TAIWANESE JOURNAL OF MATHEMATICS
Vol. 19, No. 2, pp. 419-431, April 2015
DOI: 10.11650/tjm.19.2015.4314
This paper is available online at http://journal.taiwanmathsoc.org.tw

THE SHARP LOWER BOUND FOR THE SPECTRAL RADIUS OF CONNECTED GRAPHS WITH THE INDEPENDENCE NUMBER

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Abstract. In this paper, we investigate some properties of the Perron vector of connected graphs. These results are used to characterize all extremal connected graphs which attain the minimum value among the spectral radii of all connected graphs with order $n = k\alpha$ and the independence number α. Moreover, all extremal graphs which attain the maximum value among the spectral radii of clique trees with order $n = k\alpha$ and the independence number α are characterized.

1. INTRODUCTION

Throughout this paper, we always consider simple graphs. Let $G = (V(G),\ E(G))$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. Let $A(G) = (a_{ij})$ be the $(0, 1)$ *adjacency matrix* of G with $a_{ij} = 1$ for $v_i \sim v_j$ and 0 otherwise, where " $\sim$ " stands for the adjacency relation. Let $N_G(v)$ be the neighbor set of the vertex v. The characteristic polynomial of $A(G)$ is denoted by $\psi(G, x)$ or $\psi(G)$ for short. The largest eigenvalue of $A(G)$ is called *spectral radius* of G, denoted by $\lambda(G)$. The *independence number* (also the stability number) of G, denoted by $\alpha(G)$, is the cardinality of the maximal independent sets of G, where an *independent set* is the subset of $V(G)$ such that every pair vertices of this set are not adjacent.

A classical Turán [13] theorem for the independence number stated that the *Turán graph* $T_{n,\alpha}$ which consists of α disjoint balanced cliques is the unique graph having the minimum size among all graphs of order n and the independence number α. Since the Turán graph is disconnected, Ore [10] raised a similar problem determining the minimum number of edges among all connected graphs with order n and the independence number α. Recently, this problem was settled independently by Bougard and Joret [3], and by Gitler and Valencia [5]. In spectral extremal graph theory, Nikiforov [8] presented the following result.

Received January 14, 2014, accepted July 1, 2014.
Communicated by Gerard Jennhwa Chang.
2010 *Mathematics Subject Classification*: 05C50, 05C35.
Key words and phrases: Spectral radius, Independence number, Perron vector, Clique tree.
This work is supported by National Natural Science Foundation of China (No. 11271256), Innovation Program of Shanghai Municipal Education Commission (No. 14ZZ016) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130073110075).

Theorem 1.1. [8]. *Let G be a simple graph with order n and the independence number α. Then*

$$\lambda(G) \geq \frac{n}{\alpha} - 1$$

with equality if and only if $n = k\alpha$ and G is Turán graph.

Since the above equality holds only if G is disconnected, it may be interesting to give a sharp lower bound for all connected graphs of order n with the independence number α and characterize all extremal graphs. Recently, Xu et al. [14] characterized all extremal graphs with the minimum spectral radius among all connected graphs of order n and the independence number $\alpha = 1, 2, \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, n-3, n-2,$ or $n-1$. Du and Shi [4] proved the following results.

Theorem 1.2. [4]. *Let G be a connected graph of order $n = k\alpha \geq 108$ with the independence number α. If $\alpha = 3$ or 4, then*

$$\lambda(G) \geq \lambda(P_{n,\alpha}),$$

where graph $P_{n,\alpha}$ is obtained from a path of order α by replacing each vertex to a clique of order k and has exactly $2\alpha - 2$ cut vertices.

Motivated by Theorem 1.2 and the related results, we study some properties of the extremal graphs having the minimum spectral radius among all connected graphs with the order n and the independence number α. Before stating our main results, we need some notations. Let $\mathcal{G}_{n,\alpha}$ be the set of all connected graphs of order n with the independence number α and let $\mathcal{T}_{n,\alpha}$ be the set of all clique trees of order n obtained from a tree of order α by replacing each vertex to a clique of order $\lfloor \frac{n}{\alpha} \rfloor$ or $\lceil \frac{n}{\alpha} \rceil$. A graph G of order n with the independence number α and $n \geq 2\alpha$ is called *clique path*, denoted by $P_{n,\alpha}$, if G is obtained from a path of order α by replacing each vertex to a clique of order $\lfloor \frac{n}{\alpha} \rfloor$ or $\lceil \frac{n}{\alpha} \rceil$ such that there are exactly $2\alpha - 2$ cut vertices. A graph G of order n with the independence number α is called *clique star*, denoted by $S_{n,\alpha}$, if G is obtained from star $K_{1,\alpha-1}$ by replacing each vertex to a clique of order $\lfloor \frac{n}{\alpha} \rfloor$ or $\lceil \frac{n}{\alpha} \rceil$ such that there are exactly α cut vertices. Observe that there has exactly one clique path and exactly one clique star in $\mathcal{G}_{n,\alpha}$ for $\alpha | n$, while there are many clique paths and clique stars for $\alpha \nmid n$. Moreover, let

$$\lambda_{n,\alpha} = \min\{\rho(G) : G \in \mathcal{G}_{n,\alpha}\},$$

$$\Lambda_{n,\alpha} = \max\{\rho(G) : G \in \mathcal{T}_{n,\alpha}\}.$$

The main results of this paper are stated as follows.

