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NONLINEAR SCALARIZATION CHARACTERIZATIONS OF E-EFFICIENCY IN VECTOR OPTIMIZATION

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Abstract. In this paper, two kinds of nonlinear scalarization functions are applied to characterize E-efficient solutions and weak E-efficient solutions of vector optimization problems and some nonlinear scalarization characterizations are obtained. Some examples also are given to illustrate the main results.

1. INTRODUCTION

It is well known that approximate solutions have been playing an important role in optimization theory and applications. One of the most important reasons is that approximate solutions can be obtained by using iterative algorithms or heuristic methods. During the recent years, many scholars have been introduced several concepts of approximate solutions of vector optimization problems and studied some characterizations of these approximate solutions. Especially, Gutiérrez et al. introduced a new kind of concept of approximate solutions named as $C(\epsilon)$-efficiency, which extends and unifies some known different notions of approximate solutions in [1-2]. Gao et al. introduced a new kind of approximate proper efficiency by means of co-radiant set and established some linear and nonlinear scalarization characterizations of this kind of approximate solutions in [3]. Flores-Bazán and Hernández introduced a kind of unified concept of vector optimization problems and obtained some scalarization characterizations in a unified frame in [4].

Recently, Chicoo et al. proposed the concept of E-efficiency by means of improvement sets in finite dimensional Euclidean space in [5]. E-efficiency unifies some known exact and approximate solutions of vector optimization problems. Gutiérrez et al. extended the notions of improvement sets and E-efficiency to a Hausdorff locally convex topological linear space in [6]. Furthermore, Zhao and Yang proposed a unified

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stability result with perturbations by virtue of improvement sets under the convergence of a sequence of sets in the sense of *Wijsman* in [7]. Zhao et al. established linear scalarization theorem and Lagrange multiplier theorem of weak E-efficient solutions under the nearly E-subconvexlikeness in [8]. Moreover, Zhao and Yang also introduced a kind of proper efficiency, named as E-Benson proper efficiency which unifies some proper efficiency and approximate proper efficiency, and obtained some linear scalarization characterizations of the E-Benson proper efficiency in [9].

Motivated by the works of [5-6, 8, 10-11], by making use of two kinds of nonlinear scalarization functions, we establish some nonlinear scalarization results of E-efficient solutions and weak E-efficient solutions for a class of vector optimization problems. We also give some examples to illustrate the main results.

2. Preliminaries

Let X be a linear space and Y be a real Hausdorff locally convex topological linear space. For a subset A of Y, we denote the topological interior, the closure, the boundary and the complement of A by int$A$, cl$A$, ∂A and $Y \setminus A$, respectively. The cone generated by A is defined as

$$\text{cone}A = \bigcup_{\alpha \geq 0} \alpha A.$$

A cone $A \subset Y$ is pointed if $A \cap (-A) = \{0\}$. Y^* denotes the topological dual space of Y. The positive dual cone of a subset $A \subset Y$ is defined as

$$A^+ = \{y^* \in Y^* | \langle y^*, y \rangle \geq 0, \forall y \in A\}.$$

Let K be a closed convex pointed cone in Y with nonempty topological interior. For any $x, y \in Y$, we define

$$x \leq_K y \Leftrightarrow y - x \in K.$$

Consider the following vector optimization problem:

$$\text{(VP)} \quad \begin{array}{l} \min f(x) \\ \text{s.t. } x \in S, \end{array}$$

where $f : X \to Y$ and $\emptyset \neq S \subset X$.

Definition 2.1. ([5-6]). Let $E \subset Y$. If $0 \notin E$ and $E + K = E$, then E is said to be an improvement set with respect to K. We denote the set of improvement sets in Y by $\mathfrak{T}_Y$.

Remark 2.1. Clearly, $\emptyset \in \mathfrak{T}_Y$. Moreover, from Theorem 3.1 in [9], it follows that int$E \neq \emptyset$ if $E \neq \emptyset$. In this paper, we assume that $E \neq \emptyset$.

