1,3}$ with leaves a, c, d and let z be the vertex of degree 3 on T. Let d' be a neighbor of d in S. Since G is P_5-free and $a \in S$, we get that $d(R) \in \{4, 5\}$ and b is adjacent to z. (Note that it is also possible that b is the vertex of degree 3 on T. But then there exists a Steiner tree T'' for R with leaves a, c, d and with b being the neighbor of the vertex of degree 3 on T'', such that $V(T'') = V(T) \nsubseteq N[S]$. Thus we can replace T with T''.) Without loss of generality let z also be adjacent to c. Hence the distance between z and d is one or two. First suppose that $d(R) = 5$ and let x_d be the neighbor of d on T. Note that G contains at most one edge from $\{bx_d, cx_d\}$ since T is a Steiner tree. The fact that G is P_5-free, implies that G contains at least one edge from $\{cx_d, cd'\}$ and at least one edge from $\{bx_d, bd'\}$. In all cases the vertices d', d, x_d, z, c, b induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)), a contradiction. Hence we may assume that $d(R) = 4$. Since G is P_5-free and $a \in S$, d' is adjacent to at least one vertex from $\{a, b\}$. If $a \sim d'$, then the vertices a, b, z, d, d', c induce in G one of forbidden subgraphs from $\mathcal{F}$ (shown in Figures 1(a) or (c)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on which edges (if any) from $\{bd', cd'\}$ appear in G. It remains to check the case when $b \sim d'$ and $a \nsim d'$. If $c \sim d'$ then the vertices a, b, z, d, d', c induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(e)), otherwise there exists a neighbor c' of c in S. In this case we get a contradiction in the same way as in previous case (we interchange the roles of c and d), unless $c' \sim b, c' \nsim a, c' \nsim d$. But now the vertices z, c, c', b, a, d' induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(d)), a final contradiction. ∎

Theorem 7. *If a connected graph $G = (V, E)$ is $\mathcal{F}'$-free then $N[S]$ is g_4-convex for every g_4-convex set S.*

Proof. Suppose that G is $\mathcal{F}'$-free. Let $S \subseteq V$ be an arbitrary g_4-convex set in G. For the purpose of contradiction suppose that $N[S]$ is not g_4-convex. Let $R = \{a, b, c, d\} \subseteq N[S]$ be a set with the smallest $d(R)$ such that $I(R) \nsubseteq N[S]$. By Lemma 6, $R \subseteq N[S] \setminus S$. Let T be a Steiner tree for R containing a vertex in $V \setminus N[S]$. From Lemma 4, it follows that $V(T) \cap N[S] = R$. Since the graphs from $\mathcal{F}' \setminus \mathcal{F}$ are forbidden, it follows from Theorem 5 that $N[v]$ is g_4-convex for every $v \in V$. Thus $R \nsubseteq N(v)$ for every $v \in S$.

Suppose $G[R]$ contains no edges. Since G contains no induced subgraphs from $\mathcal{F}$ (shown in Figures 1(a) and (c)) and every vertex from R has a neighbor in S, there exists a Steiner tree T' for R with $V(T') \nsubseteq N[S]$ such that T' is subdivision of $K_{1,4}$. Hence the vertices from R are the leaves of T'. Let x be the vertex of degree 4 in T' and let a' be a neighbor of a in S. Since T' is a Steiner tree and G is P_5-free, $d(x, y) \leq 2$ for every $y \in R$. First we prove that $d(R) \leq 5$. Suppose that $d(R) \geq 6$. Then there exist two vertices from R, say a and b, such that $d(a, x) = 2$ and $d(b, x) = 2$. Let x_a and x_b be the neighbors of a and b on T', respectively. Since G is P_5-free, $x_a \sim x_b$ and thus also $a' \sim b$. Now the vertices a', a, x_a, x_b, b, x induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)), a contradiction. Hence $d(R) \in \{4, 5\}$. First suppose that $d(R) = 5$ and without loss of generality let $d(a, x) = 2$ and let x_a be the neighbor of a on T'. Since T' is a Steiner tree, x_a is not adjacent to at least one vertex from $\{b, c, d\}$, say b. Therefore the vertices a', a, x_a, x, b, c either induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)) or they induce a subgraph that contains an induced P_5, which both leads to a contradiction. Thus $d(R) = 4$, i.e., all vertices from R are adjacent to x. Since G is $\mathcal{F}'$-free not all vertices from R can be adjacent to a'. We may assume that $b \nsim a'$. Let b' be a neighbor of b in S. Since G is P_5-free there is at least one edge from $\{a'b', ab'\}$ in G. First suppose that $a' \sim b'$. Then G contains a forbidden induced subgraph from $\mathcal{F}$ (shown in Figure 1(c)) or from $\mathcal{F}' \setminus \mathcal{F}$. Thus we may assume that $a' \nsim b'$ and hence $a \sim b'$. Since b' is not adjacent to all vertices from R, we may assume that $c \nsim b'$. Let c' be a neighbour of c that is in S. Now we change the role of a with b and the role of b with c and get $b \sim c', b' \nsim c'$ (in the same way as we got $a \sim b'$, $a' \nsim b'$). In this case G contains a forbidden induced subgraph from $\mathcal{F}$ (shown in Figures 1(a) or (e)), a contradiction.