Theorem 1.3. *Fixed α. Then*

$$\lim_{n \to \infty} \frac{\lambda_{n,\alpha}}{n} = \frac{1}{\alpha}.$$

Theorem 1.4. *If $n = k\alpha$ and $k > \frac{17\alpha+15}{8}$, then $P_{n,\alpha}$ is the only graph having the minimum spectral radius in $\mathcal{G}_{n,\alpha}$. In other words, for any $G \in \mathcal{G}_{n,\alpha}$, $\lambda(G) \geq \lambda(P_{n,\alpha})$ with equality if and only if G is $P_{n,\alpha}$.*

Theorem 1.5. *If $n = k\alpha$, then the clique star is the only graph having the maximum spectral radius in $\mathcal{T}_{n,\alpha}$. In other words, for any $G \in \mathcal{T}_{n,\alpha}$, $\lambda(G) \leq \lambda(S_{n,\alpha})$ with equality if and only if G is $S_{n,\alpha}$.*

Remark. Theorem 1.3 may be regarded as a spectral form of the well-known Erdös-Stone-Simonovits theorem [1], and Theorem 1.4 generalizes the result of Theorem 1.2. The rest of this paper is organized as follows. In Section 2, we present the proof of Theorem 1.3 and some relative results. In Sections 3 and 4, we present the proofs of Theorems 1.4 and 1.5, respectively.

2. A Spectral Erdös-stone-simonovits Type Theorem

In order to prove Theorem 1.3, we need the following lemmas.

Lemma 2.1. [4].

(1) *Every nonbipartite triangle free graph of order n has at most $1+\frac{(n-1)^2}{4}$ edges.*

(2) *If G is a K_{r+1}-free graph of order n with chromatic number at least $r+1 > 2$, then $|E(G)| \leq \frac{(r-1)n^2}{2r} - \frac{n}{2r} + \frac{17}{16} - \frac{1}{8r}$.*

Lemma 2.2. [11]. *If $e = uv$ is a cut edge of G, then $\psi(G) = \psi(G-e) - \psi(G-u-v)$.*

Lemma 2.3. [9]. *An $n \times n$ nonnegative matrix $T \in R^{n\times n}$ is convergent, i.e. $\lambda(T) < 1$, if and only if $(I-T)^{-1}$ exists and*

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k \geq 0,$$

where $\lambda(T)$ is the spectral radius of T.

Lemma 2.4. *If $n = k\alpha + t, \alpha > 1$, then*

$$\lambda_{n,\alpha} \leq \begin{cases} k-1+\frac{2}{k-1}, & t = 0, \\ k+\frac{2}{k}, & 1 \leq t < \alpha. \end{cases}$$

Proof. If $t = 0$, $P_{n,\alpha}$ is just as the following Fig.1.

Let $A(P_{n,\alpha}), D(P_{n,\alpha})$ be the adjacency matrix and degree diagonal matrix of $P_{n,\alpha}$. It is easy to see that $A(P_{n,\alpha})$ and $D(P_{n,\alpha})^{-1}A(P_{n,\alpha})D(P_{n,\alpha})$ have the same eigenvalues. If $v \in V_1$ or V_α, then the sum of the row corresponding to v in $D(P_{n,\alpha})^{-1}A(P_{n,\alpha})D(P_{n,\alpha})$ is at most $\max\{k-1+\frac{1}{k-1}, \frac{k-1}{k}(k-1)+1\} = k-1+\frac{1}{k-1}$.

If $v \in V_i$ and $2 \le i \le \alpha - 1$, then the sum of the row corresponding to v in the matrix $D(P_{n,\alpha})^{-1}A(P_{n,\alpha})D(P_{n,\alpha})$ is at most $\max\{k-1+\frac{2}{k-1}, k-1+\frac{2}{k}\} = k-1+\frac{2}{k-1}$. Hence $\lambda_{n,\alpha} < \lambda(P_{n,\alpha}) < k-1+\frac{2}{k-1}$. If $1 \le t < \alpha$, it is easy to see that $P_{n,\alpha}$ is a subgraph of $P_{n+\alpha-t,\alpha}$. Hence $\lambda_{n,\alpha} \le \lambda(P_{n,\alpha}) < \lambda(P_{n+\alpha-t,\alpha}) < k+\frac{2}{k}$, since $n+\alpha-t = (k+1)\alpha$. ■

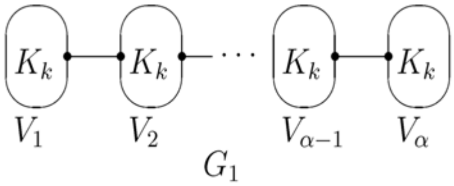

Fig. 1.

We are ready to prove Theorem 1.3.