Definition 2.2. ([6]). Let $E \in \mathfrak{T}_Y$. A feasible point $\overline{x} \in S$ is said to be an E-efficient solution of (VP) if

$$(f(\overline{x}) - E) \cap f(S) = \emptyset.$$

We denote this by $\overline{x} \in \mathrm{AE}(f, S, E)$.

Definition 2.3. ([6]). Let $E \in \mathfrak{T}_Y$. A feasible point $\overline{x} \in S$ is said to be a weak E-efficient solution of (VP) if

$$(f(\overline{x}) - \mathrm{int}E) \cap f(S) = \emptyset.$$

We denote this by $\overline{x} \in \mathrm{WAE}(f, S, E)$.

Consider the following scalar optimization problem

$$\text{(P)} \quad \min_{x \in Z} \phi(x),$$

where $\phi : X \to \mathbb{R}, \emptyset \neq Z \subset X$. Let $\epsilon \geq 0$ and $\overline{x} \in Z$. If $\phi(x) \geq \phi(\overline{x}) - \epsilon, \forall x \in Z$, then $\overline{x}$ is called an ϵ-minimal solution of (P). The set of all ϵ-minimal solutions is denoted by $\mathrm{AMin}(\phi, \epsilon)$. Moreover, If $\phi(x) > \phi(\overline{x}) - \epsilon, \forall x \in Z$, then $\overline{x}$ is called a strictly ϵ-minimal solution of (P). The set of all strictly ϵ-minimal solutions is denoted by $\mathrm{SAMin}(\phi, \epsilon)$.

Lemma 2.1. ([12]). *Let Y be a Hausdorff topological linear space and $A \subset Y$ be a convex set with nonempty interior. Then*

$$\mathit{int}A = \{y \in Y | \langle y^*, y \rangle > 0, \forall y^* \in A^+ \setminus \{0\}\}.$$

Lemma 2.2. ([12]). *Let Y be a Hausdorff topological linear space and $A \subset Y$ be a convex set. If $x \in A$ and there exists $y^* \in A^+ \setminus \{0\}$ such that $\langle y^*, x \rangle = 0$, then $x \in \partial A$.*

3. Scalarization of E-Efficiency via $\varphi_{q,E}$

In this section, we characterize E-efficient solutions and weak E-efficient solutions of (VP) via the nonlinear scalarization function $\varphi_{q,E}$ proposed by Göpfert et al. in [10]. Assume that Y be a real Hausdorff locally convex topological linear space and $E \in \mathfrak{T}_Y$ be closed.

Let $\varphi_{q,E} : Y \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$\varphi_{q,E}(y) = \inf\{s \in \mathbb{R} | y \in sq - E\}, y \in Y,$$

with $\inf \emptyset = +\infty$.

Lemma 3.1. *Let $E \in \mathfrak{T}_Y$ and $q \in intK$. Then the function $\varphi_{q,E}$ is continuous such that*

$$\{y \in Y | \varphi_{q,E}(y) < c\} = cq - intE, \forall c \in \mathbb{R},$$
$$\{y \in Y | \varphi_{q,E}(y) = c\} = cq - \partial E, \forall c \in \mathbb{R},$$
$$\varphi_{q,E}(-E) \leq 0, \qquad \varphi_{q,E}(-\partial E) = 0.$$

Proof. From $E \in \mathfrak{T}_Y$, $q \in \text{int}K$ and Proposition 2.3.4 in [10], it follows that

(i) $E + \mathbb{R}_{++}q \subset \text{int}E$;

(ii) $Y = \mathbb{R}q - E$;

(iii) $\forall y \in Y, \exists s \in \mathbb{R}$ such that $y + sq \notin E$.

Hence from (i)-(iii) and Theorem 2.3.1 in [10], the conclusion is obvious. ■

Consider the following scalar optimization problem

$$(\text{P}_{q,y}) \quad \min_{x \in S} \varphi_{q,E}(f(x) - y),$$

where $y \in Y, q \in \text{int}K$. Denote $\varphi_{q,E}(f(x)-y)$ by $(\varphi_{q,E,y} \circ f)(x)$, the set of ϵ-minimal solutions of $(\text{P}_{q,y})$ by AMin$(\varphi_{q,E,y} \circ f, \epsilon)$ and the set of strictly ϵ-minimal solutions of $(\text{P}_{q,y})$ by SAMin$(\varphi_{q,E,y} \circ f, \epsilon)$.