Let $G[R]$ contains three edges. Since $G[R]$ is disconnected we may assume that d is isolated and a, b, c induce a triangle. Then there exists a Steiner tree T' for R with $V(T') \nsubseteq N[S]$ such that T' is a path. Without loss of generality let $T' = T'[a, b, c, d]$. Let d' be a neighbor of d in S. We will first prove that $d(R) = 4$. Suppose that $d(R) > 4$ and thus $d_{T'}(c, d) \geq 3$. Lemma 4 implies, that the inner vertices of the c, d-subpath of T' are not adjacent to d'. Since G is P_5-free, $d_{T'}(c, d) = 3$ and $c \sim d'$. Thus the path a, b, c, d', d is a Steiner tree for R, contrary to the assumption that $d(R) > 4$. Hence $d(R) = 4$. Let x be a common neighbor of c and d on T'. First suppose that $c \nsim d'$. Since G is P_5-free G contains at least one edge from $\{xb, d'b\}$ and at least one edge from $\{xa, d'a\}$. If G does not contain both edges xa and xb, then G contains a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)), a contradiction. Hence we may assume that $x \sim a$ and $x \sim b$. If d' is adjacent to at least one of a or b then G contains a forbidden subgraph from $\mathcal{F}' \setminus \mathcal{F}$, a contradiction. Let c' be a neighbor of c in S. Since G is P_5-free, G contains at least one edge from $\{c'd', c'd\}$. Depending on which of those two edges appear in G, we get one of forbidden induced subgraphs from $\mathcal{F}$ (shown in Figure 1(c)) or from $\mathcal{F}' \setminus \mathcal{F}$. Thus we may assume that $c \sim d'$. If there is no edge from $\{d'a, d'b\}$ or no edge from $\{xa, xb\}$ then G contains a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(d)) or from $\mathcal{F}' \setminus \mathcal{F}$. Therefore G contains exactly one edge from $\{d'a, d'b\}$ and exactly one edge from $\{xa, xb\}$ (two edges from either set would give a forbidden subgraph from $\mathcal{F}' \setminus \mathcal{F}$). Since a and b appear symmetrically it is enough to check two cases. In the first case let x and d' have the same neighbor from $\{a, b\}$, which implies that G contains a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(e)), a contradiction. In the second case let $x \sim a, d' \sim b$. Then G contains forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)), a contradiction.

Now let $G[R]$ contain two edges. We distinguish two possibilities. First let $G[R]$

contain one isolated vertex, say d and two adjacent edges ab and bc. Let d' be a neighbor of d in S. Using Lemma 2 we get that there exists a Steiner tree T' for R, with $V(T') \nsubseteq N[S]$, which is either a path $T' = T'[a,b,c,d]$ or a path $T' = T'[c,b,a,d]$ or a subdivision of $K_{1,3}$ with leaves a,c,d in which the vertex of degree 3 is b. Since a and c are symmetrical we can assume without loss of generality that T' is either a path $T' = T'[a,b,c,d]$ or a subdivision of $K_{1,3}$ with leaves a,c,d and with b being the vertex of degree 3.