Proof. Firstly we show that $\limsup_{n\to\infty} \frac{\lambda_{n,\alpha}}{n}$ is $\frac{1}{\alpha}$. If $n = k\alpha$, by the Lemma 2.4, $\lambda_{n,\alpha} < k-1+\frac{2}{k-1}$, which implies

$$\limsup_{n\to\infty} \frac{\lambda_{n,\alpha}}{n} \le \limsup_{n\to\infty} \frac{k-1+\frac{2}{k-1}}{n} = \frac{1}{\alpha}.$$

If $n = k\alpha + t, 0 < t < \alpha$, by the Lemma 2.4, $\lambda_{n,\alpha} < k + \frac{2}{k}$, which implies

$$\limsup_{n\to\infty} \frac{\lambda_{n,\alpha}}{n} \le \limsup_{n\to\infty} \frac{k+\frac{2}{k}}{n} = \frac{1}{\alpha}.$$

Next we show that $\liminf_{n_i\to\infty} \frac{\lambda_{n,\alpha}}{n}$ is also $\frac{1}{\alpha}$. Suppose

$$\liminf_{n\to\infty} \frac{\lambda_{n,\alpha}}{n} = \frac{1}{\alpha} - 2\epsilon,\ 0 < 2\epsilon \le \frac{1}{\alpha}.$$

Then there exists an increasing sequence $\{n_i\}_{i=1}^{\infty}$, and a sequence of graphs $\{G_i\}_{i=1}^{\infty}$, where G_i is a graph of order n_i with size m_i such that $\lambda(G_i) \le (\frac{1}{\alpha} - \epsilon)n_i$. Since $\lambda(G_i) \ge \frac{2m_i}{n_i}$, we have

$$\begin{aligned}|E(G_i^c)| &\ge \frac{n_i(n_i-1)}{2} - \frac{1}{2}(\frac{1}{\alpha} - \epsilon)n_i^2 \\ &= \frac{\alpha-1}{2\alpha}n_i^2 + \frac{\epsilon n_i^2}{2} - \frac{n_i}{2} \\ &> \frac{\alpha-1}{2\alpha}n_i^2 - \frac{n_i}{2\alpha} + \frac{17}{16} - \frac{k}{8n_i}\end{aligned}$$

for large enough i. By Lemma 2.1, the chromatic number of G_i^c is at most α if i is large enough. Then G_i contains a clique of order $\lceil \frac{n_i}{\alpha} \rceil$, which implies $\lambda(G_i) > \lceil \frac{n_i}{\alpha} \rceil - 1$. Hence $\liminf_{n_i \to \infty} \frac{\lambda(G_i)}{n_i} \geq \frac{1}{\alpha}$. It is a contradiction with $\lambda(G_i) \leq (\frac{1}{\alpha} - \epsilon)n_i$ for all i. So

$$\liminf_{n_i \to \infty} \frac{\lambda_{n,\alpha}}{n} = \frac{1}{\alpha}.$$

This completes the proof. ■

Remark. It follows from Theorem 1.3 that a graph of order $n = k\alpha + t$ with spectral radius $\lambda(G) \leq (\frac{1}{\alpha} - \epsilon)n$ for positive number $\varepsilon > 0$ has an independent set with size at least α for large enough n. It is an interesting question to count how many such independent sets? Denote by $i_s(G)$ the number of s-independent set of G and $k_s(G)$ for the number of s-clique of G. It is easy to see that $k_s(G) = i_s(G^c)$. Bollobás and Nikiforov [2] gave a lower bound for $k_{r+1}(G)$ in terms of spectral radius.

Lemma 2.5. [2]. *For any graph G of order n, and $r > 1$,*

$$k_{r+1}(G) \geq \left(\frac{\lambda(G)}{n} - 1 + \frac{1}{r}\right) \frac{r(r-1)}{r+1} \left(\frac{n}{r}\right)^{r+1}.$$

By using the above lemma, we present a lower bound for $i_s(G)$.

Theorem 2.6. *Let G be a simple graph of order n and $\alpha > 1$ be an positive integer. If $\lambda(G) \leq \frac{n}{\alpha}$, then*

$$i_\alpha(G) \geq \left(\frac{1}{\alpha(\alpha-1)} - \frac{1}{n}\right) \frac{(\alpha-1)(\alpha-2)}{\alpha} \left(\frac{n}{\alpha-1}\right)^\alpha.$$

Proof. Since $\frac{2m(G)}{n} \leq \lambda(G)$, we have $m(G) \leq \frac{n^2}{2\alpha}$, which implies $m(G^c) \geq \frac{\alpha-1}{2\alpha}n^2 - \frac{n}{2}$. So $\lambda(G^c) \geq \frac{\alpha-1}{\alpha}n - 1$. By the Lemma 2.5, we can get

$$\begin{aligned} i_\alpha(G) = k_\alpha(G^c) &\geq \left(\frac{\alpha-1}{\alpha} - \frac{1}{n} - 1 + \frac{1}{\alpha-1}\right) \frac{(\alpha-1)(\alpha-2)}{\alpha} \left(\frac{n}{\alpha-1}\right)^\alpha \\ &= \left(\frac{1}{\alpha(\alpha-1)} - \frac{1}{n}\right) \frac{(\alpha-1)(\alpha-2)}{\alpha} \left(\frac{n}{\alpha-1}\right)^\alpha. \end{aligned}$$

This completes the proof. ■

Remark. It follows from Theorem 2.6 that $i_s(G)$ is about $O(n^\alpha)$ if $\lambda(G) \leq \frac{n}{\alpha}$.