Lemma 3.2. *Let $E \in \mathfrak{T}_Y$ be a convex set. Then int$(E \cap K) \neq \emptyset$.*

Proof. We first prove $E \cap K \neq \emptyset$. If $E \cap K = \emptyset$, then from E and K are both convex and by using the separation theorem, there exists $y^* \in Y^* \setminus \{0\}$ such that

$$(1) \qquad \langle y^*, e \rangle \geq \langle y^*, k \rangle, \forall e \in E, \forall k \in K.$$

Let $k = 0$ in (1), we have $\langle y^*, e \rangle \geq 0, \forall e \in E$. Hence, $y^* \in E^+$. From Proposition 2.6(a) in [6], it follows that $y^* \in K^+$, i.e.,

$$(2) \qquad \langle y^*, k \rangle \geq 0, \forall k \in K.$$

Furthermore, again from (1) and K is a cone, it follows that $\langle y^*, k \rangle \leq 0, \forall k \in K$. So, by (2), we have

$$\langle y^*, k \rangle = 0, \forall k \in K.$$

By Lemma 2.2, $K = \partial K$, which contradicts to int$K \neq \emptyset$.

Next, we prove $E \cap K$ is an improvement set with respect to K. Since $0 \notin E$ and $0 \in K$, then $0 \notin E \cap K$ and $E \cap K \subset E \cap K + K$. We only need to prove $E \cap K + K \subset E \cap K$. Since K is a convex cone, then we have

$$(3) \qquad E \cap K + K \subset K + K = K.$$

From $E \in \mathfrak{T}_Y$, we obtain

$$E \cap K + K \subset E + K = E. \tag{4}$$

It follows from (3) and (4) that $E \cap K + K \subset E \cap K$. Hence from $E \cap K \neq \emptyset$, $\text{int}K \neq \emptyset$ and Theorem 3.1 in [9], we have

$$\text{int}(E \cap K) = E \cap K + \text{int}K \neq \emptyset.$$

■

Remark 3.1. The assumption of convexity of improvement set E is only a sufficient condition to ensure int$(E \cap K) \neq \emptyset$. In fact, let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and

$$E = \mathbb{R}^2_+ \backslash \{(x_1, x_2) | 0 \leq x_1 < 1, 0 \leq x_2 < 1\}.$$

It is clear that E is a closed improvement set with respect to K and E is not a convex set. However,

$$\text{int}(E \cap K) = \{(x_1, x_2) | x_1 > 0, x_2 > 0\} \backslash \{(x_1, x_2) | 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\} \neq \emptyset.$$

According to Lemma 3.2, we can restrict $q \in \text{int}(E \cap K)$ and establish a nonlinear scalarization characterization of weak E-efficient solutions of (VP) via the nonlinear scalarization function $\varphi_{q,E}$.

Theorem 3.1. *Let $E \in \mathfrak{T}_Y$ be a closed convex set, $q \in$ int$(E \cap K)$ and*

$$\epsilon = \inf\{s \in \mathbb{R}_{++} | sq \in \textit{int}(E \cap K)\}.$$

Then

$$\overline{x} \in \textit{WAE}(f, S, E) \Leftrightarrow \overline{x} \in \textit{AMin}(\varphi_{q,E,f(\overline{x})} \circ f, \epsilon).$$

Proof. Assume that $\overline{x} \in \text{WAE}(f, S, E)$. From Lemma 3.1, it follows that

$$\{y \in Y | \varphi_{q,E}(y) < 0\} = -\text{int}E. \tag{5}$$

Since $\overline{x} \in \text{WAE}(f, S, E)$, then we have

$$(f(S) - f(\overline{x})) \cap (-\text{int}E) = \emptyset. \tag{6}$$

From (5) and (6), we deduce that

$$(f(S) - f(\overline{x})) \cap \{y \in Y | \varphi_{q,E}(y) < 0\} = \emptyset.$$