First let $T' = T'[a,b,c,d]$. Since G is P_5-free and d' cannot be adjacent to all vertices of R (otherwise G contains a subgraph of $\mathcal{F}' \setminus \mathcal{F}$), we get that $d_{T'}(c,d) = 2$. Let $x \notin N[S]$ be the neighbor of d and c on T'. Since G is P_5-free G contains at least one edge from $\{d'c, d'b, bx\}$ and at least one edge from $\{ax, bx\}$. Suppose that $x \nsim b$ and thus $x \sim a$. If $b \sim d'$ then the vertices a,b,c,x,d,d' induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on which of the the edges (if any) from $\{ad', cd'\}$ appear in G. Thus we may assume that $b \nsim d'$. Since there is at least one edge from $\{d'c, d'b, xb\}$, d' must be adjacent to c. Since G is P_5-free $a \sim d'$, otherwise the vertices a,b,c,d',d induce a P_5. But now G contains a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(e)), a contradiction. Therefore we may assume that b and x are adjacent. Since G is P_5-free there is at least one edge from $\{d'b, d'a, xa\}$. If $d' \sim a$ then G contains one of forbidden subgraphs from $\mathcal{F}$ (shown in Figure 1(c)) or from $\mathcal{F}' \setminus \mathcal{F}$, a contradiction. Thus we may assume the $d' \nsim a$. If $d' \sim b$, then G contains one of forbidden subgraphs from $\mathcal{F}$ (shown in Figures 1(d) or (e)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on which of the edges (if any) from $\{d'c, xa\}$ appear in G. Therefore also d' and b are not adjacent and thus $a \sim x$. If $d' \sim c$, then G contains a forbidden subgraph from $\mathcal{F}' \setminus \mathcal{F}$, a contradiction. Thus $d' \nsim c$ and let c' be a neighbor of c in S. Since G is P_5-free there is at least one edge from $\{d'c', c'd\}$. If $c' \sim d$, then G contains a forbidden subgraph from $\mathcal{F}' \setminus \mathcal{F}$. Therefore we may assume that $c' \nsim d$ and $c' \sim d'$. But now the vertices c', d', d, x, c, b induce a forbidden subgraph from $\mathcal{F}$ shown in Figure 1(c)), a contradiction.

Now let T' be a subdivision of $K_{1,3}$ with leaves a,c,d where the vertex of degree 3 is b. From Lemma 4 it follows that all vertices of the b,d-path in T' are from $V(G) \setminus N[S]$. Let d' be a neighbor of d in S. Since G is P_5-free and d' cannot be adjacent to all vertices from R, $d_{T'}(b,d) = 2$. Let $x \notin N[S]$ be the neighbor of d and b on T'. Again using the fact that G is P_5-free, it follows that G contains at least one edge from $\{bd', ad', xa\}$ and at least one edge from $\{bd', cd', xc\}$. If $x \sim a$ or $x \sim c$, then there exists a Steiner tree T'' which is a path and contains $x \notin N[S]$. Therefore we get a contradiction as in previous case where T' is a path. Thus we may assume that $x \nsim a, x \nsim c$. First suppose that $b \nsim d'$ and hence $d' \sim c, d' \sim a$. Then G contains a forbidden induced subgraph from $\mathcal{F}$ (shown in Figure 1(c)), a contradiction. Therefore we may assume that $b \sim d'$. In this case G contains a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(d)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on which edges (if any) from $\{ad', cd'\}$ appear in G, a contradiction.