3. Proof of Theorems 1.4 and 1.5

In order to prove Theorems 1.4 and 1.5, we need some lemmas.

Lemma 3.1. *Let $n = k\alpha$ and $k > \frac{17\alpha+15}{8}$. If a connected graph G has the minimum spectra radius among all graphs in $\mathcal{G}_{n,\alpha}$, then $G \in \mathcal{T}_{n,\alpha}$.*

Proof. By Lemma 2.4, $\lambda(G) = \lambda_{n,\alpha} \leq k - 1 + \frac{2}{k-1}$ and G does not contain K_{k+1}. Further, we claim that the chromatic number of G^c is α. Suppose that the chromatic number of G^c is at least $\alpha + 1$. By Lemma 2.1,

$$\begin{aligned}|E(G)| &\geq \frac{n(n-1)}{2} - \frac{(\alpha-1)n^2}{2\alpha} + \frac{n}{2\alpha} - \frac{17}{16} + \frac{1}{8\alpha}\\ &= \frac{kn}{2} - \frac{n-k}{2} - \frac{17}{16} + \frac{k}{8n}\\ &= \frac{(k-1)n}{2} + \frac{k}{2} - \frac{17}{16} + \frac{k}{8n}.\end{aligned}$$

By $k > \frac{17\alpha+15}{8}$, we have

$$\lambda(G) \geq \frac{2|E(G)|}{n} \geq k - 1 + \frac{1}{\alpha} - \frac{17}{8n} + \frac{k}{4n^2} > k - 1 + \frac{2}{k-1}.$$

Hence the chromatic number of G^c is α, i.e., G^c is an α-partite graph. Assume that the parts of G^c are $V_1, V_2, ..., V_\alpha$. Since G does not contain K_{k+1} and $n = k\alpha$, then $|V_1| = |V_2| = ... = |V_\alpha| = k$. Moreover, the induced subgraph in G^c by $V_i \bigcup V_j$ $(i \neq j)$ is not completely bipartite, since G is connected. Note that the spectral radius of a connected graph is an strictly increasing function with respect to adding an edge. Hence $G \in \mathcal{T}_{n,\alpha}$. ∎

Lemma 3.2. *Let G be a non-bipartite connected graph of order n and $x = (x_1, x_2, ..., x_n)^T$ be the Perron vector of $A(G)$. If $\sigma_s(v_i)$ is the number of the closed walks of length s starting at vertex v_i, $i = 1, \cdots, n$, then*

$$\lim_{s\to\infty} \frac{\sigma_s(v_i)}{\sigma_s(v_j)} \geq 1$$

with equality if and only if $x_i = x_j$.

Proof. By spectral decomposition theorem, there exist normal eigenvectors $\xi_2, \cdots, \xi_n$ corresponding to eigenvalues $\lambda_2, \cdots, \lambda_n$ such that

$$A = \lambda(A)xx^T + \lambda_2\xi_2\xi_2^T + ... + \lambda_n\xi_n\xi_n^T.$$

Then

$$A^s = \lambda(A)^s xx^T + \lambda_2^s\xi_2\xi_2^T + ... + \lambda_n^s\xi_n\xi_n^T.$$

Let e_i be the column vector whose i-th component is 1 and 0 otherwise, $i = 1, \ldots, n$. Then $\sigma_s(v_i) = e_i^T A^s e_i$. Moreover $\lambda(G) > |\lambda_i|$ for $i = 2, \ldots, n$, since G is non-bipartite and connected. Hence

$$\begin{aligned}\lim_{s\to\infty} \frac{\sigma_s(v_i)}{\sigma_s(v_j)} &= \lim_{s\to\infty} \frac{e_i^T A^s e_i}{e_j^T A^s e_j} \\ &= \lim_{s\to\infty} \frac{e_i^T x x^T e_i + \frac{\lambda_2^s}{\lambda(A)^s} e_i^T \xi_2 \xi_2^T e_i + \cdots + \frac{\lambda_n^s}{\lambda(A)^s} e_i^T \xi_n \xi_n^T e_i}{e_j^T x x^T e_j + \frac{\lambda_2^s}{\lambda(A)^s} e_j^T \xi_2 \xi_2^T e_j + \cdots + \frac{\lambda_n^s}{\lambda(A)^s} e_j^T \xi_n \xi_n^T e_j} \\ &= \frac{x_i^2}{x_j^2}.\end{aligned}$$