Thus,

$$(\varphi_{q,E,f(\overline{x})} \circ f)(x) = \varphi_{q,E}(f(x) - f(\overline{x})) \geq 0, \forall x \in S. \tag{7}$$

In addition, since $\epsilon q \in E \cap K \subset E$, then we have

$$(\varphi_{q,E,f(\overline{x})} \circ f)(\overline{x}) = \varphi_{q,E}(0) = \inf\{s \in \mathbb{R} | sq \in E\} \leq \epsilon.$$

It follows from (7) that

$$(\varphi_{q,E,f(\overline{x})} \circ f)(x) \geq (\varphi_{q,E,f(\overline{x})} \circ f)(\overline{x}) - \epsilon.$$

Therefore, $\overline{x} \in \text{AMin}(\varphi_{q,E,f(\overline{x})} \circ f, \epsilon)$.

Conversely, assume that $\overline{x} \in \text{AMin}(\varphi_{q,E,f(\overline{x})} \circ f, \epsilon)$ and $\overline{x} \notin \text{WAE}(f, S, E)$. Then there exists $\hat{x} \in S$ such that

$$f(\hat{x}) - f(\overline{x}) \in -\text{int}E. \tag{8}$$

From (8) and Lemma 3.1, it follows that for any $c \in \mathbb{R}$,

$$cq + f(\hat{x}) - f(\overline{x}) \in cq - \text{int}E = \{y \in Y | \varphi_{q,E}(y) < c\},$$

which implies that

$$\varphi_{q,E}(cq + f(\hat{x}) - f(\overline{x})) < c. \tag{9}$$

Let $c = 0$ in (9). Then

$$\varphi_{q,E}(f(\hat{x}) - f(\overline{x})) < 0. \tag{10}$$

On the other hand, $\overline{x} \in \text{AMin}(\varphi_{q,E,f(\overline{x})} \circ f, \epsilon)$ implies

$$\varphi_{q,E}(f(\hat{x}) - f(\overline{x})) \quad \geq \varphi_{q,E}(f(\overline{x}) - f(\overline{x})) - \epsilon = \varphi_{q,E}(0) - \epsilon. \tag{11}$$

We can prove

$$\begin{aligned} \varphi_{q,E}(0) &= \inf\{s \in \mathbb{R} | 0 \in sq - E\} \\ &= \inf\{s \in \mathbb{R} | sq \in E\} \\ &= \inf\{s \in \mathbb{R}_{++} | sq \in E\}. \end{aligned} \tag{12}$$

In fact, we only need to prove that for any $s \leq 0$, $sq \notin E$. Clearly, $0 \notin E$ when $s = 0$. Assume that there exists $\hat{s} < 0$ such that $\hat{s}q \in E$. Since $q \in \text{int}(E \cap K) \subset K$ and $-\hat{s}q \in K$, then we have

$$0 = \hat{s}q - \hat{s}q \in E + K = E,$$

which contradicts to $E \in \mathfrak{T}_Y$. This implies that (12) holds. Furthermore, according to the fact that $q \in \text{int}(E \cap K) \subset K$, we have for any $s \in \mathbb{R}_{++}$, $sq \in K$. It follows from (12) that

$$\varphi_{q,E}(0) = \inf\{s \in \mathbb{R}_{++} | sq \in E \cap K\}.$$

Hence,

$$\varphi_{q,E}(0) - \epsilon = \inf\{s \in \mathbb{R}_{++} | sq \in E \cap K\} - \inf\{s \in \mathbb{R}_{++} | sq \in \text{int}(E \cap K)\} = 0.$$

By (11), we have

$$\varphi_{q,E}(f(\hat{x}) - f(\overline{x})) \geq 0,$$

which contradicts to (10) and so $\overline{x} \in \text{WAE}(f, S, E)$. ■

We also can characterize E-efficient solutions of (VP) via nonlinear scalarization function $\varphi_{q,E}$ and obtain the following nonlinear scalarization characterization. The proof is similar with Theorem 3.1 and is omitted.