To conclude the case when $G[R]$ contains two edges, without loss of generality let $a \sim b, c \sim d$. Using Lemma 2 we get that there exists a Steiner tree T' for R, with $V(T') \nsubseteq N[S]$, which is a path and without loss of generality we may assume that $T' = T'[a, b, c, d]$, where all vertices of the b, c-path in T' are from $V(G) \setminus N[S]$. Let d' be a neighbor of d in S. Since G contains no graph from $\mathcal{F}$ (shown in Figures 1(a) and (c)), $d_{T'}(b, c)$=2. Let $x \notin N[S]$ be the neighbor of b and c on T'. Using the fact that G is P_5-free we get that G contains at least one edge from $\{d'c, d'b, dx\}$ and at least one edge from $\{ax, dx\}$. First suppose that $d \sim x$. Since G is P_5-free there is at least one edge from $\{d'b, d'a, ax\}$. Let first $a \sim x$. Then $a \nsim d', b \nsim d'$, otherwise G contains a forbidden subgraph from $\mathcal{F}' \setminus \mathcal{F}$. Let a' be a neighbor of a in S. Since G is P_5-free, there is at least one edge from $\{d'a', a'd\}$. Clearly $a' \nsim d$, otherwise G contains a forbidden subgraph from $\mathcal{F}' \setminus \mathcal{F}$. Thus $a' \sim d'$ and now the vertices a, a', d', d, x, c induce a forbidden induced subgraph from $\mathcal{F}$ (shown in Figure 1(c)). Therefore we may assume that $a \nsim x$ and therefore there is at least one edge from $\{d'b, d'a\}$. If $c \sim d'$, then G contains one of forbidden subgraphs from $\mathcal{F}$ (shown in Figures 1(b) or (c)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on which edges from $\{bd', ad'\}$ appear in G, a contradiction. Hence $c \nsim d'$. If at most one edge from $\{d'b, d'a\}$ appear in G, then G contains one of forbidden subgraph from $\mathcal{F}$ (shown in Figures 1(a) or (c)), a contradiction. Therefore $d' \sim b, d' \sim a$. Let c' be a neighbor of c in S. Now we interchangechange the roles of c and d from previous case and get a contradiction (the vertices a, b, c, d, x, c' induce a forbidden subgraph), unless c' is adjacent to both a and b and $d \nsim c'$. Then the vertices b, d', a, c', x, c induce a forbidden subgraph from $\mathcal{F}' \setminus \mathcal{F}$, a contradiction. Finally let $d \nsim x$ which implies that $a \sim x$. Let a' be a neighbor of a in S (it is also possible that $a' = d'$). Now we distinguish a' instead of d' and change the roles of a and d from previous case (the case where x is adjacent to d and d' is a neighbor of d in S) and gets a contradiction in the same lines.

Finally let $G[R]$ contain one edge and without loss of generality let a and b be adjacent. Since G is P_5-free there exists a Steiner tree T' for R that is a subdivision of $K_{1,3}$ with leaves a, c, d or b, c, d such that $V(T') \nsubseteq N[S]$. Without loss of generality we may also assume that the leaves of T' are a, c, d, otherwise we change the roles of a and b. Let z be the vertex of degree 3 in T' and let d' be a neighbor of d in S. Using the fact that the subgraphs from $\mathcal{F}$ (shown in Figures 1(a) and (c)) are forbidden, we get that $d(R) \in \{4, 5\}$ and we may assume that b is adjacent to z (if $b \nsim z$ then there exists a Steiner tree T'' with $V(T'') = V(T')$ such that b is adjacent to the vertex of degree 3 on T''). Without loss of generality let z also be adjacent to c. Hence the distance between z and d is 1 or 2. First suppose that $d(R) = 5$ and let x_d be the neighbor of d on T'. Note that since T' is a Steiner tree, G contains at most one edge from $\{bx_d, cx_d\}$. Using the fact that G is P_5-free, we get that G contains at least one edge from $\{cx_d, cd'\}$ and at least one edge from $\{bx_d, bd'\}$. In all cases the vertices d', d, x_d, z, c, b induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)), a contradiction. Hence we may assume that $d(R) = 4$. We distinguish two cases.

Case 1: $a \nsim z$. Since G is P_5-free, d' is adjacent to at least one vertex from $\{a, b\}$. If $a \sim d'$, then the vertices a, b, z, d, d', c induce in G one of the forbidden subgraphs from $\mathcal{F}$ (shown in Figures 1(a) or (c)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on which edges (if any) from $\{bd', cd'\}$ appear in G. It remains to check the case when $b \sim d'$ and $a \nsim d'$. If $c \sim d'$, then the vertices a, b, z, d, d', c induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(e)), otherwise there exists a neighbor c' of c in S. In this case we get a contradiction in the same way as in previous case (we interchange the roles of c and d), unless $c' \sim b, c' \nsim a, c' \nsim d$. But now the vertices z, c, c', b, a, d' induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(d)), a contradiction.