This completes the proof. ■

Lemma 3.3. *Let $n = k\alpha > 2\alpha$ and $G \in \mathcal{T}_{n,\alpha}$ be a graph obtained by joining an edge from a non-cut vertex of a graph $H \in \mathcal{T}_{n-k(l+p),\alpha-(l+p)}$ and a non-cut vertex of $P_{k(l+p),l+p}$ (see Fig.2). Let G' be the graph obtained from G by deleting the edge v_3v_4 and adding edge v_1v_4. If H has an induced subgraph $P_{kl,l}$ containing v_1, then $\lambda(G') > \lambda(G)$.*

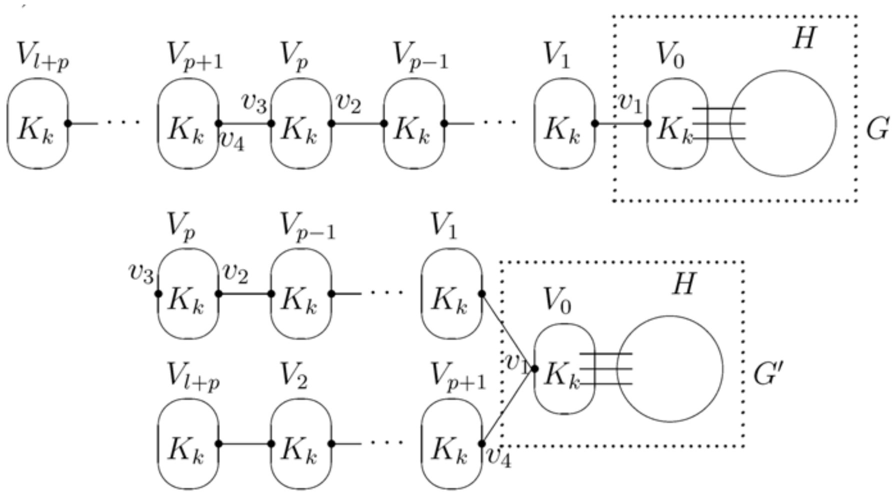

Fig. 2.

Proof. Let $x = (x(u), u \in V(G))^T$ be the Perron vector of $G \in \mathcal{T}(n,\alpha)$ and let $\mathcal{W}(s, v_i)$ be the set of all closed walks of length s starting at v_i, $i = 1, 2, 3$. We claim that there exists an injective mapping φ from $\mathcal{W}(s, v_3)$ to $\mathcal{W}(s, v_2)$. In fact, if W is a closed walk of length s starting at v_3 and containing v_2, let $\varphi(W) = W$ and the starting is the first time that v_3 goes through v_2. If W is a closed walk of length s starting at

v_3 and containing no v_2, then there exists a corresponding closed walk $W' = \varphi(W)$ of length s starting at v_2 and containing no v_3 in the subgraph $P_{k(2l+1),2l+1}$ in G, since we can consider $P_{k(2l+1),2l+1}$ as symmetry on the middle which is an edge or some vertices. Hence $\sigma_s(v_3) \leq \sigma_s(v_2)$. Similarly, there exists an injective mapping ϕ from $\mathcal{W}(s, v_2)$ to $\mathcal{W}(s, v_1)$, which implies $\sigma_s(v_2) \leq \sigma_s(v_1)$. By Lemma 3.2, $x(v_3) \leq x(v_2) \leq x(v_1)$. Hence $\lambda(G) = x^T A(G)x = x^T A(G')x - 2(x(v_1) - x(v_3))x(v_4) \leq x^T A(G')x \leq \lambda(G')$. Moreover, if $\lambda(G) = \lambda(G')$, then $x(v_1) = x(v_3)$ and x is an eigenvector of $A(G')$ corresponding to $\lambda(G)$. It is a contradiction, since $\lambda(G)x \neq A(G')x$. Therefore $\lambda(G) < \lambda(G')$. ■

Lemma 3.4. *Let $n = k\alpha > 2\alpha$ and $G \in \mathcal{T}_{n,\alpha}$ be a graph with two vertices u and v which are in clique of order k. If u is adjacent with u_1,u_2,..., u_t which belong to t vertex disjoint clique paths P_{kl_1,l_1},P_{kl_2,l_2},..., $P_{kl_t,l_t}(t > 1)$ respectively, and $d_G(u) - t = d_G(v) = k - 1$. Let G' be the graph obtained from G by deleting the edge u_1u and adding edge u_1v. Then $\lambda(G') < \lambda(G)$.*

Proof. Let x be the Perron vector of $A(G')$, then $x(u) \leq x(v)$ or $x(u) \geq x(v)$. If $x(u) \leq x(v)$, then deleting edges $uu_2, ..., uu_t$ and adding edges $vu_2, ..., vu_t$ get the graph G, then $\lambda(G) \geq x^T A(G)x \geq x^T A(G')x = \lambda(G')$ with equality if and only if x is the eigenvector of $A(G)$, but it is easy to see that x is not the eigenvector of $A(G)$, so $\lambda(G') < \lambda(G)$; If $x(u) \geq x(v)$, then deleting edges vu_1 and adding edges uu_1 get the graph G, then $\lambda(G) \geq x^T A(G)x \geq x^T A(G')x = \lambda(G')$ with equality if and only if x is an eigenvector of $A(G)$ corresponding to $\lambda(G)$. It is also easy to see that x is not an eigenvector of $A(G)$, which implies $\lambda(G') < \lambda(G)$. This completes the proof. ■