Theorem 3.2. *Let $E \in \mathfrak{T}_Y$ be a closed convex set, $q \in$ int$(E \cap K)$ and*

$$\epsilon = \inf\{s \in \mathbb{R}_{++} | sq \in \textit{int}(E \cap K)\}.$$

Then

$$\overline{x} \in \textit{AE}(f, S, E) \Leftrightarrow \overline{x} \in \textit{SAMin}(\varphi_{q,E,f(\overline{x})} \circ f, \epsilon).$$

4. Scalarization of E-Efficiency via Δ_{-K}

In this section, we characterize E-efficient solutions and weak E-efficient solutions of (VP) via the nonlinear scalarization function Δ_{-K} studied by Zaffaroni in [11]. We assume that Y be a normed space and $E \in \mathfrak{T}_Y$.

Let A be a subset of Y, $\Delta_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$\Delta_A(y) = d_A(y) - d_{Y \setminus A}(y),$$

where $d_\emptyset(y) = +\infty$, $d_A(y) = \inf\limits_{z \in A} \|z - y\|$.

Lemma 4.1. ([11]). *Let A be a proper subset of Y. Then the following statements are true:*

(*i*) $\Delta_A(y) < 0$ *for* $y \in$ *intA*, $\Delta_A(y) = 0$ *for* $y \in \partial A$, *and* $\Delta_A(y) > 0$ *for* $y \notin$ *clA;*

(*ii*) *If A is closed, then $A = \{y | \Delta_A(y) \leq 0\}$;*

(*iii*) *If A is a convex, then Δ_A is convex;*

(*iv*) *If A is a cone, then Δ_A is positively homogeneous.*

Remark 4.1. If A is a convex cone, then from Lemma 4.1(iii) and (iv), it follows that Δ_A is a sublinear function.

Consider the following scalar optimization problem:

$$(\text{P}_y) \quad \min_{x \in S} \Delta_{-K}(f(x) - y),$$

where $y \in Y$. Denote the set of ϵ-minimal solutions of (P$_y$) by AMin$(\Delta_{-K}(f(x) - y), \epsilon)$, and the set of strictly ϵ-minimal solutions of (P$_y$) by SAMin$(\Delta_{-K}(f(x)-y), \epsilon)$.

Theorem 4.1. *Let $E \in \mathfrak{T}_Y$. Then*

$$\overline{x} \in \textit{WAE}(f, S, E) \Rightarrow \overline{x} \in \textit{AMin}(\Delta_{-K}(f(x) - f(\overline{x})), d_E(0)).$$

Proof. From $\overline{x} \in \text{WAE}(f, S, E)$, it follows that

$$(f(\bar{x}) - \text{int}E) \cap f(S) = \emptyset.$$

Hence by Theorem 3.1 in [9], we have

$$(f(\bar{x}) - E - \text{int}K) \cap f(S) = \emptyset,$$

i.e.,

$$f(x) - f(\bar{x}) + e \notin -\text{int}K, \forall x \in S, \forall e \in E,$$

which implies that

$$\Delta_{-K}(f(x) - f(\bar{x}) + e) \geq 0, \forall x \in S, \forall e \in E.$$

Since K is a convex cone and by Remark 4.1, then we deduce that

$$0 \leq \Delta_{-K}(f(x) - f(\overline{x}) + e) \leq \Delta_{-K}(f(x) - f(\overline{x})) + \Delta_{-K}(e),$$

i.e.,

$$\Delta_{-K}(f(x) - f(\overline{x})) + \Delta_{-K}(e) \geq 0, \forall x \in S, \forall e \in E.$$

Therefore,

$$\Delta_{-K}(f(x) - f(\overline{x})) + \inf_{e \in E} \Delta_{-K}(e) \geq 0, \forall x \in S. \tag{13}$$