Case 2: $a \sim z$.

Case 2.1: $a \sim d'$. In this case we get forbidden induced subgraph from $\mathcal{F}$ (shown in Figure 1(e)) or from $\mathcal{F}' \setminus \mathcal{F}$ depending which edges from $\{bd', cd'\}$ appear in G, unless $b \sim d'$ and $c \nsim d'$. Thus let c' be a neighbor of c in S. Since G is P_5-free there is at least one edge from $\{dc', d'c'\}$ and at least one edge from $\{bc', d'c'\}$. If $d \nsim c'$, then the vertices c, c', d', d, z, b induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(c)). Hence $d \sim c'$. Similarly we get that $b \sim c'$. Then the vertices b, c, d, c', d', z induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(e)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on whether d' is adjacent to c' or not, a contradiction.

Case 2.2: $a \nsim d'$. Let a' be a neighbor of a in S. If $b \sim d'$, then the vertices b, a, z, d, d', c induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(d)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on whether c is adjacent to d' or not. Therefore let $b \nsim d'$. Since G is P_5-free G contains at least one edge from $\{a'd', a'd\}$. If there is just one edge from this set then G contains a forbidden subgraph from $\mathcal{F}$ (shown in Figures 1(a), (b) or (c)), depending on which of those two edges appear in G and whether b is adjacent to a' or not. Thus we may assume that $a' \sim d', a' \sim d$. It is also clear that $b \nsim a'$, otherwise G contains a forbidden subgraph from $\mathcal{F}' \setminus \mathcal{F}$. Now the vertices a', a, z, d, b, c induce a forbidden subgraph from $\mathcal{F}$ (shown in Figure 1(d)) or from $\mathcal{F}' \setminus \mathcal{F}$, depending on whether c is adjacent to a' or not, a final contradiction. ∎

Theorem 8. *If in a connected graph G, $N[S]$ is g_4-convex for every g_4-convex set S, then G is $\mathcal{F}'$-free.*

Proof. Let G be a connected graph in which $N[S]$ is a g_4-convex set for every g_4-convex set S. Let $v \in V(G)$. As $S = \{v\}$ is g_4-convex, it follows from Theorem 5 that G is $(\mathcal{F}' \setminus \mathcal{F})$-free. Suppose now that G contains a graph H from family $\mathcal{F}$ as induced subgraph. Let S be the set of vertices in H marked with square in Figure 1. Since the set S contains at most three vertices it is a g_4-convex set and it is easy to check that $N[S]$ is not a g_4-convex set, a contradiction. Thus G is also an $\mathcal{F}$-free graph. ∎

From Theorems 7 and 8 we get the main result of this section.

Corollary 9. *A connected graph G is $\mathcal{F}'$-free if and only if $N[S]$ is g_4-convex for every g_4-convex set S.*

We finish this section with some obvious remarks. It is easy to see that if in a graph $G = (V, E)$, $N[S]$ is g_4-convex for every g_4-convex set S, then $N_j[v]$ is g_4-convex for every $v \in V$ and every $j \geq 1$, see [7, Observation 1.1]. This result together with Corollary 9 give the following result.

Corollary 10. *If $G = (V, E)$ is a connected $\mathcal{F}'$-free graph, then $N_j[v]$ is g_4-convex for every $v \in V$ and every $j \geq 1$.*

Since, for example, P_5 is not $\mathcal{F}'$-free, but $N_j[v]$ is g_4-convex for every vertex v of P_5 and every $j \geq 1$, the converse of the last corollary does not hold.

5. Open Problems

We complete the paper with some open problems about local Steiner convexities.

Problem 2. Is there any characterization of those graphs G, in which, for every $j \geq 1$, a j-ball around every vertex of G is g_4-convex?

It is almost impossible to generalize Corollary 9 to g_n-convexity, since the number of forbidden substructures grow rapidly with n. But maybe another approach would solve the problem.

Problem 3. Let $n \in \mathbb{N},\ n \geq 3$. Is there a characterization of those graphs, in which the local convexity property 3 is satisfied with respect to n-Steiner convexity?

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