Lemma 3.5. *Let $n = k\alpha > 2\alpha$ and $H_{p,l} \in \mathcal{T}_{n,\alpha}$ be a graph obtained by joining two edges from two non-cut vertices of a graph $H \in \mathcal{T}_{n-k(l+p),\alpha-(l+p)}$ with a non-cut vertex of $P_{kp,p}$ and a non-cut vertex of $P_{kl,l}$, $H \neq K_k$ and $p \geq l \geq 1$(see Fig.3). Then $\psi(H_{p,l}, x) < \psi(H_{p+1,l-1}, x)$ for $x \geq \lambda(H_{p+1,l-1})$. Furhter $\lambda(H_{p,l}) > \lambda(H_{p+1,l-1})$.*

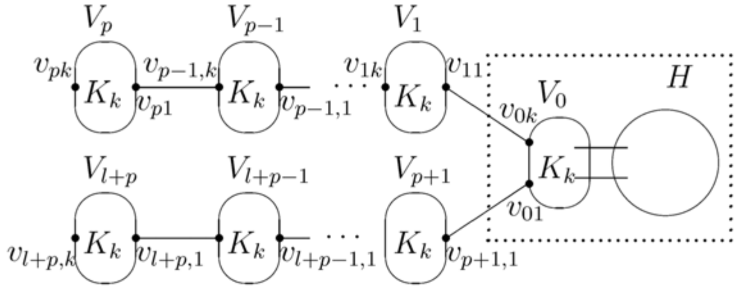

Fig. 3.

Proof. Let $H^1_{p,l}$ be the graph obtained from $H_{p,l}$ by deleting the vertex v_{pk} and the edges incident with v_{pk}, $H^2_{p,l}$ be the graph obtained from $H_{p,l}$ by deleting the vertex $v_{l+p,k}$ and the edges incident with $v_{l+p,k}$ in $H_{p,l}$ and $H^3_{p,l}$ be the graph obtained from $H_{p,l}$ by deleting the vertices v_{pk} and $v_{l+p,k}$ and the edges incident with them. Using Lemma 2.2, we can get

$$\begin{aligned}\psi(H_{p,l},x)-\psi(H_{p+1,l-1},x) &= \psi(H_{p,l-1},x)\psi(K_k,x)-\psi(H^2_{p,l-1},x)\psi(K_{k-1},x)\\ &\quad -\psi(H_{p,l-1},x)\psi(K_k,x)+\psi(H^1_{p,l-1},x)\psi(K_{k-1},x)\\ &= (\psi(H^1_{p,l-1},x)-\psi(H^2_{p,l-1},x))\psi(K_{k-1},x).\end{aligned}\tag{1}$$

Using Lemma 2.2 again, we have

$$\begin{aligned}&\psi(H^1_{p,l-1},x)-\psi(H^2_{p,l-1},x)\\ =\ &\psi(H_{p-1,l-1},x)\psi(K_{k-1},x)-\psi(H^1_{p-1,l-1},x)\psi(K_{k-2},x)\\ &-\psi(H_{p,l-2},x)\psi(K_{k-1},x)+\psi(H^2_{p,l-2},x)\psi(K_{k-2},x)\\ =\ &(\psi(H_{p-1,l-2},x)\psi(K_k,x)-\psi(H^2_{p-1,l-2},x)\psi(K_{k-1},x))\psi(K_{k-1},x)\\ &-(\psi(H_{p-1,l-2},x)\psi(K_k,x)-\psi(H^1_{p-1,l-2},x)\psi(K_{k-1},x))\psi(K_{k-1},x)\\ &-\psi(H^1_{p-1,l-2},x)\psi(K_k,x)\psi(K_{k-2},x)\\ &+\psi(H^3_{p-1,l-2},x)\psi(K_{k-1},x)\psi(K_{k-2},x)\\ &+\psi(H^2_{p-1,l-2},x)\psi(K_k,x)\psi(K_{k-2},x)\\ &-\psi(H^3_{p-1,l-2},x)\psi(K_{k-1},x)\psi(K_{k-2},x)\\ =\ &(\psi(H^1_{p-1,l-2},x)-\psi(H^2_{p-1,l-2},x))(\psi(K_{k-1},x)^2-\psi(K_{k-2},x)\psi(K_k,x))\\ =\ &(\psi(H^1_{p-1,l-2},x)-\psi(H^2_{p-1,l-2},x))(x+1)^{2k-4}.\end{aligned}\tag{2}$$