Now we calculate $\inf_{e \in E} \Delta_{-K}(e)$. From the definition of Δ_{-K}, we have

$$\Delta_{-K}(e) = d_{-K}(e) - d_{Y \backslash (-K)}(e), \forall e \in E. \tag{14}$$

We can prove $E \subset Y \backslash (-K)$. On the contrary, assume that there exists $\hat{e} \in E$ such that $\hat{e} \notin Y \backslash (-K)$, then $-\hat{e} \in K$. Hence from $E \in \mathfrak{T}_Y$, we have

$$0 = \hat{e} - \hat{e} \in E + K = E,$$

which contradicts to $0 \notin E$ and so

$$d_{Y \backslash (-K)}(e) = 0, \forall e \in E.$$

It follows from (14) that

$$\Delta_{-K}(e) = d_{-K}(e), \forall e \in E.$$

Therefore,

$$\inf_{e \in E} \Delta_{-K}(e) = \inf_{e \in E} \inf_{k \in K} ||e + k||. \tag{15}$$

Next, we prove

$$\inf_{e \in E} \inf_{k \in K} ||e + k|| = \inf_{e' \in E} ||e'||. \tag{16}$$

Since $E + K = E$, then

$$\{e + k | k \in K\} \subset E, \forall e \in E,$$

which implies that

$$\inf_{k \in K} ||e + k|| \geq \inf_{e' \in E} ||e'||, \forall e \in E.$$

So, $\inf_{e' \in E} ||e'||$ is a lower bound of $\left\{ \inf_{k \in K} ||e + k|| \right\}_{e \in E}$. Furthermore, from the definition of infimum, for any given $\epsilon > 0$, there exists $e_0 \in E$ such that

$$||e_0|| < \inf_{e' \in E} ||e'|| + \epsilon.$$

From $E + K = E$, it follows that there exist $\overline{e} \in E$ and $\overline{k} \in K$ such that

$$e_0 = \overline{e} + \overline{k}.$$

Therefore,

$$\inf_{k \in K} ||\overline{e} + k|| \leq ||\overline{e} + \overline{k}|| = ||e_0|| < \inf_{e' \in E} ||e'|| + \epsilon.$$

Hence, (16) holds and then from (15), we have

$$\inf_{e \in E} \Delta_{-K}(e) = \inf_{e' \in E} ||e'|| = d_E(0).$$

From (13), we can obtain that for any $x \in S$,

$$\Delta_{-K}(f(x) - f(\overline{x})) + d_E(0) = \Delta_{-K}(f(x) - f(\overline{x})) + \inf_{e \in E} \Delta_{-K}(e) \geq 0. \tag{17}$$

Since $0 \in \partial K$ and from Lemma 4.1(i), then

$$\Delta_{-K}(f(\overline{x}) - f(\overline{x})) = \Delta_{-K}(0) = 0. \tag{18}$$

Combine with (17) and (18), it follows that

$$\Delta_{-K}(f(x) - f(\overline{x})) \geq \Delta_{-K}(f(\overline{x}) - f(\overline{x})) - d_E(0), \forall x \in S.$$

Thus,

$$\overline{x} \in \text{AMin}(\Delta_{-K}(f(x) - f(\overline{x})), d_E(0)).$$ ■

Remark 4.2. The converse of Theorem 4.1 may not be valid. The following example can illustrates it.

Example 4.1. Let $X = Y = \mathbb{R}^2$, $\|\cdot\| = \|\cdot\|_2$, $K = \mathbb{R}^2_+$, $f(x) = x$ and

$$E = \{(x_1, x_2) | x_1 + x_2 \geq 2, x_1 \geq 0, x_2 \geq 0\},$$
$$S = \{(x_1, x_2) | -1 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}.$$