For $x \geq \lambda(H_{p+1,l-1})$, $(x+1)^{2k-4} > 0$ and $\psi(K_{k-1},x) > 0$, since K_{k-1} is a proper subgraph of $H_{p+1,l-1}$. So, for $x \geq \lambda(H_{p+1,l-1})$, $\psi(H_{p,l},x)-\psi(H_{p+1,l-1},x)$ has the same sign as $\psi(H^1_{p,l-1},x)-\psi(H^2_{p,l-1},x)$, $\psi(H^1_{p-1,l-2},x)-\psi(H^2_{p-1,l-2},x)$, ... , $\psi(H^1_{p-l+1,0},x)-\psi(H^2_{p-l+1,0},x)$, by formula (1) and (2). In addition,

$$\psi(H^1_{p-l+1,0},x)=det(xI-A(H^1_{p-l+1,0}))=det\left(xI-\begin{matrix} & v_{01}\\ v_{01} & \begin{pmatrix}A_1 & B_1\\ B_1^T & 0\end{pmatrix}\end{matrix}\right),$$

$$\psi(H^2_{p-l+1,0},x)=det(xI-A(H^2_{p-l+1,0}))=det\left(xI-\begin{matrix} & v_{p-l+1,k}\\ v_{p-l+1,k} & \begin{pmatrix}0 & B_2^T\\ B_2 & A_1\end{pmatrix}\end{matrix}\right),$$

where A_1 is the adjacent matrix of the graph H' obtained from $H_{p-l+1,0}$ by deleting the vertices $v_{01}, v_{p+1-l,k}$ and the edges which are incident with them. Then

$$\psi(H^1_{p-l+1,0}, x) = det(xI - A_1)(x - B_1^T(xI - A_1)^{-1}B_1),$$

$$\psi(H^2_{p-l+1,0}, x) = det(xI - A_1)(x - B_2^T(xI - A_1)^{-1}B_2),$$

and $\psi(H^1_{p-l+1,0}, x) - \psi(H^2_{p-l+1,0}, x) = det(xI - A_1)(B_2^T(xI - A_1)^{-1}B_2 - B_1^T(xI - A_1)^{-1}B_1)$. $\lambda(H_{p+1,l-1}) > \lambda(H')$, since H' is a proper subgraph of $H_{p+1,l-1}$. Thus $(xI - A_1)^{-1} = \frac{1}{x}(I + \frac{A_1}{x} + \frac{A_1^2}{x^2} + \frac{A_1^3}{x^3} + ...)$, for $x \geq \lambda(H_{p+1,l-1})$, by using Lemma 2.3 and $\lambda(H_{p+1,l-1}) > \lambda(H')$. It is sufficient to prove that $B_2^T A_1^t B_2 \leq B_1^T A_1^t B_1, t = 0, 1, 2, ...$ with at least one strictly inequality for some t. For $t = 0$, $B_2^T B_2 \leq B_1^T B_1$ holds, since $k - 1 = B_2^T B_2 \leq B_1^T B_1$; For $t = 1$, $B_2^T A_1 B_2 \leq B_1^T A_1 B_1$ holds, since $(k-1)(k-2) = B_2^T A_1 B_2 \leq B_1^T A_1 B_1$; For $t > 1$, $B_2^T A_1^t B_2, B_1^T A_1^t B_1$ ares the number of walks in $\mathcal{W}_2(t), \mathcal{W}_1(t)$, respectively, where $\mathcal{W}_2(t), \mathcal{W}_1(t)$ are the sets of walks of length t in H' from the vertices $N_{H_{p+1-l,0}}(v_{p+l-1,k})$ to $N_{H_{p+1-l,0}}(v_{p+l-1,k})$ and from vertices $N_{H_{p+1-l,0}}(v_{01})$ to $N_{H_{p+1-l,0}}(v_{01})$, respectively. Hence $B_2^T A_1^2 B_2 < B_1^T A_1^2 B_1$, since $H \neq K_k$. For $t > 2$, there exists a injective map φ_t from $\mathcal{W}_2(t)$ to $\mathcal{W}_1(t)$. In fact, let $W \in \mathcal{W}_2(t)$. If $V(W) \cap V(H) = \phi$, then there exists a walk W' in $H'[\cup_{i=0}^{p-l+1} V_i - \{v_{01}, v_{p-l+1,k}\}]$ such that $W' \in \mathcal{W}_1(t)$, since we can consider the graph $H'[\cup_{i=0}^{p-l+1} V_i - \{v_{01}, v_{p-l+1,k}\}]$ as symmetry on the middle which is an edge or some vertices, and let $\varphi_k(W) = W'$. If $V(W) \cap V(H) \neq \phi$, let $W = uP_1 v_{p-l+1,1} P_2 v_{0k} P_3 v_{0k} P_4 v_{p-l+1,1} P_5 v$, where $u, v \in N_{H_{p+1-l,0}}(v_{p+l-1,k})$, $V(P_2) \cap V_{p-l+1} = \phi$, $V(P_2) \cap V_0 = \phi$, $V(P_4) \cap V_{p-l+1} = \phi$ and $V(P_4) \cap V_0 = \phi$. Then there is a walk $W' = u' P_1' v_{0k} P_3 v_{0k} P_4 v_{p-l+1,1} P_2 v_{0k} P_5' v'$ such that $\varphi_t(W) = W' \in \mathcal{W}_1(t)$, where P_1', P_5' are obtained by the symmetry of the graph $H'[\cup_{i=0}^{p-l+1} V_i - \{v_{01}, v_{p-l+1,k}\}]$. By the definition, φ_t is an injective map for $t = 3, 4, \cdots$. Then $\psi(H^1_{p-l+1,0}, x) - \psi(H^2_{p-l+1,0}, x) < 0$ for $x \geq \lambda(H_{p+1,l-1})$. This completes the proof. ∎

Corollary 3.6. *Let $n = k\alpha > 2\alpha$ and $H_{p,l} \in \mathcal{T}_{n,\alpha}$ be a graph satisfied the condition of Lemma 3.5. Then $\lambda(H_{p,l}) > \lambda(H_{p+l,0})$.*

Proof. By Lemma 3.5, $\lambda(H_{p,l}) > \lambda(H_{p+1,l-1}) > ... > \lambda(H_{p+l,0})$. ∎

Now we are ready to prove the Theorem 1.4.

Proof. It is sufficient to prove that $P_{n,\alpha}$ is the unique graph with the minimum spectral radius in $\mathcal{T}_{n,\alpha}$. Suppose $\lambda(G) = \lambda_{n,\alpha}$, by Lemma 3.1, $G \in \mathcal{T}_{n,\alpha}$. Next consider the following cases to prove the assertion:

Case 1. If there are two vertices u and v each of which has at least two pendent clique paths adjacent with. Suppose that u is adjacent with pendent clique paths P_1,

P_2 and v is adjacent with pendent clique paths P_3, P_4. Let l_1, l_2, l_3, l_4 be the lengths of the pendent clique paths P_1, P_2, P_3, P_4, respectively. Without loss of generality, let $l_1 \geq l_3 \geq l_4$. Then deleting the edge incident with P_4 and v, and adding it to the end of P_3 get a new graph G'. By Lemma 3.3, $\lambda(G) > \lambda(G')$, which is a contradiction with $\lambda(G) = \lambda_{n,\alpha}$.

Case 2. If there is a vertex u which has at least two pendent clique paths adjacent with, and there is not another vertex which has at least two pendent clique paths adjacent with. Suppose that u is adjacent with pendent clique paths P_1, P_2,..., P_t. Assume u is in the clique G_1 the size of which is k. The degree of $V(G_1)\backslash\{u\}$ is $k-1$, suppose $v \in V(G_1)\backslash\{u\}$. Then deleting some edge uw which is not in G_1 and adding edge vw gets the new graph G'. It is easy to see that $G' \in \mathcal{G}(n, \alpha)$. By Lemma 3.4, $\lambda(G) > \lambda(G')$, which contradicts with $\lambda(G) = \lambda_{n,\alpha}$.

Case 3. If there is not a vertex which has at least two pendent clique paths adjacent with. By Corollary 3.6, G must be a clique path.

By **Cases 1**, **2** and **3**, the assertion holds. This completes the proof. ■

4. Proof of Theorems 1.5

Proof. Let $G \in \mathcal{T}(n, \alpha)$ and x be the Perron vector of G. Then we consider the following cases:

Case 1. There is a clique G_1 of order k in G which has two vertices u and v whose degrees are both larger than k. Without loss of generality, let $x(u) \geq x(v)$. Then deleting the edges incident with w_k not in G_1 and adding them to u to get a new graph G', then $\lambda(G') \geq x^T A(G')x \geq x^T A(G)x = \lambda(G)$, by Rayleigh quotient principle, with equality holding if and only if x is the eigenvector of $A(G')$. It is easy to see that x is not the eigenvector of $A(G')$, which implies $\lambda(G') > \lambda(G)$.

Case 2. For each clique G_1 of order k in G, there is only one vertex in G_1 whose degree is larger than $k-1$. For any graph H, let $E_1(H) = \{e = uv \in E(H) | d(u) > k, d(v) > k\}$ and $n(H) = |E_1(H)|$. Let $uv \in E_1(G)$, without loss of generality, suppose $x(u) \geq x(v)$ and $vu, vv_1, ..., vv_t$ are all the edges incident with v which are not in any clique of order k. Then deleting the edges $vv_1, ..., vv_t$ and adding edges $uv_1, ..., uv_t$ to get a new graph G', obviously $G' \in \mathcal{T}(n, \alpha)$ and $\lambda(G') \geq x^T A(G')x^T \geq x^T A(G)x^T = \lambda(G)$ with equality holding if and only if x is an eigenvector of $A(G')$. It is easy to find that x is not the eigenvector of $A(G')$, so $\lambda(G') > \lambda(G)$ and $n(G') < n(G)$.

Since $G \in \mathcal{T}(n, \alpha)$, then by **Cases 1** and **2**, it is easy to see that $\lambda(G) \leq \lambda(S_{n,\alpha})$ with equality holding if and only if $G = S_{n,\alpha}$. ■

ACKNOWLEDGMENT

The authors are grateful to the anonymous referee for the useful comments and suggestion to the earlier version of this paper.

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