Clearly, $E \in \mathfrak{T}_Y$ and $d_E(0) = \sqrt{2}$. Let $\overline{x} = (1, 1) \in S$. Since

$$\Delta_{-K}(f(x) - f(\overline{x})) \geq -1 > -\sqrt{2} = -d_E(0), \forall x \in S,$$

then

$$\overline{x} \in \text{AMin}(\Delta_{-K}(f(x) - f(\overline{x})), d_E(0)).$$

However,

$$\begin{aligned}(f(\overline{x}) - \text{int}E) \cap f(S) &= \{(x_1, x_2) | x_1 < 1, x_2 < 1, x_1 + x_2 < 0\} \cap f(S) \\ &= \{(x_1, x_2) | x_1 + x_2 < 0, -1 \leq x_1 < 0, x_2 \geq 0\} \neq \emptyset,\end{aligned}$$

which implies that

$$\overline{x} \notin \text{WAE}(f, S, E).$$

However, under suitable conditions, we can prove the converse of Theorem 4.1 is valid.

Theorem 4.2. *Let $E \subset K$, $E \in \mathfrak{T}_Y$ and $\epsilon = \inf_{e \in E} d_{\partial K}(e)$. Then*

$$\overline{x} \in \textit{AMin}(\Delta_{-K}(f(x) - f(\overline{x})), \epsilon) \Rightarrow \overline{x} \in \textit{WAE}(f, S, E).$$

Proof. Assume that $\overline{x} \notin \text{WAE}(f, S, E)$. Then there exists $\hat{x} \in S$ such that $f(\hat{x}) - f(\overline{x}) \in -\text{int}E$. From $E \in \mathfrak{T}_Y$ and Theorem 3.1 in [9], there exists $\hat{e} \in E$ such that $f(\hat{x}) - f(\overline{x}) + \hat{e} \in -\text{int}K$. It follows from Lemma 4.1(i) that $\Delta_{-K}(f(\hat{x}) - f(\overline{x}) + \hat{e}) < 0$. From Remark 4.1,

$$\Delta_{-K}(f(\hat{x}) - f(\overline{x})) \leq \Delta_{-K}(f(\hat{x}) - f(\overline{x}) + \hat{e}) + \Delta_{-K}(-\hat{e}).$$

Hence,

$$\begin{aligned} \Delta_{-K}(f(\hat{x}) - f(\overline{x})) &< \Delta_{-K}(-\hat{e}) = -d_{Y\setminus(-K)}(-\hat{e}) \\ &= -d_{-\partial K}(-\hat{e}) = -d_{\partial K}(\hat{e}) \le -\inf_{e\in E} d_{\partial K}(e) = -\epsilon. \end{aligned} \tag{19}$$

On the other hand, $\overline{x} \in \text{AMin}(\Delta_{-K}(f(x) - f(\overline{x})), \epsilon)$ implies that

$$\Delta_{-K}(f(\hat{x}) - f(\overline{x})) \ge \Delta_{-K}(f(\overline{x}) - f(\overline{x})) - \epsilon = -\epsilon,$$

which contradicts to (19) and so $\overline{x} \in \text{WAE}(f, S, E)$. ∎

We also can obtain a nonlinear scalarization characterizations of E-efficient solutions of (VP) by means of the nonlinear scalarization function Δ_{-K}. The proofs are similar with Theorem 4.1 and Theorem 4.2 and are omitted.

Theorem 4.3. *Let $E \in \mathfrak{T}_Y$. Then*

$$\overline{x} \in AE(f, S, E) \Rightarrow \overline{x} \in SAMin(\Delta_{-K}(f(x) - f(\overline{x})), d_E(0)).$$

Theorem 4.4. *Let $E \subset K$, $E \in \mathfrak{T}_Y$ and $\epsilon = \inf_{e\in E} d_{\partial K}(e)$. Then*

$$\overline{x} \in SAMin(\Delta_{-K}(f(x) - f(\overline{x})), \epsilon) \Rightarrow \overline{x} \in AE(f, S, E).$$

5. Concluding Remarks

E-efficient solutions and weak E-efficient solutions unify some known exact and approximate solutions in vector optimization. In this paper, we employ two kinds of classic nonlinear scalarization functions to characterize E-efficient solutions and weak E-efficient solutions of vector optimization problems and obtain some nonlinear scalarization characterizations. It remains one interesting question how to weaken or drop the convexity of improvement set E in Theorem 3.1.

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